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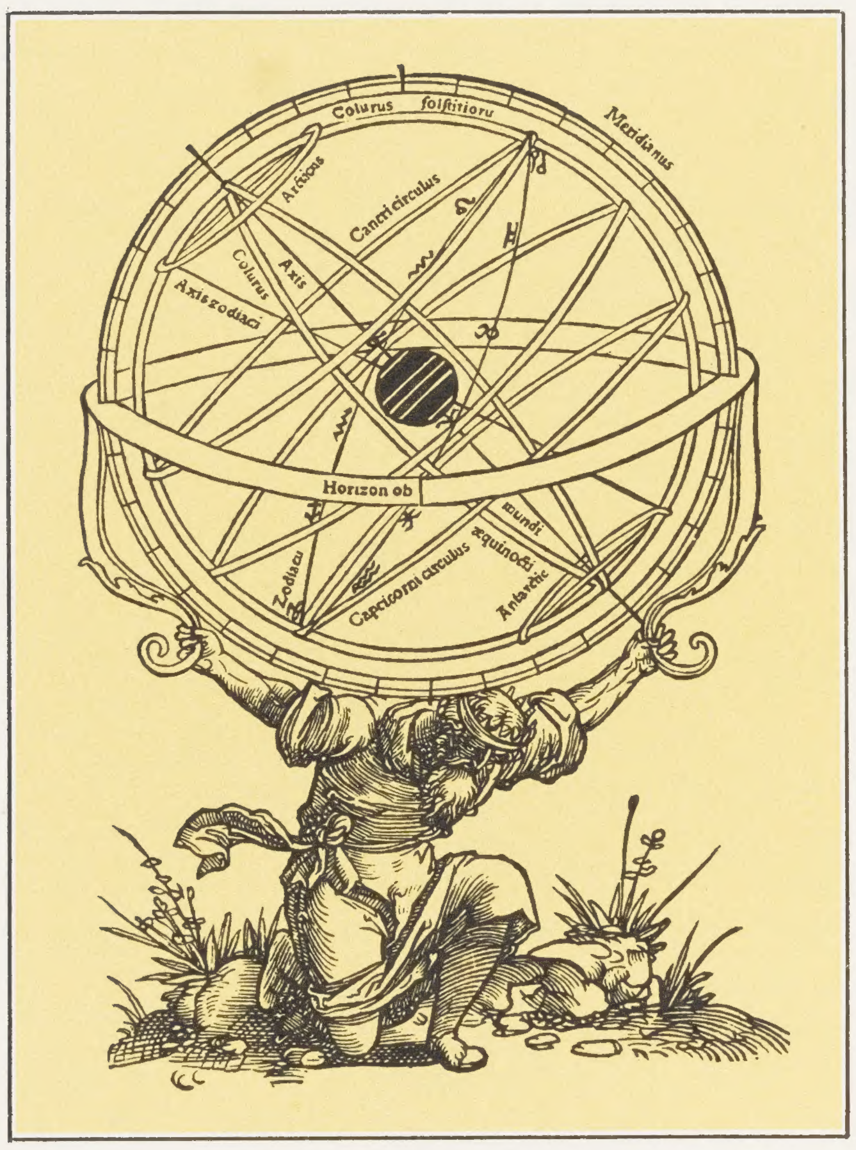

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March 6, 2018

ANALYTICAL INSTITUTIONS.

ANALYTICAL INSTITUTIONS,

IN FOUR BOOKS:

ORIGINALLY WRITTEN IN ITALIAN,

BY

DONNA MARIA GAETANA AGNESI,

PROFESSOR OF THE MATHEMATICKS AND PHILOSOPHY IN THE UNIVERSITY OF BOLOGNA.

TRANSLATED INTO ENGLISH

BY THE LATE

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AND LUCASIAN PROFESSOR OF THE MATHEMATICKS IN THE UNIVERSITY OF CAMBRIDGE.

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ANALYTICAL INSTITUTIONS.

BOOK II.

THE ANALYSIS OF QUANTITIES INFINITELY SMALL.

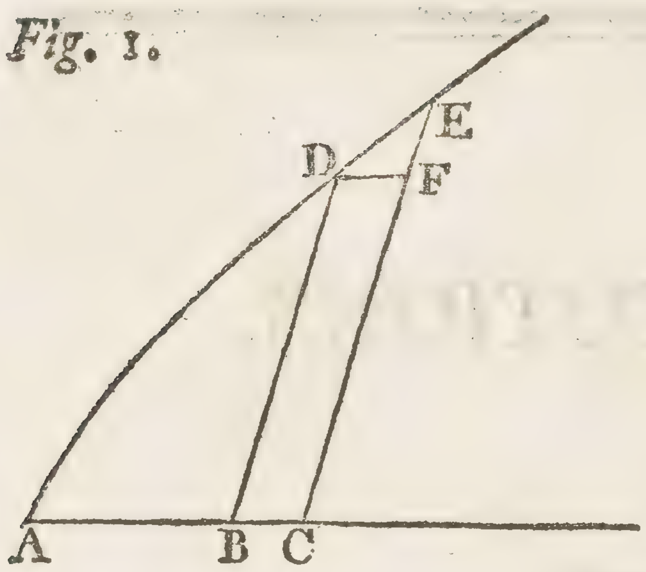
THE Analyſis of infinitely ſmall Quantities, which is otherwiſe called the *Differential Calculus*, or the *Method of Fluxions*, is that which is converſant about the differences of variable quantities, of whatever order thoſe differences may be. This Calculus contains the methods of finding the Tangents of Curve-Lines, of the *Maxima* and *Minima* of Quantities, of Points of Contrary Flexure, and of the Regreſſion of Curves, of the *Radii* of Curvature, &c.; and therefore we ſhall divide it into ſeveral Sections, as the nature of the ſeveral ſubjects may require. Introduction.

SECT. I.

Of the Notion or Notation of Differentials *of ſeveral Orders, and the Method of calculating with the ſame.*

1. By the name of *Variable Quantities* we underſtand ſuch, as are capable of continual increaſe or decreaſe, while others continue the ſame. They are to be conceived as *Flowing Quantities*, or as generated (as it were) by a continual motion. Variable quantities, what.

Fig. 1.



For inſtance, in Fig. 1, let there be a right line ABC, which is conceived as generated by the motion of the point A, and is produced *in infinitum*. Upon this, at any inclination, let another right line BD inſiſt, and let it be conceived that, whilſt the point B moves from B to C, carrying with it the line BD from the place BD to CE, always remaining parallel to itſelf, the point D ſhall deſcribe the line FE in ſuch a manner, as to paſs through all the points of the curve ADE. It is plain that the abſciſſes AB, AC, as alſo the ordinates BD, CE, and likewiſe the arches AD, AE, will be quantities continually increaſing and decreaſing, and therefore are called *Variable Quantities*, or *Fluents*, or *Flowing Quantities*.

Conſtant quantities, what.

2. *Conſtant Quantities* are ſuch, which neither increaſe nor diminiſh, but are conceived as invariable and determinate, while others vary. Such are the parameters, diameters, axes, &c. of curve-lines.

Conſtant quantities are repreſented by the firſt letters of the alphabet, *a, b, c, d,* &c. and variable quantities by the laſt letters, *z, y, x, v,* &c. juſt as is uſually done in the common Algebra, in reſpect to known and unknown quantities.

A fluxion or difference, what.

3. Any infinitely little portion of a variable quantity is called it's *Difference* or *Fluxion*; when it is ſo ſmall, as that it has to the variable itſelf a leſs proportion than any that can be aſſigned; and by which the ſame variable being either increaſed or diminiſhed, it may ſtill be conceived the ſame as at firſt.

Fig. 2.

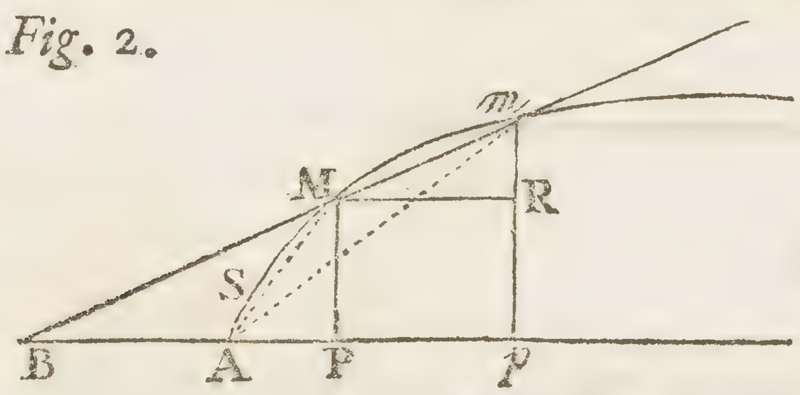
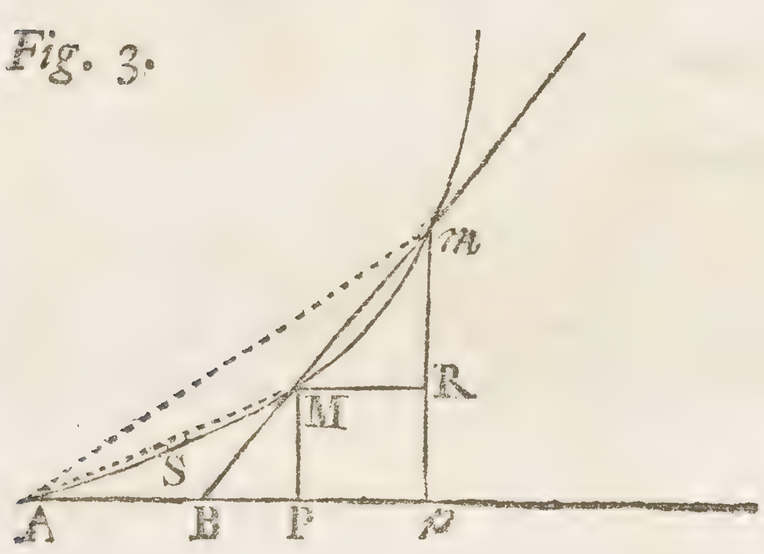


Fig. 3.

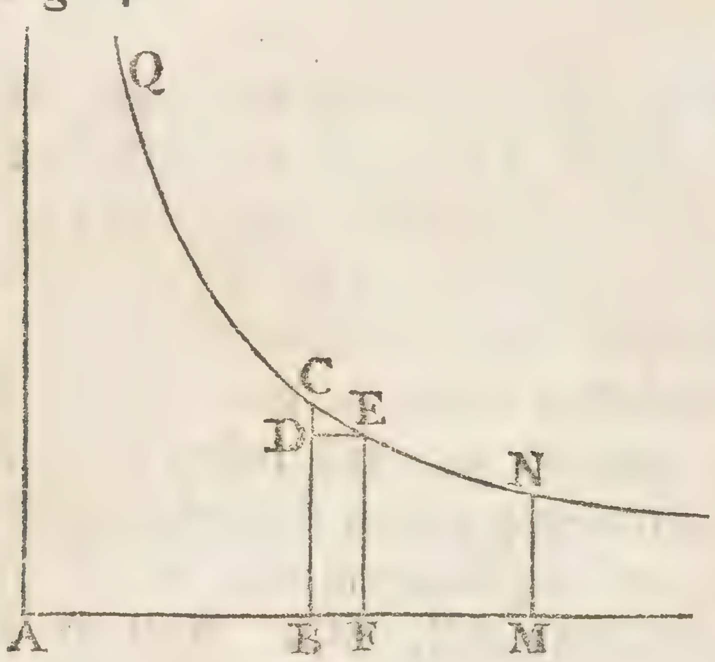


Let AM (Fig. 2, 3.) be a curve whoſe axis or diameter is AP; and if, in AP produced, we take an infinitely little portion P*p*, it will be the difference or fluxion of the abſciſs AP, and therefore the two lines AP, A*p*, may ſtill be conſidered as equal, there being no aſſignable proportion between the finite quantity AP, and the infinitely little portion P*p*. From the points P, *p*, if we raiſe the two parallel ordinates PM, *pm*, in any angle, and draw the chord *m*M produced to B, and the right line MR parallel to AP; then, becauſe the two triangles BPM, MR*m*, are ſimilar, it will be BP . PM :: MR . R*m*. But the two quantities BP, PM, are finite, and MR is infinitely little; then,

then alſo Rm will be infinitely little, and is therefore the fluxion of the ordinate PM. For the ſame reaſon, the chord Mm will be infinitely little; but (as will be ſhown afterwards,) the chord Mm does not differ from it's little arch, and they may be taken indifferently for each other; therefore the arch Mm will be an infinitely little quantity, and conſequently will be the fluxion or difference of the arch of the curve AM. Hence it may be plainly ſeen, that the ſpace PMmp likewiſe, contained by the two ordinates PM, pm, by the infiniteſimal Pp, and by the infinitely little arch Mm, will be the fluxion of the area AMP, comprehended between the two co-ordinates AP, PM, and the curve AM. And drawing the two chords AM, Am, the mixtilinear triangle MAm will be the fluxion of the ſegment AMS, comprehended by the chord AM, and by the curve ASM.

4. The mark or characteriſtic by which Fluxions are uſed to be expreſſed, is by putting a point over the quantity of which it is the fluxion. Thus, if the abſciſs AP $= x$, then will it be Pp or MR $= \dot{x}$. And, in like manner, if the ordinate PM $= y$, then it will be Rm $= \dot{y}$. And making the arch of the curve ASM $= s$, the ſpace APMS $= t$, the ſegment AMS $= u$, it will be Mm $= \dot{s}$, PMmp $= \dot{t}$, AMm $= \dot{u}$. And all theſe are called *Firſt Fluxions*, or *Differences* of the *firſt Order*. And it may be obſerved, that the foregoing fluxions are written with the affirmative ſign + if their flowing quantities increaſe, and with the negative ſign − if they decreaſe. Thus, in the curve NEC, (Fig. 4.) becauſe AB $= x$, BF $= \dot{x}$, BC $= y$, it will be DC $= -\dot{y}$, the negative fluxion of y. How fluxions are repreſented, and what are their ſeveral orders.

Fig. 4.



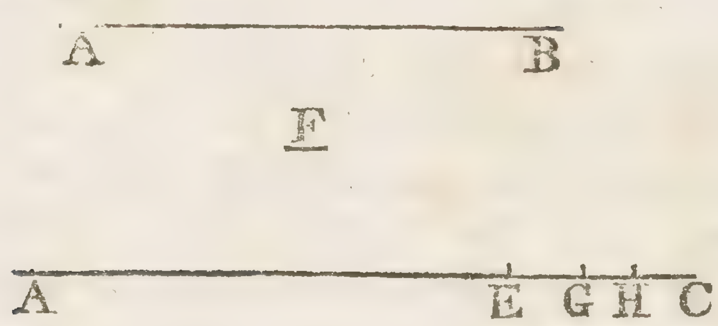
That theſe differential quantities are real things, and not merely creatures of the imagination, (beſides what is manifeſt concerning them, from the methods of the Ancients, of polygons inſcribed and circumſcribed,) may be clearly perceived from only conſidering that the ordinate MN (Fig. 4.) moves continually approaching towards BC, and finally coincides with it. But it is plain, that, before theſe two lines coincide, they will have a diſtance between them, or a difference, which is altogether inaſſignable, that is, leſs than any given quantity whatever. In ſuch a poſition let the lines BC, FE, be ſuppoſed to be, and then BF, CD, will be quantities leſs than any that can be given, and therefore will be *inaſſignable*, or *differentials*, or *infiniteſimals*, or, finally, *fluxions*.

Thus, by the common Geometry alone, we are aſſured that not only theſe infinitely little quantities, but infinite others of inferior orders, really enter the compoſition of geometrical extenſion. If incommenſurable quantities exiſt in Geometry, which are infinites in their kind, as is well known to Geometricians

 and

and Analyſts, then infiniteſimal magnitudes of various orders muſt neceſſarily be admitted.

Fig. 5.



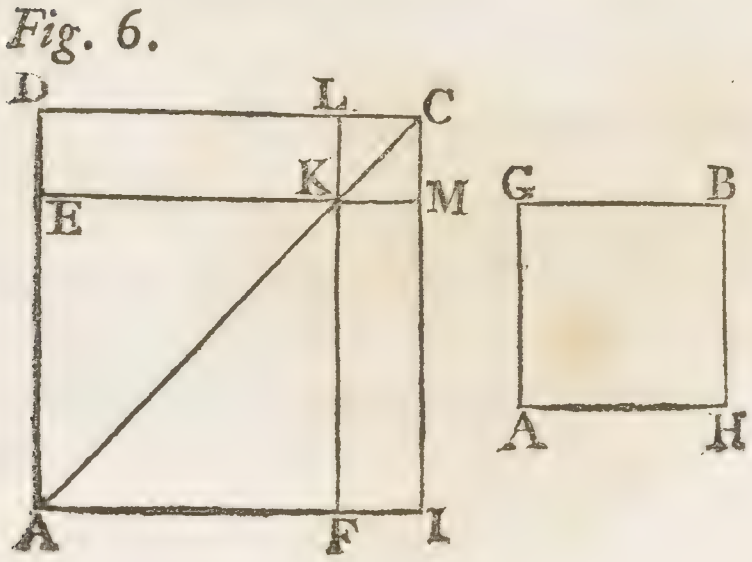
For the ſake of an example, let AB be the ſide of a ſquare, and AC it's diagonal or diameter; which two lines (by the laſt propoſition of the tenth Book of *Euclid*,) are incommenſurable to each other. Now it may be proved that this aſymmetry of their's does not proceed from any little finite line CE, how ſmall ſoever it may be taken, but from another which is infinitely leſs than it, and therefore of the infiniteſimal order.

Let it be ſuppoſed then, if poſſible, that it is the finite line CE which is the cauſe of the aſymmetry or incommenſurability between the two lines AB, AC; conſequently the remaining line AE will be commenſurable to the ſide AB. Let the right line F be their common meaſure, which can never be equal to EC, for then the diameter and ſide would be commenſurable. It muſt therefore be either greater or leſs than it.

In the firſt caſe, let F be ſubtracted from CE as often as can be done, and let the remainder be CG. Now, becauſe F meaſures AB, AE, and alſo EG, the two right lines AB, AG, will have to each other a rational proportion; and therefore it was not the magnitude CE that made the lines AB, AC, incommenſurable, but ſome quantity leſs than it, ſuppoſe GC, which therefore is finite, the finite line F being once or oftener ſubtracted from the finite line CE. Let F be biſected, and each part biſected again, and ſo on, till there ariſe an aliquot part of F which is leſs than CG, and which being taken from CG, there will remain CH. But this, by the ſame way of argumentation, is not the quantity that cauſes the incommenſurability of the lines AB, AC. And as the ſame way of reaſoning obtains in all other finite magnitudes, we may thence fairly conclude that the incommenſurability proceeds from an inaſſignable quantity, or which is leſs than any that can be given. The ſame may be alſo proved in the other caſe, or when the common meaſure F is greater than CE.

From hence I ſhall proceed, further, to take notice, that the ſquares upon the right lines AB, AC, which are to each other as one to two, notwithſtanding that their ſides are irrational, are nevertheleſs commenſurable to each other; and that this proceeds from an infinitely little quantity of the ſecond order. The two ſquares AB, AC, being propoſed, (Fig. 6.) let the two quantities ED, FI, equal and infiniteſimal, be thoſe which render the ſides AD, AG, AI, AH, incommenſurable; and the conſtruction being completed as in the figure, it is known that the two rectangles DK, IK, are incommenſurable

Fig. 6.



menfurable to the fquare AB. But the whole fquare AC is to the other AB in a rational proportion: therefore the fquare AC is made fo by the infinitefimal fquare KC, a quantity of the fecond order, by which it exceeds the faid incommenfurable gnomon.

It may be obferved, that cubes upon the lines AI, AH, are incommenfurable, although their bafes are rational; and it may be eafily proved, that they are made fuch by means of an inaffignable magnitude of the third order, and we may go on in like manner as far as we pleafe.

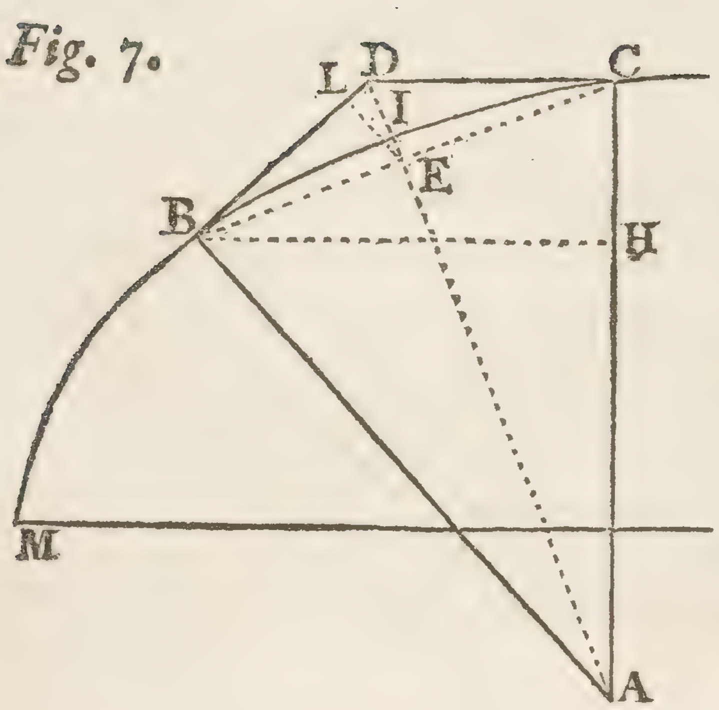
5. After the fame manner that firft differences or fluxions have no affignable proportion to finite quantities; fo differences or fluxions of the fecond order have no affignable proportion to firft differences, and are infinitely lefs than they: fo that two infinitely little quantities of the firft order, which differ from each other only by a quantity of the fecond order, may be affumed as equal to each other. The fame is to be underftood of third differences or fluxions in refpect of the fecond; and fo on to higher orders. How higher orders of fluxions are reprefented.

Second fluxions are ufed to be reprefented by two points over the letter, third fluxions by three points, and fo on. So that the fluxion of $\dot{x}$, or the fecond fluxion of x, is written thus, $\ddot{x}$; where it may be obferved, that $\ddot{x}$ and $\dot{x}^2$ are not the fame, the firft fignifying (as faid before,) the fecond fluxion of x, and the other fignifying the fquare of $\dot{x}$. The third fluxion of x will be $\dddot{x}$, and fo on. Thus, $\ddot{y}$ will be the fluxion of $\dot{y}$, or the fecond fluxion of y; and fo of others.

But, to give a juft idea of fecond, third, &c. fluxions, the following Theorems will be convenient.

THEOREM I.

Fig. 7.



6. Let there be any curve MBC, and BC an infinitely little portion of it of the firft order. From the points B, C, let the right lines BA, CA, be drawn perpendicular to the curve, and meeting in A. I fay, the lines BA, CA, may be affumed as equal to each other. Infinitefimals proved to exift.

Let the tangents BD, CD, be drawn, and the chord BC. If the two lines BA, CA, be unequal, let one of them, as CA, be the greater, and to this let the perpendicular BH be

be drawn. The difference between the lines BA, CA, will be lefs than the intercepted line CH, which is lefs than the chord CB, becaufe of the right angle at H. But the chord BC is an infinitefimal of the firft order, the arch being fuppofed an infinitefimal; therefore the difference between BA and CA, at leaft, will not be greater than an infinitefimal of the firft order, and therefore thofe lines BA and CA may be affumed as equal.

Coroll. I. Therefore the triangle BAC will be equicrural, and thence the angles at the bafe ABC, ACB, will be equal; and being fubtracted from the right angles ABD, ACD, will leave the two angles BCD, DBC, equal to each other, and confequently the two tangents BD, CD, will be equal.

Coroll. II. The right line DA being drawn, the two triangles ADB, ADC, will be equal and fimilar; and that line will bifect the angles BAC, BDC. And, becaufe the two triangles AEB, AEC, are fimilar and equal, the fame line AD will be perpendicular to BC, and will divide it into equal parts in E.

Coroll. III. And the two triangles DAC, EDC, being fimilar, the angle DCE will be equal to the angle DAC; and the two angles DCE, DBE, being taken together, will be equal to the angle BAC.

Coroll. IV. From hence it follows, that any infinitefimal arch BC, of any curve whatever, will have the fame affections and properties as the arch of a circle, defcribed on the centre A, with the radius AB or AC.

Coroll. V. The two triangles AEB, BED, being fimilar, we fhall have AE . [illegible] :: EB . ED. But AE is a finite line, and EB an infinitefimal of the firft order; therefore ED will be an infinitefimal of the fecond order, and it's value will be $= \frac{EBq}{AE}$. But the rectangle of twice AE into EI is equal to the fquare of EB, from the property of the circle. Therefore $EBq = 2AE \times EI = AE \times ED$, and confequently 2AE . AE :: ED . EI. But the firft term of the analogy is double to the fecond, therefore the third is double to the fourth. Confequently the two lines EI, DI, of the fecond order will be equal.

Coroll. VI. And therefore the difference between the femichord BE, and the tangent BD, is an infinitefimal of the third degree; for as much as from the centre B, and with the diftance BE, drawing the arch of a circle EL, a magnitude of the fecond clafs, which coincides with it's fine; the two triangles BDE, EDL, will be fimilar, which, befides the right angles at E and L, have a common angle in D. Thence it will be BD . DE :: DE . DL. But BD is a firft fluxion, DE is a fecond fluxion by the foregoing corollary, and therefore DL will be a third fluxion. Wherefore the arch of the curve BI being greater

 than

than the femichord BE, and lefs than the tangent BD, it cannot differ from either of them but by a magnitude of the third order.

THEOREM II.

Fig. 8.

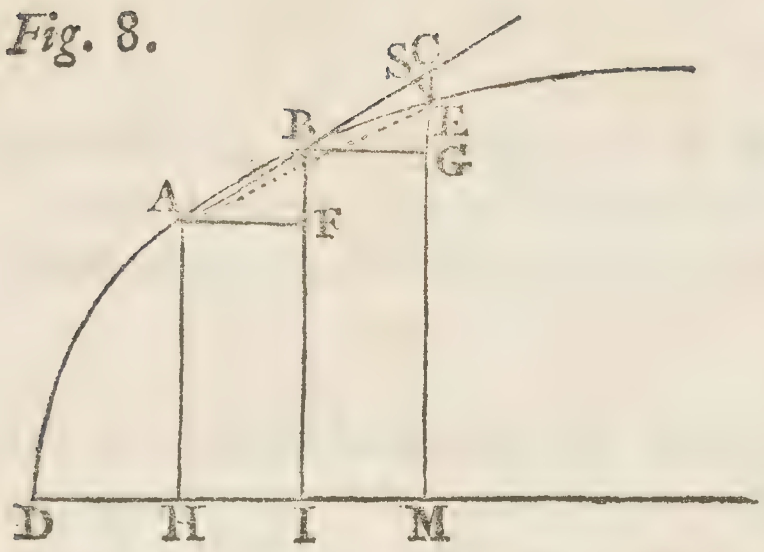
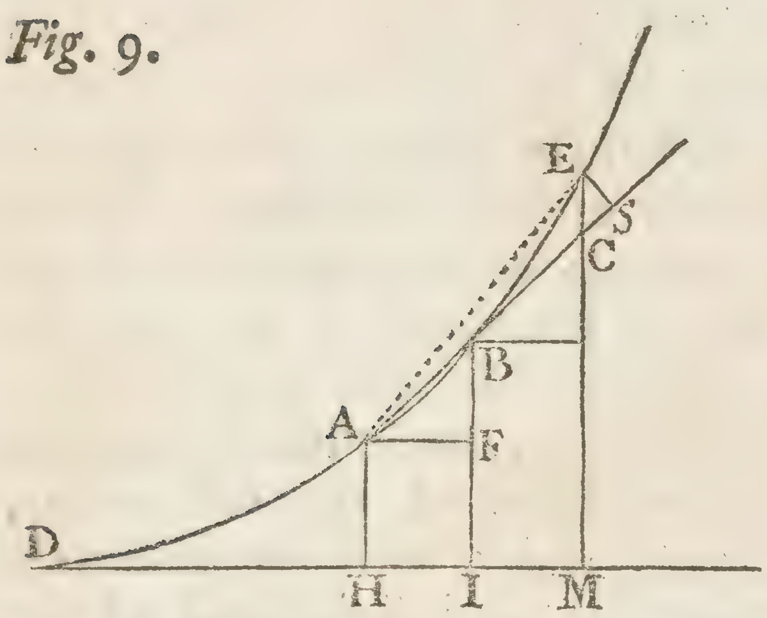


Fig. 9.



7. Let there be any curve whatever, DAE (Fig. 8, 9.), in whofe axis are taken two equal infinitefimal portions of the firft order HI, IM; let parallel ordinates HA, IB, ME, be drawn, which in the given curve fhall cut off the little arches AB, BE, which are likewife infinitefimals of the firft order. Let there be drawn the chord ABC, which fhall meet the ordinate produced, ME, in the point C. I fay, that the intercepted line CE, between the curve and the chord AB produced, fhall be an infinitefimal of the fecond order.

Let the chord AE be drawn. If the right line IM were a finite and affignable quantity, then the triangle ACE would alfo be finite. But ME continually approaching, [from a finite diftance,] to the ordinate HA, [while IB remains fixed,] fo that IM may alfo become a fluxion, or may be an infinitefimal of the firft order; the angle ACE always continuing the fame, the angle AEC increafes, making the angle CAE always lefs and lefs, till at laft it becomes lefs than any given angle, that is, an infinitefimal. In this cafe, as the fine of an infinitely little angle of the firft order, having a finite and affignable radius, is an infinitefimal quantity of the firft order; fo the fine of an infinitefimal angle, CAE, of the firft order, with a radius AE or AC, which is an infinitefimal quantity of the firft order, fhall be an infinitefimal quantity of the fecond order. But in triangles the fides are proportional to the fines of the oppofite angles, and therefore the right line CE fhall be an infinitefimal of the fecond order.

Wherefore, calling DH $= x$, HA $= y$, HI $=$ IM $= \dot{x}$; then FB $=$ GC $= \dot{y}$, and EC $= -\ddot{y}$; the negative fign being prefixed, becaufe $\dot{y}$ does not increafe but diminifh (Fig. 8.). And thus, on the contrary, it will have the pofitive fign if $\dot{y}$ increafe, that is, if the curve be convex in this point to the axis DM (Fig. 9.).

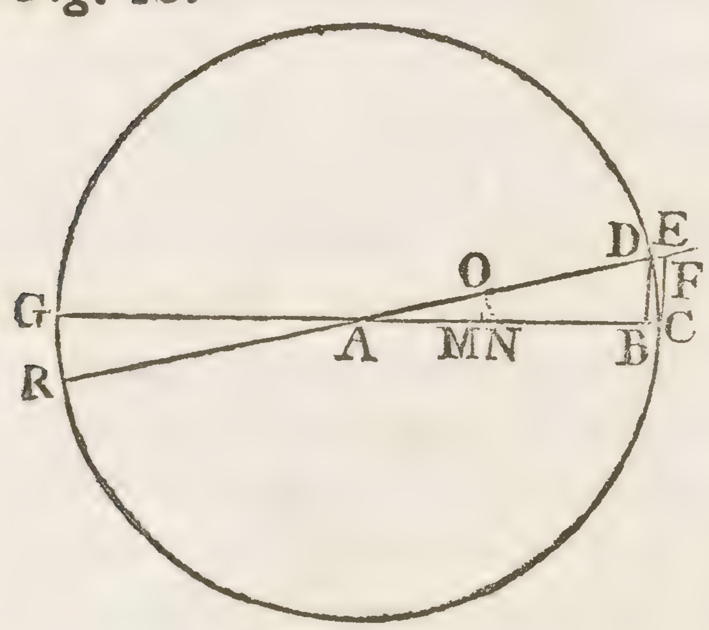
Coroll.

Coroll. If from the point E the normal ES be drawn to BC, then alſo ES, CS, will be the fluxions of the ſecond order; for each of them is leſs than EC.

THEOREM III.

8. If in the circle be taken an arch which is an infiniteſimal of the firſt order, I ſay, that it's verſed ſine ſhall be an infiniteſimal of the ſecond order; and the difference between the right ſine and the tangent ſhall be an infiniteſimal of the third order.

Fig. 10.



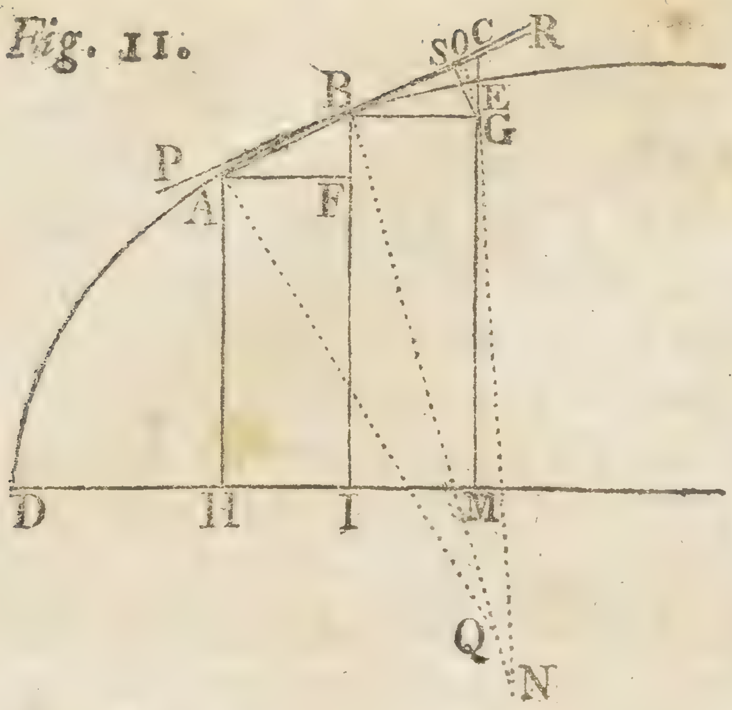
Let the arch DC be an infiniteſimal of the firſt order, DB it's right ſine, CE the tangent, and let DF be drawn parallel to AC. From the nature of the circle, it is GB . BD :: BD . BC. But GB is a finite quantity, and BD an infiniteſimal of the firſt order. Therefore, as GB is infinitely greater than BD, ſo BD will be infinitely greater than BC. Therefore BC or DF will be an infiniteſimal of the ſecond order. By the ſimilitude of the triangles ABD, DEF, it will be AB . BD :: DF . FE. But AB, a finite quantity, is infinitely greater than BD, an infiniteſimal of the firſt order, and therefore DF, an infiniteſimal of the ſecond order, will be infinitely greater than FE, which is therefore a third fluxion, or an infiniteſimal of the third order.

9. *Coroll.* I. And whereas the tangent is always greater than it's arch, the arch greater than it's chord, and the chord greater than the right ſine, the tangent and the right ſine may be aſſumed as equal, they not differing but by an infiniteſimal of the third order. Alſo, theſe following may be aſſumed as equal, the tangent, the arch, the chord, and the right ſine.

10. *Coroll.* II. If we conceive the radius of the circle AN to be an infiniteſimal of the firſt order, the arch NO and it's right ſine OM will be infiniteſimals of the ſecond; and therefore the verſed ſine MN will be an infiniteſimal of the third order.

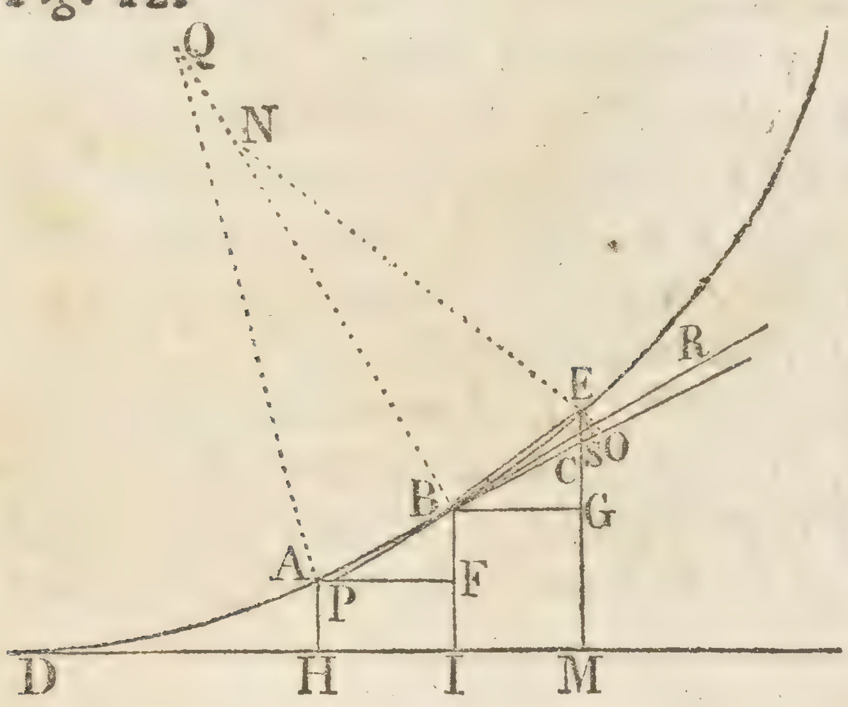
11. *Coroll.*

Fig. 11.



11. *Coroll.* III. In the axis DM (Fig. 11, 12.) let there be two first differences HI, IM, equal to each other, to which correspond the two infinitesimal arches AB, BE, of the curve DABE; and let be drawn the two chords BE, AB, of which this is produced till it meets in C the ordinate ME, produced also if necessary. Let ES be drawn perpendicular to BC, and from centre B, with radius BE, let the arch EO be drawn. By the corollary of Theor. II. CS is an infinitesimal of the second degree, and, by the foregoing, OS is an infinitesimal of the third degree. Then CO is also an infinitesimal of the second degree, because an infinitesimal of the third degree being added to, or subtracted from, an infinitesimal of the second degree, makes no alteration in it. Now, because HI = IM, or AF = BG, and, because of equal and similar triangles AFB, BGC, it will be also AB = BC. But the arches may be assumed equal to their chords; then CO will be the difference of the two arches AB, BE; and therefore, if the arch DA = s, it will be AB = BC = $\dot{s}$, and CO = — $\ddot{s}$ with a negative sign, because AB decreases when BE is less than AB, as in Fig. 11. And, on the contrary, with a positive sign, as in Fig. 12.

Fig. 12.



SCHOLIUM.

12. In determining the second differences (or fluxions) of the ordinate, and of the arch of the curve, I have supposed, both in Theor. II. and in this last corollary, that the first differences HI, IM, are equal; that is to say, that the first difference of the abscis does not alter, but remains constant, in which case the second difference of the abscis is none at all. So that, calling the abscis x, it's first difference will be $\dot{x}$, and it's second $\ddot{x} = 0$.

Wherefore we may further make these two other conclusions, one of which is, that if the first difference of the ordinate be constant, those of the abscis and of the curve will be variable. The other is, that if the first difference of the curve be constant, those of the abscis and ordinate will be variable.

 Now,

Fig. 13.

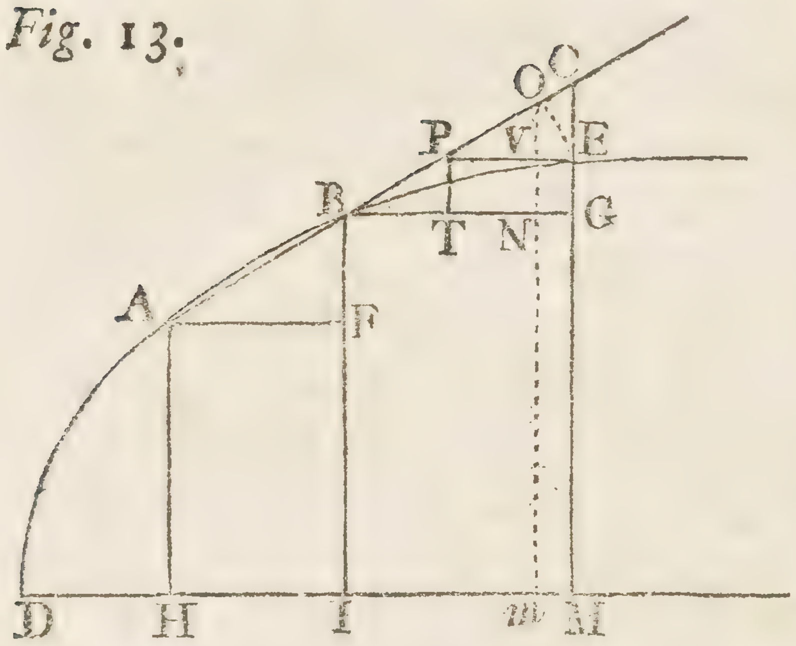
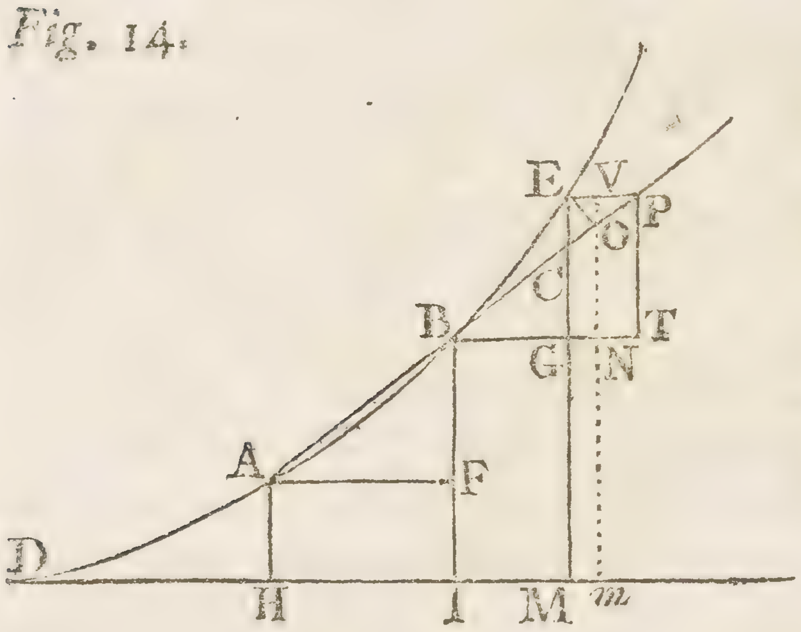


Fig. 14.



Now, theſe things being premiſed, we may eaſily proceed to theſe two other hypotheſes. Suppoſing what has been already advanced, let BF (Fig. 13, 14.) be equal to EG; that is, let the fluxion of the ordinate be conſtant; and let EP be drawn parallel to BG, and PT perpendicular to it. Then will BF = PT, and therefore AF = BT, AB = BP, and GT or EP will be the difference between HI and IM. And with centre B, diſtance BE, deſcribing the arch EO, PO will be the difference between the arch AB and the arch BE, becauſe the chords may be aſſumed inſtead of the infiniteſimal arches. But, becauſe of the ſimilar triangles BTP, CEP, we ſhall have PT . TB :: CE . EP, PT . PB :: CE . CP; and PT, TB, BP, are firſt fluxions, and CE is a ſecond fluxion; therefore EP, CP, and much more OP, will be ſecond fluxions. Whence, if DH $= x$, DA $= s$, it will be TG = PE $= \ddot{x}$, PO $= \ddot{s}$, in Fig. 13, and PE $= -\ddot{x}$, PO $= -\ddot{s}$, in Fig. 14, and $\ddot{y} = 0$.

Let the firſt differential of the curve be conſtant, that is, AB = BE. From the point O let fall ON parallel to TP. Becauſe, by ſuppoſition, it is AB = BE = BO, it will be alſo AF = BN. Then VE or NG will be the difference between HI and IM. But it will be alſo FB = NO; then VO will be the difference between BF and EG. But it is plain that, EC being a fluxion of the ſecond order, EV and VO will be ſo too. Then, if it be DH $= x$, HA $= y$, it will be NG $= \ddot{x}$, OV $= -\ddot{y}$, in Fig. 13, and NG $= -\ddot{x}$, OV $= \ddot{y}$, in Fig. 14, and $\ddot{s} = 0$.

The ſuppoſition of a conſtant firſt fluxion makes calculations more ſhort and eaſy, as will be ſeen in applying it to uſe. However, on many occaſions, for the ſake of greater univerſality, we ſhall proceed from firſt to ſecond differences, without making the ſuppoſition of any conſtant firſt fluxion, which it will be always eaſy to determine.

Let HI, IM, (Fig. 15, 16.) be firſt fluxions of the abſciſs DH, though not preciſely equal to each other, and let their difference be ML, a ſecond fluxion. Let the reſt be as above, and draw the ordinate LN, and Ei parallel to BG. Therefore, LM being the difference of HI and IM, it will be HI = IL; that is, AF = BR; and therefore the triangles ABF, BRN, will be ſimilar and equal.

Fig. 15.

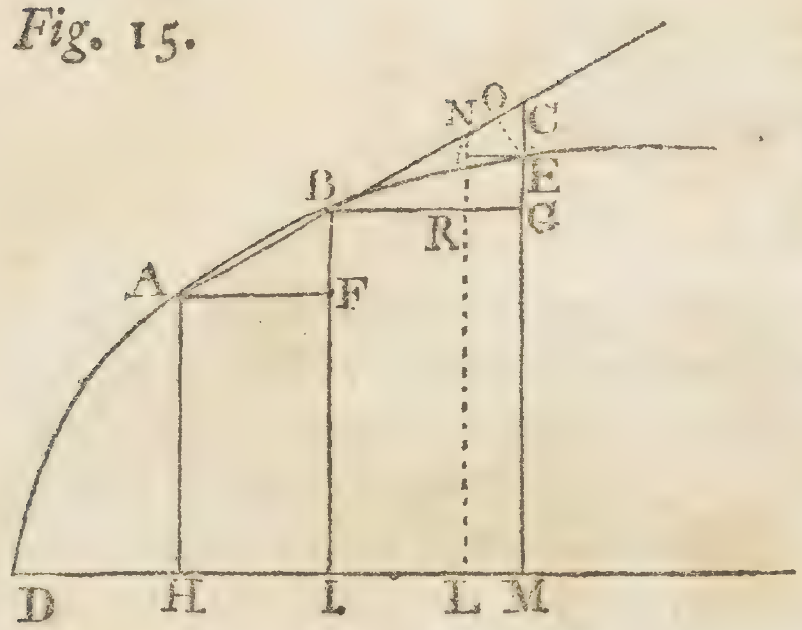
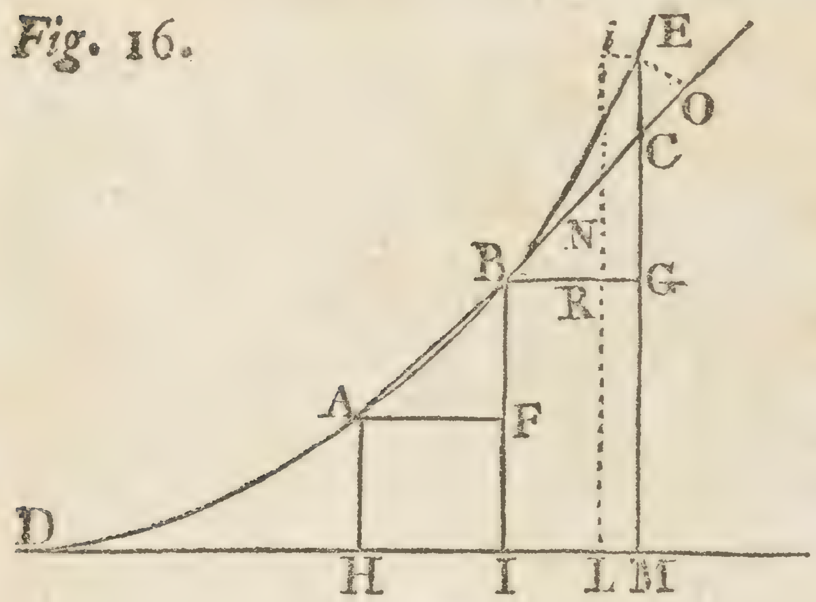


Fig. 16.



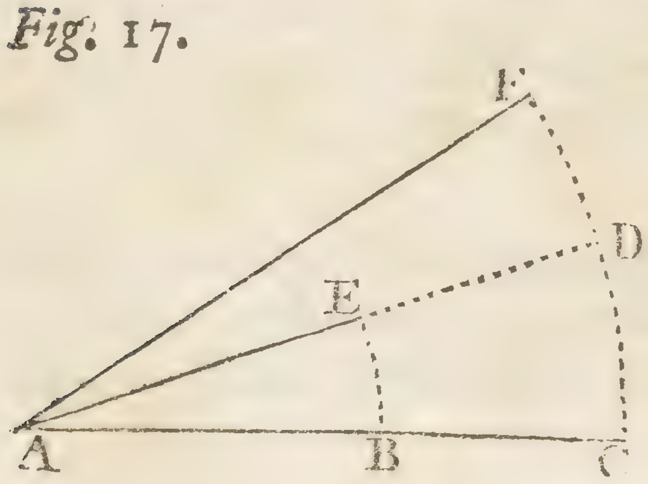
equal. Consequently BF = NR, and N*i* will be the difference between BF and EG; that is, the difference of BF, or the second difference of AH. In like manner, it will be AB = BN, and therefore NO will be the difference between the arch AB and the arch BE; and therefore the difference of the arch AB, or the second difference of the arch DA. Wherefore it is plain that N*i*, NO, are fluxions of the second degree. The same things will obtain, if, instead of supposing IM greater than HI by a second differential, we should suppose it less.

13. It is to be observed here, that the foregoing determinations do not require any restrictions concerning the angles of the co-ordinates, though the figures may seem to insinuate that they are at right angles; for the conclusions will be all the same, whatever the angles may be.

LEMMA.

14. Right-lined angles are to one another in a ratio compounded of the direct ratio of their arches, and the inverse ratio of their *radii*.

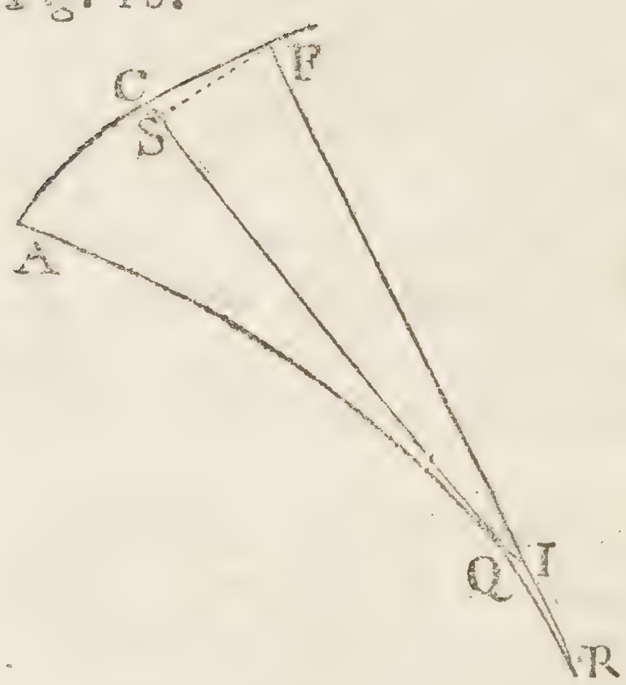
Fig. 17.



Let there be two angles EAB, FAC (Fig. 17.). Producing AE to D, from the similitude of the sectors ABE, ACD, it will be AB . BE :: AC . CD; therefore $CD = \frac{BE \times AC}{AB}$. But the angle EAB, or DAC, is to the angle FAC, as CD to CF; therefore the angle EAB will be to the angle FAC, as $\frac{BE \times AC}{AB}$ to CF; that is, as $\frac{BE}{AB}$ to $\frac{CF}{AC}$.

THEOREM IV.

Fig. 18.



15. Taking the arch CF, an infiniteſimal of the firſt degree, in any curve whatever ACF, and drawing CI, FI, perpendicular to the curve; with centre I, and radius IF, if we deſcribe the circular arch FS, I ſay that it will fall all within the curve ACF, towards C, and the intercepted line CS will be an infiniteſimal quantity of the third degree.

Upon the curve AQR a thread may be conceived to be ſtretched, ſo as that, being fixed in any point below, as in R, and taken by it's end in the point A, it may continually recede from the curve, but in ſuch a manner as to be always equally ſtretched, and with it's point A to deſcribe the curve ACF. Now, the thread being in the poſition CQ, it will be a tangent to the curve AQR in the point Q; and in the poſition FR, which I ſuppoſe to be infinitely near to CQ, it will be a tangent in R; then producing CQ, it will meet FR in I. Now, ſince, by the generation of the curve ACF, the right line QC is equal to the curve QA, and the right line RF to the curve RQA, and the two infinitely little tangents QI, RI, are together greater than the element QR; therefore CI, IR, taken together, will be greater than the curve RQA, or than the right line FR. Then, taking away the common IR, IC will be greater than IF, and therefore the circular arch FS, deſcribed with centre I and radius IF, will fall within the curve. But, by Theor. I. and III., the two tangents QI, RI, do not exceed the arch QR but by a third fluxion. Therefore the curve AQ, together with the right lines QI, IR, exceed the curve AQR, or the right line FR, by the ſame quantity. Then taking away the common IR, AQ, together with QI, that is, IC, will be greater than IF by an infiniteſimal of the third order.

16. *Coroll.* Therefore we may conceive the circular arch FS as coinciding with the arch of the curve FC; and one may be taken for the other indifferently. And the tangent RF will be perpendicular to the curve ACF in the point F, and QC in the point C.

The curve AQR is called the *Evolute*, the curve ACF is the *Involute*, or curve generated by the evolute; that is, produced by the unwinding of the ſtring or thread AQR; and the circle FS, deſcribed with centre I and radius IF, is the *Oſculating* or equicurved circle; alſo, IF is called the *Radius of Curvature* of the curve ACF in the point F.

THE-

THEOREM V.

17. If in the curve DABE (Fig. 11, 12.), at the points A, B, E, infinitely near, (that is, the arches AB, BE, being infinitesimals of the first order,) be drawn the perpendiculars QA, QB, and NE, which meet BQ in the point N; I say, that the angles AQB, BNE, may be assumed as equal.

For, by the foregoing Lemma, the angle AQB is to the angle BNE, as $\frac{AB}{AQ}$ is to $\frac{EB}{BN}$, that is, as $AB \times BN$ is to $EB \times AQ$. But the rectangle $EB \times AQ$ is not less than the rectangle $AB \times BN$, but only by the rectangle $BE \times QN$, and by the rectangle of BN into the difference of the arches AB, BE. And, as QN, BE, are infinitesimal quantities of the first degree, their rectangle will be an infinitesimal of the second degree; as also, the difference of the arches AB, BE, being an infinitesimal of the second degree, the rectangle of these into BN will be an infinitesimal of the second degree. Therefore the two rectangles $AB \times BN$ and EB into AQ do not differ from each other, but by two infinitesimal rectangles of the second degree, and therefore may be assumed as equal, and consequently the angles AQB, BNE.

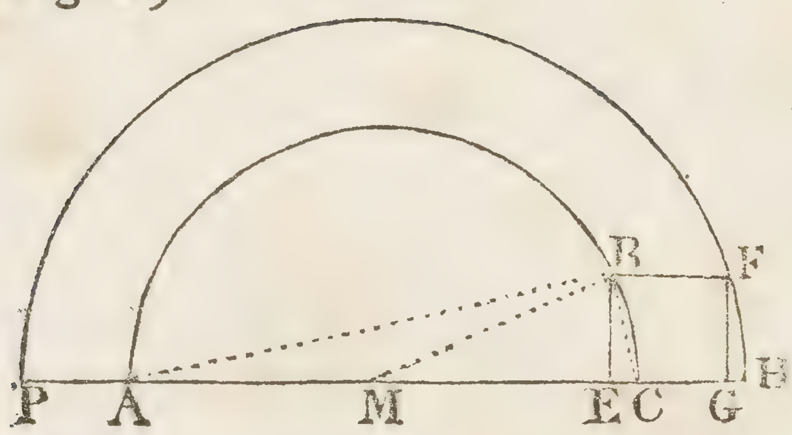
18. *Coroll.* I. If PBR be drawn a tangent at the point B, it will bisect the angle CBE, made by the two chords ABC and BE. For, by Theor. I. Coroll. III. the angle BQA being double to the angle PBA, to which the angle CBR is equal; thence the angle BNE shall be double to the angle CBR. But, by the same Corollary, the angle BNE is double to the angle RBE. Therefore the angles CBR, RBE, are equal.

19. *Coroll.* II. Therefore the angle CBE will be equal to the angle BNE, and thence the sector BNE will be similar to the sector EBO.

THEOREM VI.

20. If in two circles, the diameters of which exceed each other by a first infinitesimal, be taken two right sines equal to each other, and infinitesimals of the first degree, the difference of their versed signs shall be an infinitesimal of the third degree.

Fig. 19.

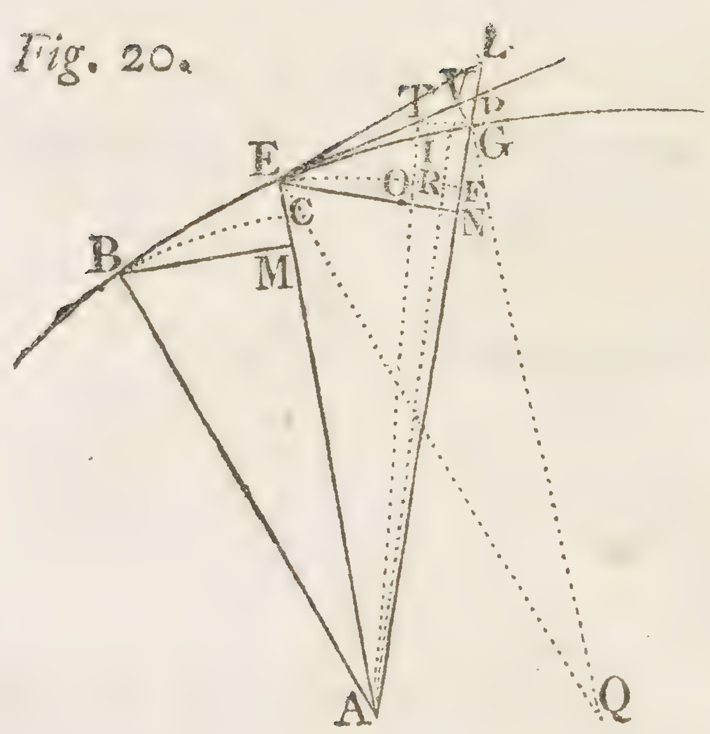


Let the two circles be ABC, PFH, and let the equal right ſines be BE, FG, infiniteſimals of the firſt degree, and the verſed ſines EC, GH. Let the chords AB, BC, be drawn. The ſine BE, and therefore the arch BC, being a firſt fluxion, the angle BME will be an infiniteſimal of the firſt order, and therefore alſo the angle BAC, which is the half of it, and the angle EBC, which is equal to this. Therefore, ſince the angle EBC, and the ſides EB, BC, are firſt infiniteſimals, the verſed ſine EC will be a ſecond infiniteſimal.

The ſame obtains of the verſed ſine GH. But the verſed ſine EC (by the property of the circle,) is found to be $\frac{EBq}{AE}$, and the verſed ſine GH $= \frac{GFq}{PG} = \frac{EBq}{PG}$. Therefore we ſhall have this analogy, EC . GH :: PG . AE. But PG, a finite quantity, exceeds AE, a finite quantity, by an infiniteſimal quantity in reſpect of itſelf, that is, of the firſt order, by hypotheſis. Therefore EC, an infiniteſimal quantity of the ſecond order, will exceed GH, an infiniteſimal of the ſecond order, by an infiniteſimal quantity in reſpect of itſelf, that is, of the third order.

THEOREM VII.

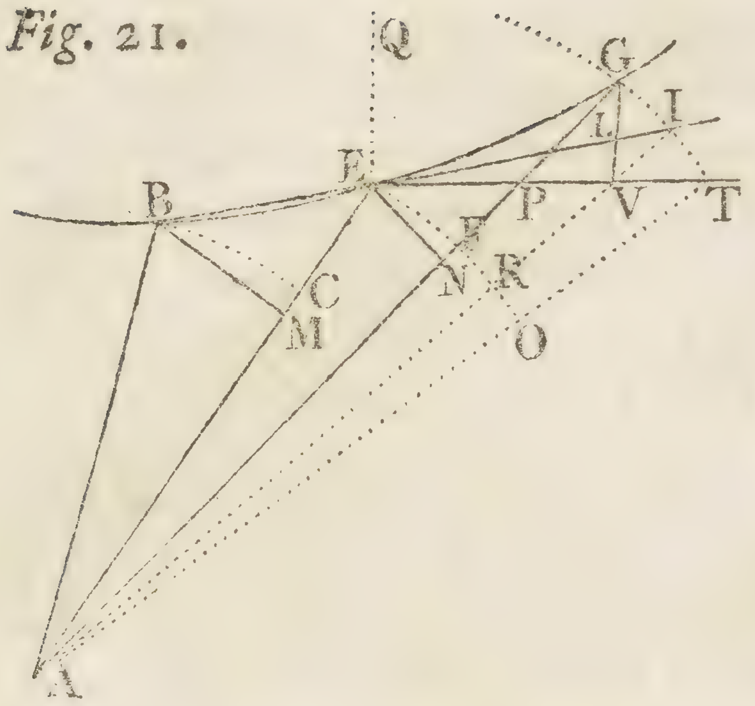
Fig. 20.



21. Let the curve BEG (Fig. 20, 21.) be referred to a *focus*, that is, ſuch, that all the ordinates proceed from a given point, which is called the *Focus*, and let this point be A. From hence let be drawn three ordinates, which are infinitely near, AB, AE, AG, which contain the two infinitely little arches of the firſt degree, BE, EG; and draw the chord BE, which, produced, meets the ordinate AG (produced if need be,) in the point L. With centre A let the arches BC, EF, be deſcribed, and let BM, EN, be their right ſines. Laſtly, make the angle NEP equal to the angle MBE. I ſay, that the intercepted line GP ſhall be the infinitely little difference of the ſecond order of the ordinate AB.

Let

Fig. 21.



Let the chord EG be drawn. Since the angles MBE, NEP, are equal by conſtruction, and the angles at M and N are right ones, the triangles EBM, PEN, will be ſimilar; then taking the ſine BM for conſtant, that is, ſuppoſing it equal to EN, the foreſaid triangles will alſo be equal. Therefore it will be ME = NP. But, ſuppoſing BM = EN, by the foregoing Theorem the difference of the verſed ſines MC, NF, is infiniteſimal in reſpect of them. Therefore, alſo, CE, FP, will be equal, and thence GP will be the difference between CE and FG. But the right lines EQ, QG, being drawn perpendicular to the curve in the points E, G, the angle LEG will be equal to the angle EQG, by Theor. V. Coroll. II. [which is true whether the curve be referred to an axis, or to a *focus*.] And the angle EQG is infinitely little. Therefore, alſo, the angle LEG will be infinitely little. And, becauſe the right lines EG, EL, are infiniteſimals of the firſt order, GL will be an infiniteſimal of the ſecond order; and much more GP, reſpect being had to Fig. 20.

By Theor. III. Coroll. I. the line BM is equal to the arch BC. Then, inſtead of the ſine, taking the arch for conſtant, and making it $= \dot{x}$, AB $= y$, CE $= \dot{y}$, it will be GP $= -\ddot{y}$. And with centre E, and diſtance EG, deſcribing the arch GV, it will be VP $= -\ddot{s}$, if BE $= \dot{s}$.

22. *Coroll.* The angle LEP will be equal to the angle EAG. For the angle EPA, by conſtruction, is equal to the angle BEA; but the external angle EPA is equal to the two internal angles L and LEP; and the other, BEA, is equal to the two, L and EAG. Then, taking away the common L, there will remain the two equal angles LEP, EAG. Wherefore this will be true, whether the curve be concave towards the point A, (Fig. 20.) or whether it be convex, (Fig. 21.) as it is eaſy to perceive. In the ſame Fig. 21, the angle LEP will be an infiniteſimal, and therefore LP is an infiniteſimal of the ſecond order. But it has been ſeen, that GL is alſo an infiniteſimal of the ſecond order. Therefore the whole, GP, will be ſo alſo, which will be $= \ddot{y}$; and with centre E, diſtance EG, the arch GV being deſcribed, it will be PV $= \ddot{s}$.

If we ſuppoſe $\dot{y}$ to be conſtant, with centre A, and diſtance AG, let the arch GT be deſcribed, and from the point T let the right line TOA be drawn. Becauſe FG = EC, by hypotheſis, the triangle TEO will be ſimilar and equal to the triangle EBC; and therefore BC = EO, and BE = ET. Then OF $= \ddot{x}$, and TV $= \ddot{s}$, in Fig. 20. But OF $= -\ddot{x}$, and TV $= -\ddot{s}$, in Fig. 21.

Taking

Fig. 22.

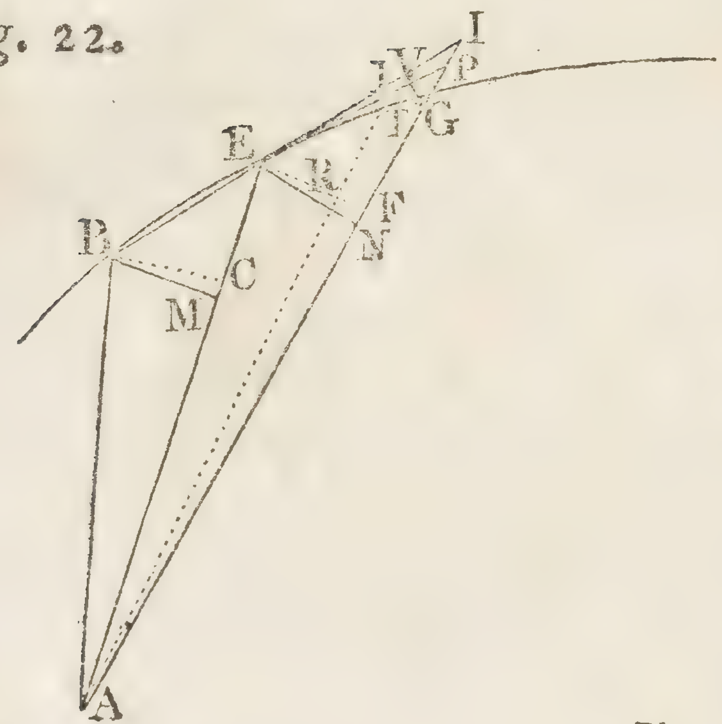
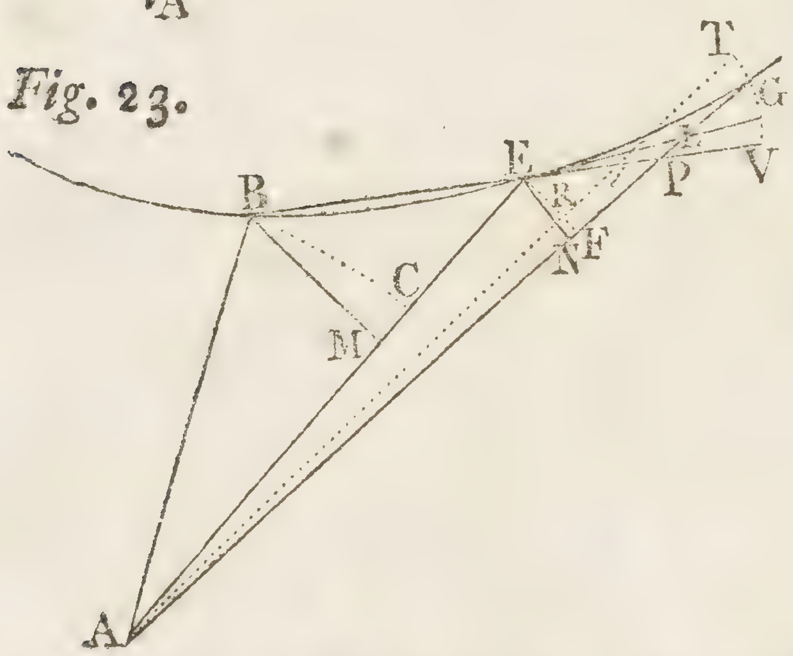


Fig. 23.



Taking $\dot{s}$ for constant, and drawing the right line VRA, it will be EG = EV = BE; and therefore the triangles EBC, EVR, are equal and similar; thence is BC = ER, and CE = RV. Whence RF = $\ddot{x}$, VI = $-\ddot{y}$, in Fig. 20. But RF = $-\ddot{x}$, and VI $= \ddot{y}$, in Fig. 21.

If no first fluxion be taken for constant, let EG be greater than BC, (Fig. 22, 23.) by the second fluxion RF; let the right line ART be drawn; with centre A, distance AG, draw the arch GT; and with centre E, distance EG, draw the arch GV. Therefore, since BC = ER, it will be also CE = RI, and BE = EI. Therefore TI will be the difference between CE and FG, and VI the difference between BE and EG.

SCHOLIUM I.

23. It may not be beside our purpose to obviate a difficulty, which seems likely to arise. And this is, that in the foregoing Theorem the lines CE, FP, are assumed as equal, in virtue of Theor. VI.; which Theorem supposes as equal the sines BM, EN. Whence it may seem, that the determinations concerning second differentials can only take place in the case when we make a supposition of a constant fluxion BC, and in no other. But, to remove this difficulty, it will be sufficient to consider, that, though BC be supposed variable, the difference will be an infinitesimal of the second degree, which does not hinder the equality of the first fluxions BC, EF, nor of the sines BM, EN.

SCHOLIUM II.

24. In the foregoing Theorems are contained the principles, by which infinitesimals of any order may be managed, and which prepare the way to make a right

right uſe of the Method of Fluxions, whether direct or inverſe; and beſides, to apply the ſyntheſis of the ancients to infinitely little magnitudes of all degrees; and to make uſe of the ſtricteſt Geometry, which proceeds with a particular ſimplicity and elegance.

Now, to avoid paralogiſms, into which it is but too eaſy to fall, it will be needful to reflect, that infinitely little lines of any order, (agreeably to what obtains likewiſe in thoſe that are finite,) have two important circumſtances to be conſidered, which are their magnitude and their poſition. And as to their magnitude, I think they cannot be rejected except by thoſe, who fancy ſuch infiniteſimal quantities to be mere nullities.

Now, although quantities, by diminiſhing *ad infinitum*, may paſs from one order to another, the proportions in every order continue the ſame. And, becauſe of three lines of any the ſame order a triangle may be formed, it may be conſidered, that if, by leſſening proportionally the ſides, ſo as to paſs from one degree to another, the angles are not thereby changed, the ſides muſt always preſerve the ſame ratio to one another; that is, infiniteſimals with the finite, and infiniteſimals of the ſecond order with thoſe of the firſt, and with finite; and ſo on.

But if two magnitudes, of any order whatever, ſhall differ by a magnitude which in reſpect of them ſhall be inaſſignable, then with the utmoſt ſecurity, and without any danger of error, one of them may be taken for the other; nor need it be apprehended that ſuch a compariſon will introduce the leaſt error.

Therefore it is neceſſary to be much upon our guard, when the poſition of lines and angles is concerned; for, to confound them when they ought to be nicely diſtinguiſhed, muſt needs lead us into unavoidable paralogiſms.

25. The principal foundations of this calculus being thus laid, I ſhall paſs on to the methods or rules of finding the fluxions or differences of analytical formulas or expreſſions. And, firſt, let us take the differences of various quantities added together, or ſubtracted from one another; for example, of $a + x + z + y - u$. As the fluxion of x is $\dot{x}$, of z is $\dot{z}$, &c; and as the conſtant quantity a has no fluxion; then, conceiving every variable to be increaſed by it's fluxion, according to it's ſign, the formula propoſed will be changed into this other, $a + x + \dot{x} + z + \dot{z} + y + \dot{y} - u - \dot{u}$; from which ſubtracting the firſt, the remainder will be $\dot{x} + \dot{z} + \dot{y} - \dot{u}$, which is exactly that quantity by which the propoſed quantity is increaſed, that is to ſay, it's difference or fluxion.

Hence we derive this general rule, that, to find the fluxion of any aggregate of analytical quantities of one dimenſion, it will be ſufficient to take the fluxion of every one of the variable quantities with it's ſign, and the aggregate of theſe fluxions ſhall be the fluxion of the quantity propoſed. So, the fluxion of

$b - s - z$ will be $-\dot{s} - \dot{z}$. The fluxion of $aa - 4bz + by$ will be $-4b\dot{z} + b\dot{y}$. And ſo of others.

26. But if the quantity propoſed to be differenced ſhall be the product of ſeveral variables, as xy; becauſe x becomes $x + \dot{x}$, and y becomes $y + \dot{y}$, and xy becomes $xy + y\dot{x} + x\dot{y} + \dot{x}\dot{y}$, which is the product of $x + \dot{x}$ into $y + \dot{y}$; from this product ſubtracting, therefore, the propoſed quantity xy, there will remain $y\dot{x} + x\dot{y} + \dot{x}\dot{y}$. But $\dot{x}\dot{y}$ is a quantity infinitely leſs than either of the other two, which are the rectangle of a finite quantity into an infiniteſimal. But $\dot{x}\dot{y}$ is the rectangle of two infiniteſimals, and therefore is infinitely leſs, and muſt be ſuppoſed entirely to vaniſh. The fluxion, therefore, of xy will be $x\dot{y} + y\dot{x}$.

Let us difference xyz by this rule. The product of $x + \dot{x}$ into $y + \dot{y}$ into $z + \dot{z}$ is $xyz + yz\dot{x} + xz\dot{y} + xy\dot{z} + z\dot{x}\dot{y} + y\dot{x}\dot{z} + x\dot{y}\dot{z} + \dot{x}\dot{y}\dot{z}$; which, ſubtracting the quantity propoſed, will give the remainder $yz\dot{x} + xz\dot{y} + xy\dot{z} + z\dot{x}\dot{y} + y\dot{x}\dot{z} + x\dot{y}\dot{z} + \dot{x}\dot{y}\dot{z}$. But the firſt, ſecond, and third terms are each the product of two finite quantities and one infiniteſimal; the fourth, fifth, and ſixth are the products of one finite quantity and two infiniteſimals, and therefore every one of theſe is infinitely leſs than any one of thoſe, and therefore will vaniſh: and much more the laſt, which is the product of three infiniteſimals. Therefore let all theſe terms vaniſh, beginning at the fourth, and then $yz\dot{x} + xz\dot{y} + xy\dot{z}$ will be the fluxion of xyz.

Hence ariſes this rule, that, to take the fluxions of the product of ſeveral quantities multiplied together, we muſt take the ſum of the products of the fluxion of every one of thoſe quantities into the products of the others. Thus, the fluxion of $bxzt$ will be $bxz\dot{t} + bxt\dot{z} + btz\dot{x} + xzt \times 0$; becauſe the fluxion of the conſtant quantity b is nothing. That is, the fluxion of $bxzt$ will be $bxz\dot{t} + bxt\dot{z} + btz\dot{x}$. The fluxion of $\overline{a+x} \times \overline{b-y}$ will be $\dot{x} \times \overline{b-y} - \dot{y} \times \overline{a+x}$, that is, $b\dot{x} - y\dot{x} - a\dot{y} - x\dot{y}$.

27. Let the formula to be differenced be a fraction, ſuppoſe $\frac{x}{y}$. If we put $\frac{x}{y} = z$, it will be then $x = zy$. And therefore their differences will alſo be equal, that is, $\dot{x} = \dot{z}y + z\dot{y}$. Wherefore $\dot{z} = \frac{\dot{x} - z\dot{y}}{y}$. But $z = \frac{x}{y}$; therefore, ſubſtituting this value inſtead of z, it will be $\dot{z} = \frac{\dot{x}}{y} - \frac{x\dot{y}}{yy} = \frac{\dot{x}y - x\dot{y}}{yy}$. But if $z = \frac{x}{y}$, then $\dot{z}$ will be the differential of $\frac{x}{y}$, and therefore the differential of $\frac{x}{y}$ will be $\frac{y\dot{x} - x\dot{y}}{yy}$.

Now

Now the rule will be, that the differential of a fraction will be another fraction, the numerator of which will be the product of the difference of the numerator into the denominator, subtracting the product of the difference of the denominator into the numerator of the proposed fraction; and the denominator must be the square of the denominator of the same proposed fraction.

Therefore the difference or fluxion of $\frac{a}{x}$ will be $-\frac{a\dot{x}}{xx}$. The fluxion of $\frac{a+x}{x}$ will be $\frac{x\dot{x} - a\dot{x} - x\dot{x}}{xx}$, that is, $-\frac{a\dot{x}}{xx}$. The fluxion of $\frac{y}{b-y}$ will be $\frac{b\dot{y} - y\dot{y} + y\dot{y}}{\overline{b-y}|^2}$, that is, $\frac{b\dot{y}}{\overline{b-y}|^2}$. The fluxion of $\frac{3xy}{a-x}$ will be $\frac{\overline{3x\dot{y} + 3y\dot{x}} \times \overline{a-x} + \dot{x} \times 3xy}{\overline{a-x}|^2}$, that is, $\frac{3ax\dot{y} + 3ay\dot{x} - 3xx\dot{y}}{\overline{a-x}|^2}$.

28. Now let us find the fluxions of powers, and, first, of perfect and positive powers, that is, whose exponents are positive integer numbers; for example, of x^2. But xx is the product of x into x, and therefore, by the rule of products, it's fluxion will be $\dot{x}x + x\dot{x}$, that is, $2x\dot{x}$. To find the fluxion of x^3. Now this is the product of x into x into x, and therefore the fluxion will be $xx\dot{x} + xx\dot{x} + xx\dot{x}$, that is, $3xx\dot{x}$. And, as we may proceed in the same manner *in infinitum*, the fluxion of x^m, m being any positive integer, will be $mx^{m-1}\dot{x}$.

If the exponent be negative, suppose ax^{-2}, or $\frac{a}{x^2}$, the fluxion, by the rule of fractions, will be the product of the fluxion of the numerator into the denominator, subtracting the product of the fluxion of the denominator into the numerator, the whole being divided by the square of the denominator. But the fluxion of the denominator is $2x\dot{x}$; so that the fluxion of ax^{-2} or $\frac{a}{xx}$ will be $-\frac{2ax\dot{x}}{x^4}$, that is, $-\frac{2a\dot{x}}{x^3}$. The fluxion of x^{-3}, or $\frac{1}{x^3}$, will be $-\frac{3xx\dot{x}}{x^6}$, or $\frac{3\dot{x}}{x^4}$. And, in general, the fluxion of $\frac{ax^{-m}}{b}$, or $\frac{a}{bx^m}$, will be $-\frac{mab\dot{x}x^{m-1}}{bbx^{2m}}$, that is, $-\frac{ma\dot{x}x^{-m-1}}{b}$.

Let it be an imperfect power, and, first, let it be positive; that is, let the exponent be an affirmative fraction, as $\sqrt[n]{x^m}$, or $x^{\frac{m}{n}}$, where $\frac{m}{n}$ stands for any positive fraction. Make $x^{\frac{m}{n}} = z$, and, raising each part to the power n, it

 will

will be $x^m = z^n$, of which taking the fluxions, we ſhall have $m\dot{x}x^{m-1} = n\dot{z}z^{n-1}$, whence $\dot{z} = \frac{m\dot{x}x^{m-1}}{nz^{n-1}}$. But, becauſe $x^m = z^n$, and thence $z^{n-1} = x^{m-\frac{m}{n}}$, which being ſubſtituted, it will be $\dot{z} = \frac{m\dot{x}x^{m-1}}{nx^{m-\frac{m}{n}}}$, that is, $\dot{z} = \frac{m}{n}\dot{x}x^{\frac{m}{n}-1}$.

If the exponent were negative, as $\frac{1}{\sqrt[n]{x^m}}$, that is, $x^{-\frac{m}{n}}$, or elſe $\frac{1}{x^{\frac{m}{n}}}$, the fluxion, by the rule of fractions, would be $-\frac{\frac{m}{n}\dot{x}x^{\frac{m}{n}-1}}{x^{\frac{2m}{n}}}$, or $-\frac{m}{n}\dot{x}x^{-\frac{m}{n}-1}$.

Therefore the general rule is, that the fluxion of any power whatever, whether perfect or imperfect, poſitive or negative, will be the product of the exponent of the power into the quantity raiſed to a power leſs by an unit than the given power, and this multiplied into the fluxion of the quantity.

Let it be required to find the fluxion of $x^{\frac{3}{2}}$; it will be $\frac{3}{2}x^{\frac{3}{2}-1}\dot{x}$, that is, $\frac{3}{2}x^{\frac{1}{2}}\dot{x}$, or elſe $\frac{3}{2}\dot{x}\sqrt{x}$.

Let be given $x^{\frac{5}{4}}$; it's fluxion will be $\frac{5}{4}x^{\frac{5}{4}-1}\dot{x}$, that is, $\frac{5}{4}x^{\frac{1}{4}}\dot{x}$, or $\frac{5}{4}\dot{x}\sqrt[4]{x}$.

Let be given $\frac{1}{x^{\frac{3}{2}}}$, that is, $x^{-\frac{3}{2}}$; the fluxion will be $-\frac{3}{2}\dot{x}x^{-\frac{3}{2}-1}$, or $-\frac{3}{2}\dot{x}x^{-\frac{5}{2}}$, or, laſtly, $-\frac{3\dot{x}}{2x^{\frac{5}{2}}}$.

The fluxion of $\overline{ax+xx}|^2$ will be $2 \times \overline{ax+xx} \times \overline{a\dot{x}+2x\dot{x}}$, that is, $2aax\dot{x} + 6ax^2\dot{x} + 4x^3\dot{x}$.

The fluxion of $\overline{xy+ax}|^3$ will be $3 \times \overline{xy+ax}|^2 \times \overline{x\dot{y}+y\dot{x}+a\dot{x}}$, that is, $3x^3y^2\dot{y} + 6ax^3y\dot{y} + 3a^2x^3\dot{y} + 3y^3x^2\dot{x} + 9ay^2x^2\dot{x} + 9a^2yx^2\dot{x} + 3a^3x^2\dot{x}$.

The fluxion of $\frac{1}{\overline{ax-yy}|^2}$, or $\overline{ax-yy}|^{-2}$, will be $-2 \times \overline{ax-yy}|^{-3} \times \overline{a\dot{x}-2y\dot{y}}$, or $\frac{-2a\dot{x}+4y\dot{y}}{\overline{ax-yy}|^3}$.

The

The fluxion of $\sqrt{ax-xx}$, or $\overline{ax-xx}|^{\frac{1}{2}}$, will be $\frac{1}{2} \times \overline{ax-xx}|^{-\frac{1}{2}} \times \overline{a\dot{x}-2x\dot{x}}$, that is, $\frac{a\dot{x}-2x\dot{x}}{2 \times \overline{ax-xx}|^{\frac{1}{2}}}$.

The fluxion of $\sqrt{xx+xy}$, or $\overline{xx+xy}|^{\frac{1}{2}}$, will be $\frac{1}{2} \times \overline{xx+xy}|^{-\frac{1}{2}} \times \overline{2x\dot{x}+\dot{x}y+\dot{y}x}$, that is, $\frac{2x\dot{x}+\dot{x}y+\dot{y}x}{2 \times \overline{xx+xy}|^{\frac{1}{2}}}$.

The fluxion of $\sqrt[3]{ax-xx}$, or $\overline{ax-xx}|^{\frac{1}{3}}$, will be $\frac{1}{3} \times \overline{ax-xx}|^{-\frac{2}{3}} \times \overline{a\dot{x}-2x\dot{x}}$, that is, $\frac{a\dot{x}-2x\dot{x}}{3 \times \overline{ax-xx}|^{\frac{2}{3}}}$.

The fluxion of $\frac{1}{\sqrt[3]{ay+xy}}$, or $\frac{1}{\overline{ay+xy}|^{\frac{1}{3}}}$, or $\overline{ay+xy}|^{-\frac{1}{3}}$, will be $-\frac{1}{3} \times \overline{ay+xy}|^{-\frac{4}{3}} \times \overline{a\dot{y}+x\dot{y}+y\dot{x}}$, or $-\frac{a\dot{y}+x\dot{y}+y\dot{x}}{3 \times \overline{ay+xy}|^{\frac{4}{3}}}$.

The fluxion of $\overline{a-x}\sqrt[3]{a+x}$, or $\overline{a-x} \times \overline{a+x}|^{\frac{1}{3}}$, is $-\dot{x} \times \overline{a+x}|^{\frac{1}{3}} + \frac{1}{3} \times \overline{a-x} \times \overline{a+x}|^{-\frac{2}{3}} \times \dot{x}$, or $-\dot{x}\sqrt[3]{a+x} + \frac{a\dot{x}-x\dot{x}}{3 \times \overline{a+x}|^{\frac{2}{3}}}$.

The fluxion $\sqrt{ax+xx+\sqrt[4]{a^4-x^4}}$, or $\overline{ax+xx+\overline{a^4-x^4}|^{\frac{1}{4}}}|^{\frac{1}{2}}$ will be $\frac{1}{2} \times \overline{a\dot{x}+2x\dot{x}+\frac{1}{4}\times -4\dot{x}x^3 \times \overline{a^4-x^4}|^{-\frac{3}{4}}} \times \overline{ax+x^2+\overline{a^4-x^4}|^{\frac{1}{4}}}|^{-\frac{1}{2}}$, or $\frac{a\dot{x}+2x\dot{x}-\frac{\dot{x}x^3}{\overline{a^4-x^4}|^{\frac{3}{4}}}}{2 \times \overline{ax+xx+\overline{a^4-x^4}|^{\frac{1}{4}}}|^{\frac{1}{2}}}$, or $\frac{\overline{a+2x} \times \dot{x}\sqrt[4]{\overline{a^4-x^4}|^3} - \dot{x}x^3}{2\sqrt{ax+xx+\sqrt[4]{a^4-x^4}} \times \sqrt[4]{\overline{a^4-x^4}|^3}}$.

The fluxion of $\frac{aa+xx}{\sqrt{ax+xx}}$, or $\overline{aa+xx} \times \overline{ax+xx}|^{-\frac{1}{2}}$, will be $2\dot{x}x \times \overline{ax+xx}|^{-\frac{1}{2}} - \frac{1}{2} \times \overline{a\dot{x}+2x\dot{x}} \times \overline{ax+xx}|^{-\frac{3}{2}} \times \overline{aa+xx}$, that is, $\frac{2\dot{x}x \times \overline{ax+xx} - \frac{1}{2} \times \overline{a\dot{x}+2x\dot{x}} \times \overline{aa+xx}}{\overline{ax+xx}|^{\frac{3}{2}}}$, or $\frac{3a\dot{x}x^2+2\dot{x}x^3-a^3\dot{x}-2aa\dot{x}x}{2\sqrt{\overline{ax+xx}|^3}}$.

The fluxion of $\frac{x\sqrt{ax+xx}}{a\sqrt{ay-xy}}$ will be $\frac{3a^2yx\dot{x}+2ayx^2\dot{x}-3yx^3\dot{x}-a^2x^2\dot{y}+x^4\dot{y}}{2a \times \sqrt{ax+xx} \times \sqrt{\overline{ay-xy}|^3}}$.

29. After the same manner as the fluxions of finite quantities are found, so are found the fluxions of infinitesimal quantities of the first order, and the fluxions of infinitesimal quantities of the second order, and so on successively, making use of the same rules which have now been explained.

Here

Here it muſt be conſidered, whether any firſt fluxion be aſſumed as conſtant, and which it is; for then it's fluxion will be nothing, and ſo ought to be omitted in taking the fluxion.

Let the formula $y\dot{x} - x\dot{y}$ be propoſed, to find it's difference or fluxion. Let no fluxion at preſent be ſuppoſed to be conſtant, and it's fluxion will be $\dot{x}\dot{y} + y\ddot{x} - \dot{x}\dot{y} - x\ddot{y}$, that is, $y\ddot{x} - x\ddot{y}$. Now let the fluxion $\dot{x}$ be aſſumed as conſtant; then the difference will be $\dot{x}\dot{y} - \dot{x}\dot{y} - x\ddot{y}$, or $- x\ddot{y}$. Let the fluxion $\dot{y}$ be conſtant, then the difference will be $\dot{x}\dot{y} + y\ddot{x} - \dot{x}\dot{y}$, that is, $y\ddot{x}$.

Let the quantity be $\frac{y\dot{x}}{\dot{y}}$, in which no firſt fluxion is taken for conſtant. The fluxion will be $\frac{\dot{x}\dot{y}^2 + y\dot{y}\ddot{x} - y\dot{x}\ddot{y}}{\dot{y}\dot{y}}$, or $\dot{x} + \frac{y\ddot{x}}{\dot{y}} - \frac{y\dot{x}\ddot{y}}{\dot{y}\dot{y}}$. Here, taking $\dot{x}$ for conſtant, it will be $\dot{x} - \frac{y\dot{x}\ddot{y}}{\dot{y}\dot{y}}$. Taking $\dot{y}$ for conſtant, it will be $\dot{x} + \frac{y\ddot{x}}{\dot{y}}$.

Let the formula be $\frac{y\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}}{\dot{z}}$, and let $\dot{z}$ be conſtant. The fluxion will be $\frac{\dot{y}\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}}{\dot{z}} + y \times \frac{\dot{x}\ddot{x} + \dot{y}\ddot{y}}{\dot{z}\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}}$, that is, $\frac{\dot{x}\dot{x}\dot{y} + \dot{y}^3 + y\dot{x}\ddot{x} + y\dot{y}\ddot{y}}{\dot{z}\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}}$. Taking $\dot{y}$ for conſtant, it will be

$$\frac{\dot{y}\dot{z}\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}} + \frac{y\dot{z}\dot{x}\ddot{x}}{\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}} - y\ddot{z}\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}}{\dot{z}\dot{z}},$$

that is,

$$\frac{\dot{x}\dot{x}\dot{y}\dot{z} + \dot{y}^3\dot{z} + y\dot{z}\dot{x}\ddot{x} - y\dot{x}\dot{x}\ddot{z} - y\dot{y}\dot{y}\ddot{z}}{\dot{z}\dot{z}\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}}.$$

Taking $\dot{x}$ for conſtant, it will be

$$\frac{\dot{y}\dot{z}\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}} + \frac{y\dot{z}\dot{y}\ddot{y}}{\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}} - y\ddot{z}\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}}{\dot{z}\dot{z}}, \text{ that is, } \frac{\dot{x}\dot{x}\dot{y}\dot{z} + \dot{y}^3\dot{z} + y\dot{z}\dot{y}\ddot{y} - y\dot{x}\dot{x}\ddot{z} - y\dot{y}\dot{y}\ddot{z}}{\dot{z}\dot{z}\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}}.$$

And, laſtly, if no fluxion be conſtant, the differential will be

$$\frac{\dot{y}\dot{z}\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}} + y\dot{z} \times \frac{\dot{x}\ddot{x}+\dot{y}\ddot{y}}{\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}} - y\ddot{z}\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}}{\dot{z}\dot{z}},$$

that is,

$$\frac{\dot{x}\dot{x}\dot{y}\dot{z} + \dot{y}^3\dot{z} + y\dot{z}\dot{x}\ddot{x} + y\dot{z}\dot{y}\ddot{y} - y\dot{x}\dot{x}\ddot{z} - y\dot{y}\dot{y}\ddot{z}}{\dot{z}\dot{z}\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}}.$$

Now in this, if we expunge all the terms in which $\ddot{z}$ is found, that is, if we aſſume the hypotheſis of $\dot{z}$ being conſtant, this expreſſion will be changed into the firſt. And if we cancel thoſe in which $\ddot{y}$ is found, it will be changed into the ſecond. And, by expunging thoſe in which $\ddot{x}$ is found, it will become the third, as is manifeſt.

Let

Let be given $\frac{x\dot{x} + y\dot{y}}{\sqrt{xx + yy}}$, and let $\dot{x}$ be conſtant. Then the fluxion will be

$$\frac{\dot{x}\dot{x} + \dot{y}\dot{y} + y\ddot{y} \times \sqrt{\overline{xx + yy}} - \frac{x\dot{x} + y\dot{y}}{\sqrt{xx + yy}} \times \overline{x\dot{x} + y\dot{y}}}{xx + yy}, \text{ or}$$

$\frac{x^2\dot{y}^2 + x^2y\ddot{y} + y^2\dot{x}^2 + y^3\ddot{y} - 2xy\dot{x}\dot{y}}{\overline{xx + yy}|^{\frac{3}{2}}}$. Taking $\dot{y}$ for conſtant, it will be

$$\frac{\overline{\dot{x}^2 + x\ddot{x} + \dot{y}^2} \times \sqrt{\overline{xx + yy}} - \frac{\overline{x\dot{x} + y\dot{y}} \times \overline{x\dot{x} + y\dot{y}}}{\sqrt{xx + yy}}}{xx + yy}, \text{ that is,}$$

$\frac{x^3\ddot{x} + x^2\dot{y}^2 + y^2\dot{x}^2 + y^2x\ddot{x} - 2xy\dot{x}\dot{y}}{\overline{xx + yy}|^{\frac{3}{2}}}$. And laſtly, taking neither of the fluxions for conſtant, it will be $\frac{\overline{\dot{x}^2 + x\ddot{x} + \dot{y}^2 + y\ddot{y}} \times \sqrt{\overline{xx + yy}} - \frac{x\dot{x} + y\dot{y}}{\sqrt{xx + yy}} \times \overline{x\dot{x} + y\dot{y}}}{xx + yy}$, that is, $\frac{x^3\ddot{x} + x^2\dot{y}^2 + x^2y\ddot{y} + y^2\dot{x}^2 + y^2x\ddot{x} + y^3\ddot{y} - 2xy\dot{x}\dot{y}}{\overline{xx + yy}|^{\frac{3}{2}}}$.

Let it be required to find the fluxion of this differential formula of the ſecond degree, $\frac{\overline{\dot{x}^2 + \dot{y}^2} \times \sqrt{\overline{\dot{x}^2 + \dot{y}^2}}}{-\dot{x}\ddot{y}}$, or of this, $\frac{\overline{\dot{x}^2 + \dot{y}^2}|^{\frac{3}{2}}}{-\dot{x}\ddot{y}}$, taking $\dot{x}$ for conſtant. The fluxion will be $\frac{3\dot{y}\ddot{y} \times \overline{\dot{x}^2 + \dot{y}^2}|^{\frac{1}{2}} \times -\dot{x}\ddot{y} + \dot{x}\dddot{y} \times \overline{\dot{x}^2 + \dot{y}^2}|^{\frac{3}{2}}}{\dot{x}^2\ddot{y}^2}$. The hypotheſis of $\dot{y}$ being conſtant, cannot take place in this formula, becauſe here is already found $\ddot{y}$. Taking neither of the fluxions as conſtant, the differential will be

$$\frac{3 \times \overline{\dot{x}\ddot{x} + \dot{y}\ddot{y}} \times \overline{\dot{x}^2 + \dot{y}^2}|^{\frac{1}{2}} \times -\dot{x}\ddot{y} + \overline{\ddot{x}\ddot{y} + \dot{x}\dddot{y}} \times \overline{\dot{x}^2 + \dot{y}^2}|^{\frac{3}{2}}}{\dot{x}^2\ddot{y}^2}.$$

In a like method we muſt proceed in all other caſes, ſtill more compounded.

SECT.

SECT. II.

The Method of Tangents.

Fig. 24.

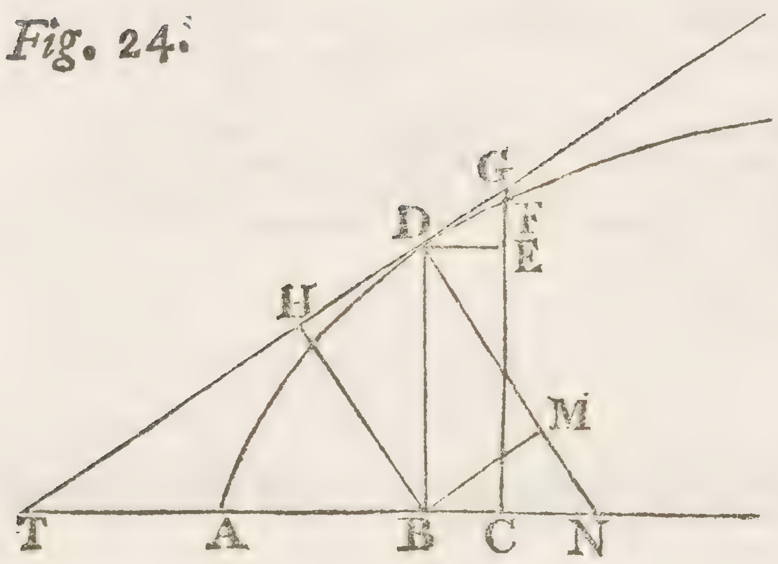
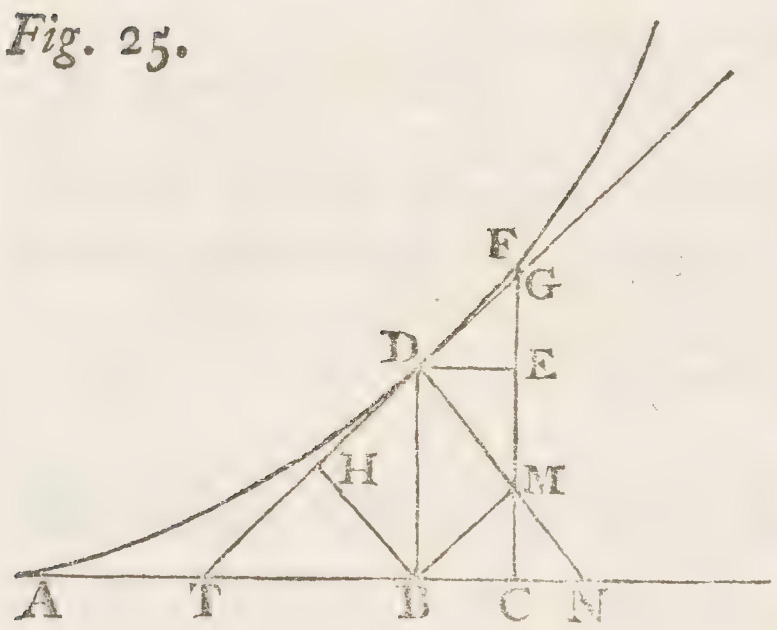


Fig. 25.



30. Let the right line TDG (Fig. 24, 25.) be a tangent to the curve ADF in any point D, and the ordinate BD be perpendicular to the axis AB in the point B, to which let CF be infinitely near, which produced (if need be,) ſhall meet the tangent in the point G, and let DE be drawn parallel to the axis AB. By what has been already demonſtrated in the foregoing Theorems, and their Corollaries, GF will be an infiniteſimal in reſpect of EF, and alſo the difference between DF and DG will be an infiniteſimal in reſpect of the little arch DF. Therefore we may aſſume as equal the two lines EF, EG, as alſo the two, DF, DG; and therefore, if AB $= x$, BD $= y$, it will be EF $=$ EG $= \dot{y}$, DF $=$ DG $= \sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}$. But the ſimilar triangles GED, DBT, give us this analogy, GE . ED :: DB . BT; that is, in analytical terms, $\dot{y} . \dot{x} :: y .$ BT, and therefore BT $= \frac{y\dot{x}}{\dot{y}}$; and this will be a general formula for the ſubtangent of any curve.

Wherefore, in the caſe of any given curve, in order to have the ſubtangent, nothing elſe is required to be done, but to find the fluxion of the equation, and to ſubſtitute the value of $\dot{x}$ or $\dot{y}$ in the general formula $\frac{y\dot{x}}{\dot{y}}$, by which the differentials will vaniſh, and we ſhall have the value of the ſubtangent expreſſed in finite terms. This will belong to the curve in any point whatever; and if we would have it at a determinate point, inſtead of the unknown quantities we are to ſubſtitute ſuch as ſhall belong to the given points.

31. Becauſe

31. Becaufe we may affume EF $=$ EG, and DF $=$ DG, it will follow, that we may confider the point G as coinciding with F, that is, that the tangent DG, the arch DF, and it's chord, are all confounded together, or that curves may be confidered as polygons of an infinite number of infinitely little fides. This conclufion obtains only when we confine ourfelves to firft fluxions; but when we are to proceed to fecond fluxions, the point G muft not then be confounded with the point F, for GF will then be a fecond fluxion. Now, whereas, in the Method of Tangents, there is no occafion for fecond fluxions, it may be fafely fuppofed that the tangent coincides with the little arch and it's chord.

32. The fame triangle GDE will fupply formulas for the other lines, which are analogous to the fubtangent.

Becaufe the triangles GED, DBT, are fimilar, it will be GE . GD :: DB . DT; that is, $\dot{y} \,.\, \sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}} :: y \,.\, \mathrm{DT}$, and therefore $\mathrm{DT} = \frac{y\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}}{\dot{y}}$; which is a general formula for the tangent.

Let DN be perpendicular to the curve in the point D. The triangles GDE, DBN, will be fimilar, whence it will be DE . EG :: DB . BN; that is, $\dot{x} \,.\, \dot{y} :: y \,.\, \mathrm{BN}$, and therefore $\mathrm{BN} = \frac{y\dot{y}}{\dot{x}}$, a general formula for the fubnormal.

It will be alfo DE . DG :: DB . DN, or $\dot{x} \,.\, \sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}} :: y \,.\, \mathrm{DN}$; therefore $\mathrm{DN} = \frac{y\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}}{\dot{x}}$, a general formula for the normal.

From the point B draw BM perpendicular to DN, and BH perpendicular to DT. The triangle GDE will be fimilar to DBM, whence GD . GE :: DB . BM, or $\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}} \,.\, \dot{y} :: y \,.\, \mathrm{BM} = \frac{y\dot{y}}{\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}}$, a general formula for the line BM.

The fame triangle GDE will alfo be fimilar to DBH; whence it will be GD . DE :: DB . BH, or $\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}} \,.\, \dot{x} :: y \,.\, \mathrm{BH} = \frac{y\dot{x}}{\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}}$, a general formula for the line BH.

33. The fimilitude of the two triangles GED, DBT, will alfo be a means of difcovering the angle, which the tangent makes with the axis at any point of the curve at pleafure. For, becaufe the angle DTB is known, therefore the ratio of the right fine DB to the fine of the complement BT will be known alfo; that is, the ratio of GE to ED, or that of $\dot{y}$ to $\dot{x}$.

Therefore, the equation of the curve being given, if it's fluxions be found and resolved into an analogy, of which two terms are $\dot{y}$ and $\dot{x}$, we may have the ratio of the sines of the angle DTB, and consequently the angle will be known.

Fig. 26.

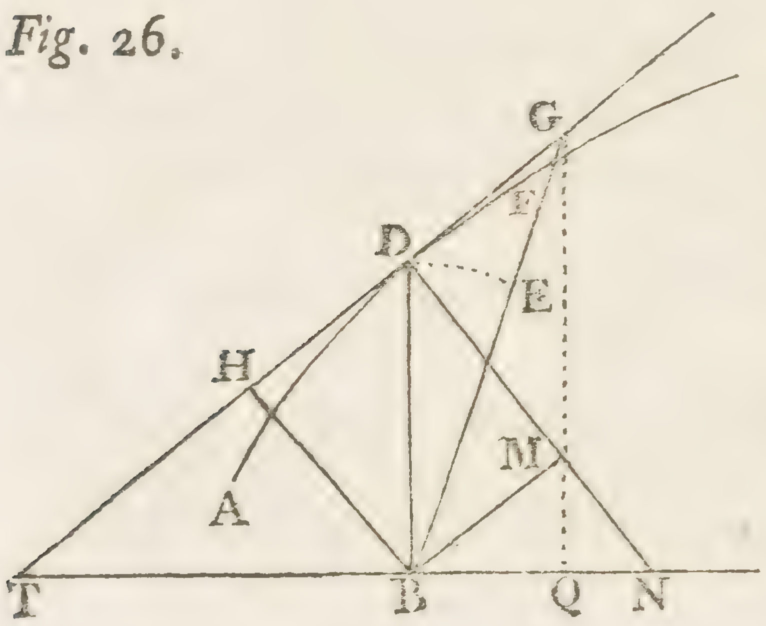
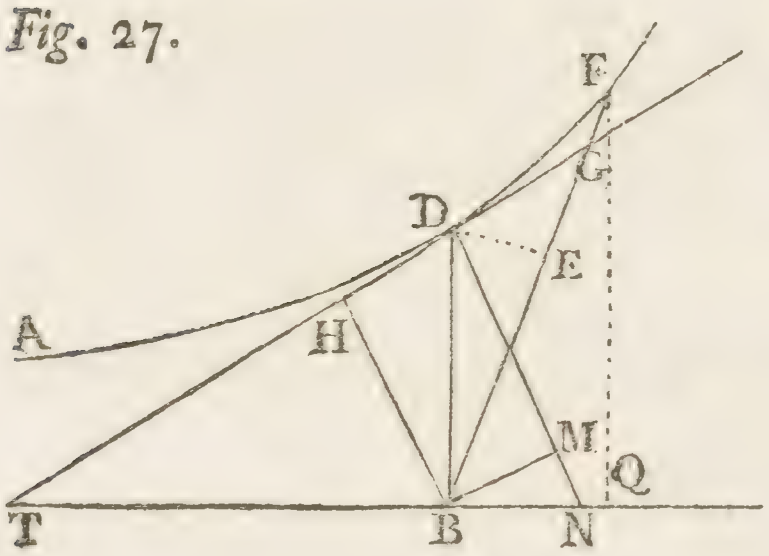


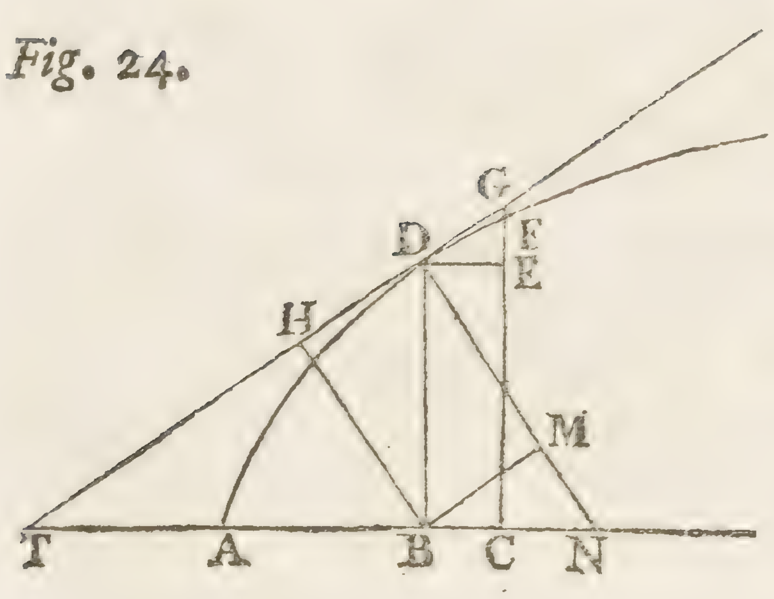
Fig. 27.



34. By the same way of argumentation, the same formulas may be derived for such curves as are referred to a *focus*, (Fig. 26, 27.) if we only consider, that, drawing from the *focus* B the right line BT perpendicular to the ordinate BD, meeting the tangent in T; the triangles DTB, DGE, will be similar, because the angles TBD, DEG, are right angles, and the angle TDB is not greater than the angle DGE, except by an infinitely little angle DBG, which is plainly seen by drawing GQ perpendicular to TB. Therefore the two angles TDB, DGE, may be assumed as equal, and consequently the two, BTD, GDE; therefore the two triangles DTB, GDE, are similar. But GF is an infinitesimal in respect of EF; therefore, &c.

EXAMPLE I.

Fig. 24.



35. Let the curve ADF be the *Apollonian* parabola, whose equation is $ax = yy$. Taking the fluxions, it will be $a\dot{x} = 2y\dot{y}$, or $\dot{x} = \frac{2y\dot{y}}{a}$. Wherefore, substituting this value instead of $\dot{x}$, in the general formula for the subtangent $\frac{y\dot{x}}{\dot{y}}$, we shall have $\frac{2yy}{a}$, or $2x$, putting, instead of yy, it's value ax, given

given by the equation of the curve. Therefore the ſubtangent in the parabola is double to the abſciſs; ſo that, taking AT = AB, and from the point T drawing the right line TD to the point D, it ſhall be a tangent to the curve at the point D. Inſtead of the value of $\dot{x}$, given from the equation of the curve, if we ſubſtitute the value of $\dot{y}$, or $\frac{a\dot{x}}{2y}$, in the general formula $\frac{y\dot{x}}{\dot{y}}$, it will be alſo $\frac{2yy}{a}$, as before; which may ſuffice to obſerve in this Example.

In the ſame parabola, if we require the ſubnormal BN; the general formula of the ſubnormal is $\frac{y\dot{y}}{\dot{x}}$. But, by the equation of the curve, it is $\dot{x} = \frac{2y\dot{y}}{a}$; ſo that, making the ſubſtitution, the ſubnormal in the parabola will be $= \frac{1}{2}a$, that is, half of the parameter; and therefore, making BN $= \frac{1}{2}a$, and from the point N drawing the right line ND to the point D, this ſhall be perpendicular to the curve in D.

If we ſeek the tangent DT, the general formula of which is $\frac{y\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}}{\dot{y}}$, by the equation of the curve we have $\dot{x} = \frac{2y\dot{y}}{a}$. Then, ſubſtituting this value inſtead of $\dot{x}$ in the formula, we ſhall have $\frac{y\sqrt{4yy\dot{y}\dot{y} + aa\dot{y}\dot{y}}}{a\dot{y}} = \frac{y}{a}\sqrt{4yy + aa} = \sqrt{4xx + ax}$, (putting, inſtead of yy, it's value ax from the given equation,) which will be the tangent required.

If we would have the normal DN, ſubſtituting the value of $\dot{x} = \frac{2y\dot{y}}{a}$ in the general formula $\frac{y\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}}{\dot{x}}$, it will be $\frac{y\sqrt{4yy\dot{y}\dot{y} + aa\dot{y}\dot{y}}}{2y\dot{y}} = \frac{\sqrt{4yy + aa}}{2} = \frac{\sqrt{4ax + aa}}{2}$, putting, inſtead of yy, it's value from the given equation.

If we would have the right line BM; ſubſtituting the value of $\dot{x} = \frac{2y\dot{y}}{a}$ in the general formula $\frac{y\dot{y}}{\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}}$, it will be $\frac{ay\dot{y}}{\sqrt{4yy\dot{y}\dot{y} + aa\dot{y}\dot{y}}} = \frac{ay}{\sqrt{4yy + aa}} = \frac{a\sqrt{ax}}{\sqrt{4ax + aa}}$.

If we would have the right line BH; ſubſtituting the value of $\dot{x}$ in the general formula $\frac{y\dot{x}}{\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}}$, it will be $\frac{2yy\dot{y}}{\sqrt{4yy\dot{y}\dot{y} + aa\dot{y}\dot{y}}} = \frac{2yy}{\sqrt{4yy + aa}} = \frac{2ax}{\sqrt{4ax + aa}}$.

 Having

Having found the ſubtangent, there is no need of any formulas for finding the other lines, though here, by way of exerciſe, I have made uſe of them. For, when BT is known, the triangle TDB, right-angled at B, will furniſh us with the tangent TD, and the ſimilar triangles TBD, DBN, DMB, DHB, with all the other lines. So that, in the following examples, I ſhall apply the method to finding the ſubtangents only.

If we would have the angle which is made by the tangent of the parabola with it's axis; taking the fluxional equation $a\dot{x} = 2y\dot{y}$, and reſolving it into an analogy, it will be $\dot{y} \,.\, \dot{x} :: a \,.\, 2y$. That is, that the right ſine BD is to the ſine of the complement BT, as the parameter is to the double of the ordinate; whence is determined the point D. And if we would determine the tangent to any certain point, for example, to the point D, to which correſponds the abſciſs AB $= x = \frac{1}{4}a$; from the equation of the curve finding the ordinate y, correſponding to $x = \frac{1}{4}a$, which, in this caſe, is $y = \frac{1}{2}a$, we ſhall have the analogy, $\dot{y} \,.\, \dot{x} :: a \,.\, a$; that is, the angle DTB will be half a right angle, when it is $y = \frac{1}{2}a$, or $x = \frac{1}{4}a$.

At the vertex A it is $y = 0$, and therefore the analogy for the angle of the tangent at the vertex will be $\dot{y} \,.\, \dot{x} :: a \,.\, 0$; that is, the ratio of $\dot{y}$ to $\dot{x}$ is infinite, which is as much as to ſay, that the ſine of the complement will be nothing at all, or that, at the vertex, the tangent is perpendicular to the axis.

EXAMPLE II.

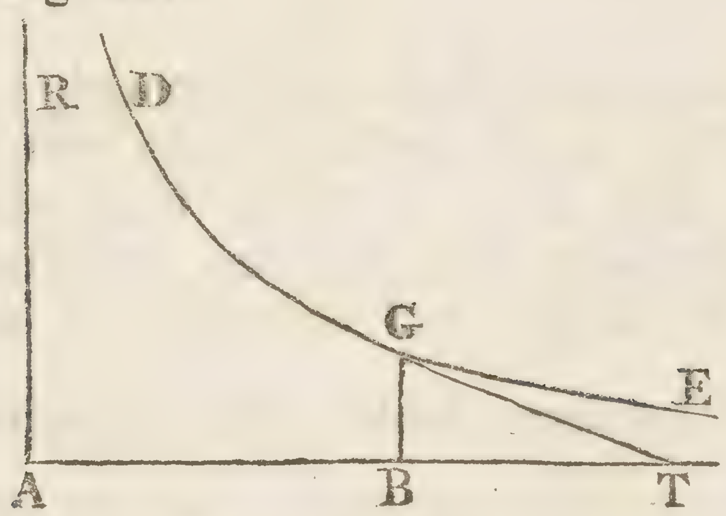
36. Let the equation be $x = y^m$, which is a general equation to all parabolas of any degree whatever; where m ſtands for any poſitive number, integer, or fraction, and unity ſupplies any dimenſions that are wanting. By taking the fluxions, it will be $\dot{x} = m\dot{y}y^{m-1}$; and, ſubſtituting this value inſtead of $\dot{x}$ in the general formula $\frac{y\dot{x}}{\dot{y}}$, the ſubtangent will be $my^m = mx$. Let $m = 3$, that is, let it be the firſt cubic parabola $x = y^3$; it's ſubtangent will be $3x$. Let $m = \frac{3}{2}$, that is, let it be the ſecond cubic parabola $xx = y^3$; the ſubtangent will be $\frac{3}{2}x$, &c.

The fluxional equation of the curve $\dot{x} = m\dot{y}y^{m-1}$ gives this analogy, $\dot{y} \,.\, \dot{x} :: 1 \,.\, my^{m-1}$. But, putting $y = 0$, if m be greater than unity, the analogy will be $\dot{y} \,.\, \dot{x} :: 1 \,.\, 0$; or the ratio of $\dot{y}$ to $\dot{x}$ will be infinite, and therefore the tangent at the vertex is perpendicular to the axis. And if m be leſs than unity, the

the analogy will be $\dot{y} \,.\, \dot{x} :: 1 \,.\, \frac{m}{y^{1-m}}$; that is, making $y = 0$, $\dot{y} \,.\, \dot{x} :: 1 \,.\, \frac{m}{0}$, which is as much as to ſay, that the ratio of $\dot{y}$ to $\dot{x}$ is infinitely little, and therefore, at the vertex, the tangent is parallel to the axis.

EXAMPLE III.

Fig. 28.



37. Let the curve be DCE, of which we deſire the ſubtangent, the equation of which is $xy = aa$, being the hyperbola between it's aſymptotes. By taking the fluxions, we ſhall have $x\dot{y} + y\dot{x} = 0$, or $\dot{x} = -\frac{x\dot{y}}{y}$. Wherefore, ſubſtituting this value of $\dot{x}$ in the formula of the ſubtangent $\frac{y\dot{x}}{\dot{y}}$, the ſubtangent will be $-x$ with a negative value, which is as much as to ſay, that the ſubtangent BT muſt be taken on the contrary part of the abſciſs.

Therefore, taking BT = BA, and drawing the right line TC to the point C, it ſhall be a tangent to the curve at the point C.

Now, becauſe in the curve DCE, as the axis increaſes, the ordinate y will decreaſe, in taking the fluxion we might have put $\dot{y}$ negative; but becauſe, for the ſame reaſon, we ought to have taken the ſame $\dot{y}$ negative alſo in the general formula, I have omitted to do it in both places, becauſe it comes to the ſame thing, without incumbering ourſelves with changing ſigns; and what is now mentioned may be underſtood on other like occaſions.

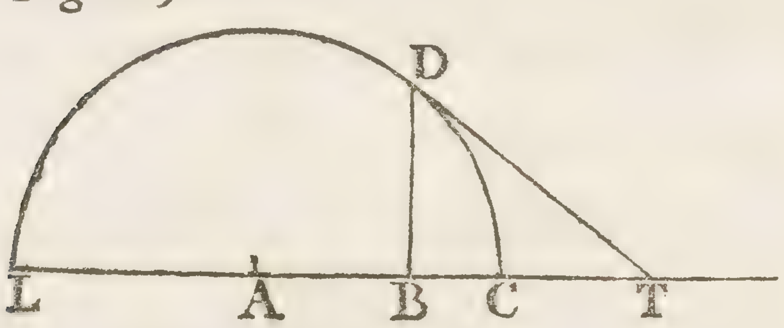
Let $x = \frac{1}{y^m}$ be a general equation to all hyperbolas *ad infinitum*, between their aſymptotes, where m ſtands for any poſitive number, integer, or fraction. By taking the fluxions, we ſhall have $\dot{x} = -\frac{m\dot{y}y^{m-1}}{y^{2m}} = -\frac{m\dot{y}}{y^{m+1}}$. And, ſubſtituting this value in the general formula $\frac{y\dot{x}}{\dot{y}}$, the ſubtangent will be $-\frac{m}{y^m}$, or $-mx$, by the equation of the curve.

EX-

EXAMPLE IV.

38. Let the curve ADF (Fig. 24.) be a circle whose diameter is $2a$, AB $= x$, BD $= y$; the equation will be $2ax - xx = yy$, whose fluxion is $2a\dot{x} - 2x\dot{x} = 2y\dot{y}$, and therefore $\dot{x} = \frac{y\dot{y}}{a - x}$. Then, substituting this value in the formula $\frac{y\dot{x}}{\dot{y}}$, the subtangent will be $\frac{yy}{a - x}$, that is, $\frac{2ax - xx}{a - x}$, by putting, instead of yy, it's value from the given equation. Therefore the subtangent in the circle will be a fourth proportional to $a - x$, $2a - x$, and x.

Fig. 29.



But if the circle shall be denoted by this equation, $aa - xx = yy$, in which the abscis AB $= x$ is taken from the centre; by taking the fluxions, we shall have $x\dot{x} = -y\dot{y}$, and therefore $\dot{x} = -\frac{y\dot{y}}{x}$. Wherefore, substituting this value in the formula, the subtangent will be $= -\frac{yy}{x}$, that is, a third proportional to AB and BD, but negative; that is to say, it must be taken from B towards T.

EXAMPLE V.

39. Let the curve ADF (Fig. 24.) be an ellipsis, with this equation $ax - xx = \frac{ayy}{b}$; taking the abscis from the vertex A. The fluxional equation will be $a\dot{x} - 2x\dot{x} = \frac{2ay\dot{y}}{b}$, and therefore $\dot{x} = \frac{2ay\dot{y}}{b \times \overline{a - 2x}}$. Now, substituting this value in the general formula $\frac{y\dot{x}}{\dot{y}}$, then $\frac{2ayy}{b \times \overline{a - 2x}}$ will be the subtangent; or else, $\frac{2ax - 2xx}{a - 2x}$, instead of $\frac{ayy}{b}$, putting it's value $ax - xx$ from the given equation.

Making $x = \frac{1}{2}a$, half the transverse axis, in the value of the subtangent, it will be $\frac{2aa}{0}$, that is, infinite. Therefore the tangent will be parallel to the transverse

tranfverfe axis in that point, in which the conjugate axis meets the curve. And this we fhall find to be true alfo, if we inquire what is that angle, which the tangent itfelf makes with the fame axis.

Let the equation, in general, to ellipfes of any degree be this; $\frac{ay^{m+n}}{b} = x^m \times \overline{a-x}|^n$, where m and n reprefent any pofitive numbers, whether integers or fractions. The fluxion of this will be $\frac{m+n}{b} \times a\dot{y}y^{m+n-1} = m\dot{x}x^{m-1} \times \overline{a-x}|^n - n\dot{x}x^m \times \overline{a-x}|^{n-1}$; and therefore $\dot{x} = \frac{\overline{m+n} \times a\dot{y}y^{m+n-1}}{bmx^{m-1} \times \overline{a-x}|^n - bnx^m \times \overline{a-x}|^{n-1}}$. And, fubftituting this value in the general formula, it will be $\frac{\overline{m+n} \times ay^{m+n}}{bmx^{m-1} \times \overline{a-x}|^n - bnx^m \times \overline{a-x}|^{n-1}}$. Then, inftead of $\frac{ay^{m+n}}{b}$, putting it's value from the given equation, the fubtangent will be $\frac{\overline{m+n} \times x^m \times \overline{a-x}|^n}{mx^{m-1} \times \overline{a-x}|^n - nx^m \times \overline{a-x}|^{n-1}}$. And, dividing the numerator and denominator by $x^{m-1} \times \overline{a-x}|^{n-1}$, it will be, finally, $\frac{\overline{m+n} \times \overline{ax-xx}}{ma-mx-nx}$.

Make $m = 1$, $n = 1$, that is, let it be the ellipfis of *Apollonius*; then the fubtangent will be $\frac{2ax-2xx}{a-2x}$, as before. Make $m = 3$, $n = 2$; then the equation is $\frac{ay^5}{b} = x^3 \times \overline{a-x}|^2$, and the fubtangent will be $\frac{5ax-5xx}{3a-5x}$. And fo of others.

If the equation were $\frac{ay^{m+n}}{b} = x^m \times \overline{a+x}|^n$, it would exprefs all hyperbolas of any degree, when referred to their axis; taking, in the fame manner, the beginning of the axis from the vertex A. Then, by a like operation, we fhould find the fubtangent to be $\frac{\overline{m+n} \times \overline{ax+xx}}{ma+mx+nx}$, which differs from the foregoing only in it's figns; as alfo, the equation, from whence it is derived, differs only in it's figns.

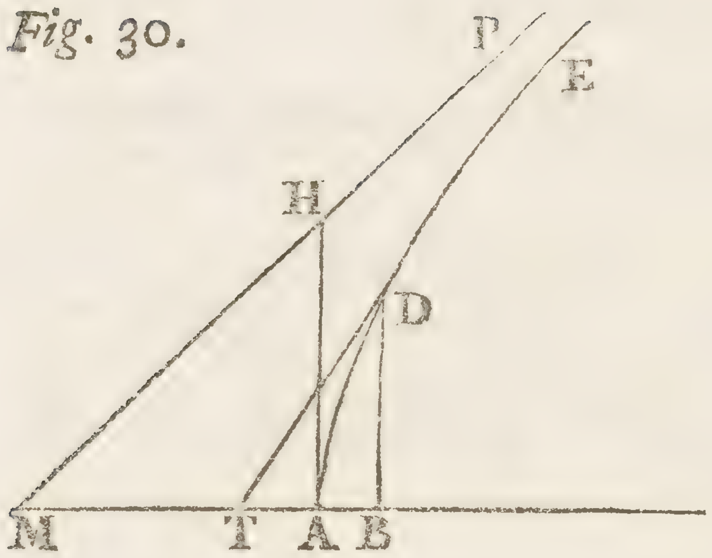
Make $m = 1$, $n = 1$, which is the *Apollonian* hyperbola. The fubtangent will be $\frac{2ax+2xx}{a+2x}$. Make $m = 3$, $n = 2$, then the equation will be $\frac{ay^5}{b} = x^3 \times \overline{a+x}|^2$; and the fubtangent will be $\frac{5ax+5xx}{3a+5x}$, &c.

40. From

Afymptotes. 40. From this method of tangents may be further derived a way of difcovering whether curves propofed have afymptotes, and the manner of drawing them, when they are inclined to the axis. For, as to the more fimple cafes, in which they are either perpendicular or parallel to the axes, fufficient has been faid in the firft Part, Sect. V.

EXAMPLE I.

Fig. 30.



41. Let the curve be ADE, with the equation $\frac{ay^{m+n}}{b} = x^m \times \overline{a+x}|^n$, as above, the fubtangent of which is TB $= \frac{\overline{m+n} \times \overline{ax+xx}}{ma+mx+nx}$. Then the intercepted line AT $= \frac{\overline{m+n} \times \overline{ax+xx}}{ma+mx+nx} - x$, that is, $\frac{nax}{ma+mx+nx}$.

It is plain that the tangent TD will become an afymptote, when touching the curve at an infinite diftance; that is, when the abfcifs AB $= x$ becomes infinite, the intercepted line AT fhall remain finite. Now, putting x infinite in the expreffion of AT, the firft term ma of the denominator is infinitely lefs than the others, and therefore vanifhes. Whence, in this cafe, it will be $\frac{nax}{mx+nx}$, or $\frac{na}{m+n}$, which is a finite quantity. Wherefore the curve has an afymptote, which will begin from the point M, making AM $= \frac{na}{m+n}$. Now, to draw it, let AH be raifed perpendicular to AB, and let it be, for example, MHP. This being fuppofed, if we take x infinite, it will be $\dot{x} \,.\, \dot{y}$ :: MA . AH, and, in the fuppofition of x being infinite, the equation of the curve $\frac{ay^{m+n}}{b} = x^m \times \overline{a+x}|^n$, ($a$ being nothing in refpect of x, will be changed into this other, $\frac{ay^{m+n}}{b} = x^{m+n}$. Or, extracting the root, and, for convenience, making $m+n=t$, it will be $y\sqrt[t]{a} = x\sqrt[t]{b}$; and, taking the fluxions, $\dot{y}\sqrt[t]{a} = \dot{x}\sqrt[t]{b}$; fo that $\dot{x} \,.\, \dot{y}$ :: $\sqrt[t]{a} \,.\, \sqrt[t]{b}$. Whence MA . AH :: $\sqrt[t]{a} \,.\, \sqrt[t]{b}$. And, becaufe MA

$MA = \frac{na}{t}$, it will be $\frac{na}{t} \cdot AH :: \sqrt[t]{a} \cdot \sqrt[t]{b}$, or $AH = \frac{na}{t} \times \sqrt[t]{\frac{b}{a}}$. If, therefore, we take $AM = \frac{na}{t}$, and raiſing the perpendicular $AH = \frac{na}{t} \times \sqrt[t]{\frac{b}{a}}$, and drawing the indefinite right line MHP; this will be the aſymptote of the curve ADE.

Make $m = 1$, $n = 1$, that is, let the curve be the *Apollonian* hyperbola, whoſe equation is $\frac{ayy}{b} = ax + xx$; it will be $t = 2$, and therefore $AM = \frac{1}{2}a$, $AH = \frac{a}{2} \times \sqrt{\frac{b}{a}} = \frac{1}{2}\sqrt{ab}$. That is, AM is half the tranſverſe axis, and AH half the conjugate, juſt as it ſhould be from the Conic Sections.

EXAMPLE II.

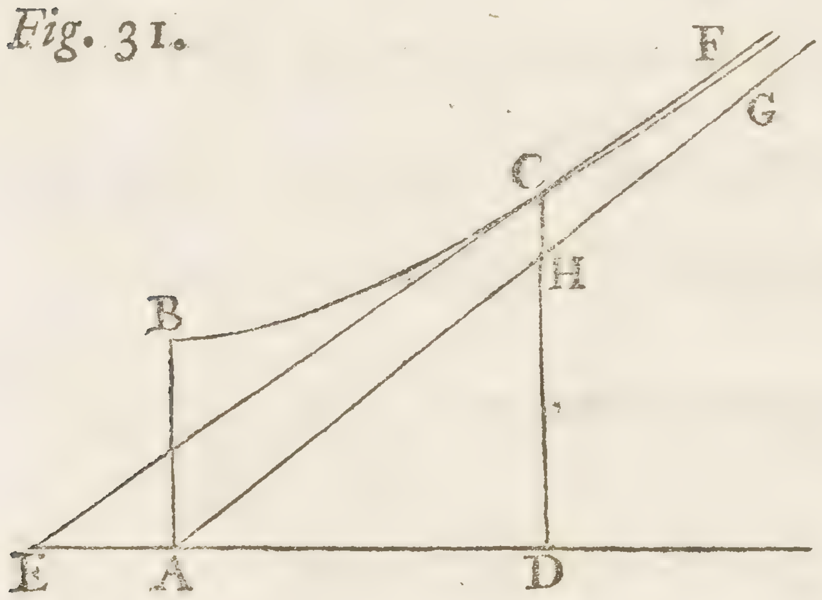
42. Let ADE (Fig. 30.) be a curve whoſe equation is $y^3 - x^3 = axy$; making $AB = x$, $BD = y$. By taking the fluxions, we ſhall have $3y^2\dot{y} - 3x^2\dot{x} = ax\dot{y} + ay\dot{x}$, and therefore $\frac{y\dot{x}}{\dot{y}} = \frac{3y^3 - axy}{3xx + ay}$; and $AT = \frac{y\dot{x}}{\dot{y}} - x = \frac{3y^3 - 3x^3 - 2axy}{3xx + ay}$. Or, inſtead of $3y^3 - 3x^3$, putting it's value $3axy$ from the equation of the curve, it will be $AT = \frac{axy}{3xx + ay}$. And, making x infinite, that is, in caſe of an aſymptote, in which AT becomes AM, the term ay is nothing in reſpect of $3xx$, ſo that it will be $AM = \frac{axy}{3xx} = \frac{ay}{3x}$.

But, becauſe, in the propoſed equation, the indeterminates cannot be ſeparated, nor, conſequently, can the value of AM be determined; if we put $AM = \frac{ay}{3x} = t$, (which expedient may alſo be uſed in other like caſes,) it will be $y = \frac{3tx}{a}$; which value being ſubſtituted in the propoſed equation, it will be $\frac{27t^3x^3}{a^3} - x^3 = 3tx^2$, or $\frac{27t^3x}{a^3} - x = 3t$. But, as x is infinite, the laſt term will be nothing in compariſon of the others, ſo that it will be $\frac{27t^3x}{a^3} - x = 0$, or $t = \frac{1}{3}a$. Taking, therefore, $AM = \frac{1}{3}a$, the aſymptote muſt be drawn

drawn from the point M. Moreover, it muft be MA . AH :: $\dot{x}$. $\dot{y}$, and the propofed equation $y^3 - x^3 = axy$, or $y^3 = x^3 + axy$, will be reduced to $x^3 = y^3$, or $x = y$, when x is infinite, and therefore $\dot{x} = \dot{y}$. Therefore, making MA = AH, if from the point M, through the point H, a right line be drawn, it will be an afymptote to the curve.

I add further, that the line AT muft neceffarily approach to a certain limit, beyond which it cannot pafs, and that the aforefaid limit is then an infinitefimal, or nothing. Here follows a plain Example of this.

Fig. 31.



Let BCF be an equilateral hyperbola; and making AB = a, AD = x, DC = y, we fhall have the equation $aa + xx = yy$, the fluxion of which is $x\dot{x} = y\dot{y}$. Thence the fubtangent will be ED $= \frac{y\dot{x}}{\dot{y}} = \frac{yy}{x}$ $= \frac{aa + xx}{x}$, and confequently ED — AD $= \frac{aa + xx}{x} - x = \frac{aa}{x} =$ AE.

Putting $x = 0$, AE will become infinite, and the tangent at the point B will be parallel to the axis AD. And, making $x = \infty$, it will be AE = 0. Wherefore the point E defcribes the whole line AE infinitely produced, and finifhes it's courfe at it's origin A, beyond which it paffes not, though the curve turns it's convexity towards the axis. Therefore the afymptote AG proceeds from the point A, and makes half a right angle with the line of the abfciffes; forafmuch as, in the equation of the *locus* $aa + xx = yy$, making $x = \infty$, the conftant quantity aa will vanifh, and it becomes $xx = yy$, or $x = y$.

Fig. 32.

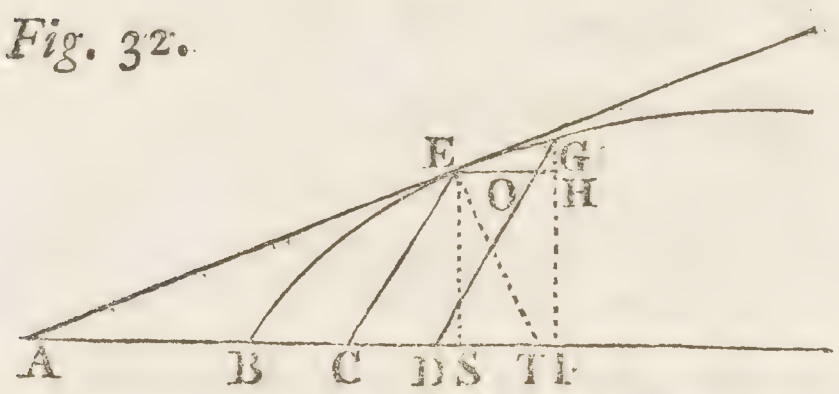
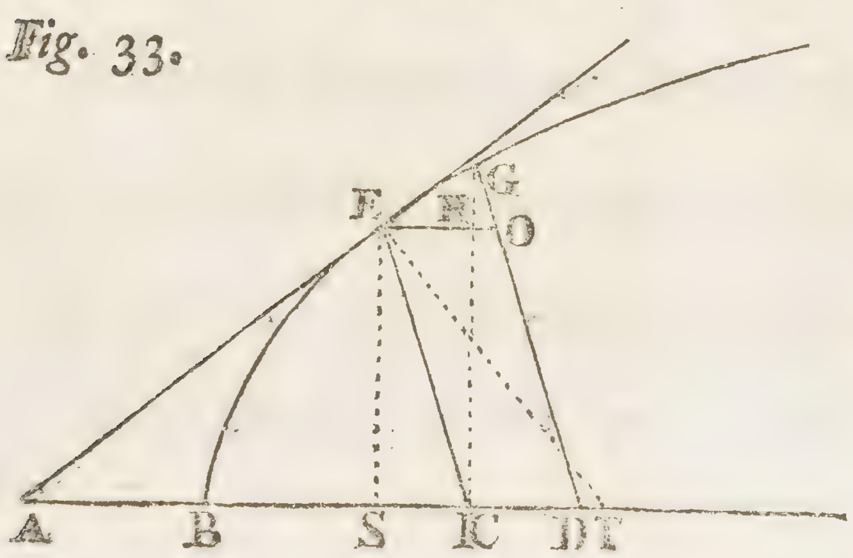


Fig. 33.



43. Hitherto I have fuppofed that the angle of the co-ordinates is a right angle; but, if it were obtufe or acute, making, as before, BC = x, CE = y, CD = $\dot{x}$, OG = $\dot{y}$, (Fig. 32, 33.) the fubtangent will be neither more nor lefs than $\frac{y\dot{x}}{\dot{y}}$, for the two triangles GEO, EAC, will be ftill fimilar; but the other formulas will have need of fome reformation.

In the triangle EOG, the angle at O, equal to the angle ACE, is fuppofed to be known; therefore, from the point G letting fall GI perpendicular to AD, and producing

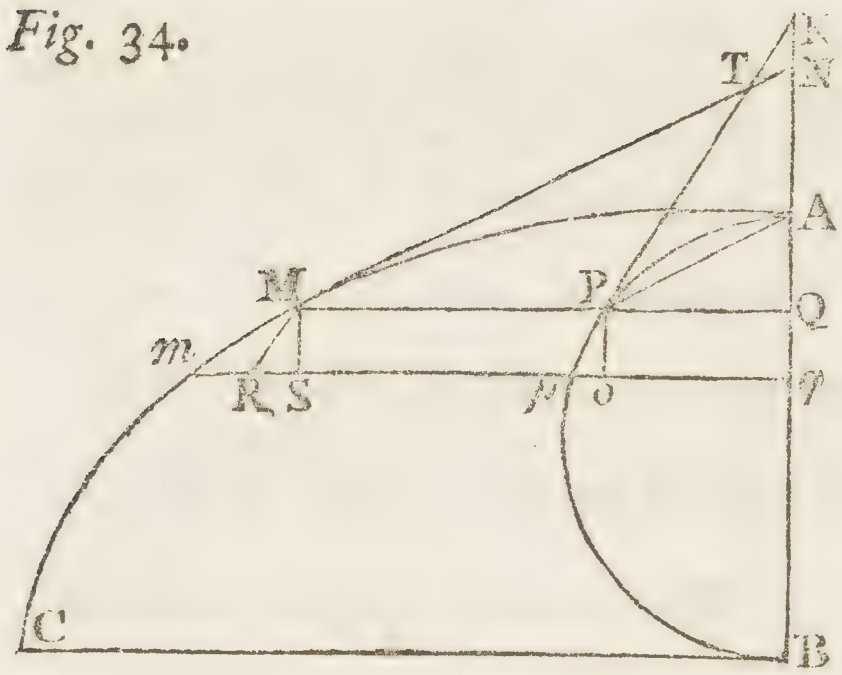
producing EO to H, if there be occafion, in the triangle GOH the angle GOH will be known, and the angle at H be a right angle. Wherefore the angle OGH is known, and confequently the triangle OGH is given *in fpecie*, that is, the ratio of GO to GH is given. Let this be the fame as a to m, and therefore it will be $a \,.\, m :: \dot{y} \,.\, \mathrm{GH} = \frac{m\dot{y}}{a}$. Alfo, the ratio of GO to OH will be given, which may therefore be as a to n; and confequently $a \,.\, n :: \dot{y} \,.\, \mathrm{OH} = \frac{n\dot{y}}{a}$. Then $\mathrm{EH} = \dot{x} \pm \frac{n\dot{y}}{a}$, (where the fign muft be affirmative in Fig. 32, and negative in Fig. 33.) Wherefore $\mathrm{EG}q = \frac{aa\dot{x}\dot{x} \pm 2an\dot{x}\dot{y} + nn\dot{y}\dot{y} + mm\dot{y}\dot{y}}{aa}$. But if OG be expreffed by a, GH by m, OH by n, then it will be $aa = mm + nn$, and $aa\dot{y}\dot{y} = mm\dot{y}\dot{y} + nn\dot{y}\dot{y}$, which, being fubftituted in this value of $\mathrm{EG}q$, will make $\mathrm{EG}q = \frac{aa\dot{x}\dot{x} \pm 2an\dot{x}\dot{y} + aa\dot{y}\dot{y}}{aa}$, and $\mathrm{EG} = \dot{s} = \sqrt{\frac{a\dot{x}^2 \pm 2n\dot{x}\dot{y} + a\dot{y}^2}{a}}$, the expreffion of the element or fluxion of the curve. This being fuppofed, by the fimilitude of the triangles EGO, AEC, it will be $\mathrm{GO} \,.\, \mathrm{GE} :: \mathrm{EC} \,.\, \mathrm{EA}$, that is, $\dot{y} \,.\, \sqrt{\frac{a\dot{x}\dot{x} \pm 2n\dot{x}\dot{y} + a\dot{y}\dot{y}}{a}} :: y \,.\, \mathrm{EA}$; or $\mathrm{EA} = \frac{y}{\dot{y}}\sqrt{\frac{a\dot{x}\dot{x} \pm 2n\dot{x}\dot{y} + a\dot{y}\dot{y}}{a}}$, which will be the formula of the tangent.

Let TE be perpendicular to the curve, and ES to the diameter AI. Then, by fimilar triangles GOH, ECS, we fhall have $\mathrm{ES} = \frac{my}{a}$, and $\mathrm{CS} = \frac{ny}{a}$. And, by the fimilar triangles GEH, EST, we fhall have EH . HG :: ES . ST. That is, $\frac{a\dot{x} \pm n\dot{y}}{a} \,.\, \frac{m\dot{y}}{a} :: \frac{my}{a} \,.\, \mathrm{ST} = \frac{mmy\dot{y}}{a \times \overline{a\dot{x} \pm n\dot{y}}}$. And therefore $\mathrm{CT} = \frac{mmy\dot{y}}{a \times \overline{a\dot{x} \pm n\dot{y}}} \pm \frac{ny}{a} = \frac{mmy\dot{y} + nny\dot{y} \pm any\dot{x}}{a \times \overline{a\dot{x} \pm n\dot{y}}} = \frac{ay\dot{y} \pm ny\dot{x}}{a\dot{x} \pm n\dot{y}}$, which is the formula of the fubnormal.

In a like manner, the other formulas may be reduced, which it is fufficient only to take notice of here.

44. But the curves, whofe tangents we defire, may be *Tranfcendent* or *Mechanical*, that is, are not expreffible by any Algebräical equation, but may depend on the rectification of other curves, which are not rectifiable. Let the curve Tangents to tranfcendent curves.

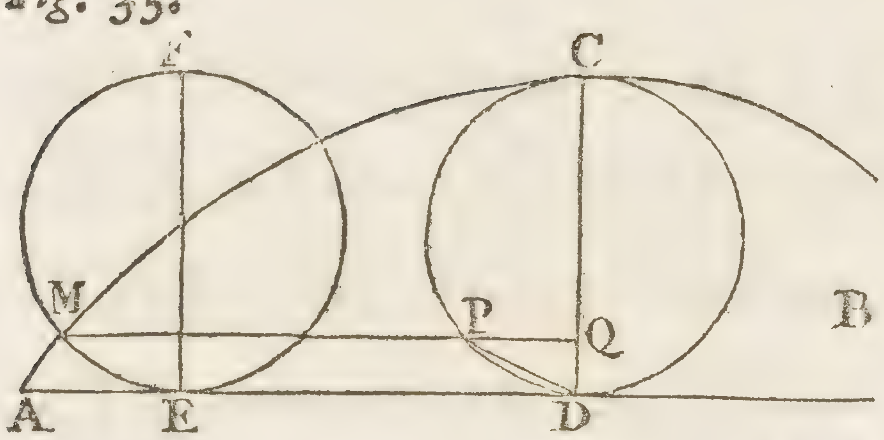
Fig. 34.



curve be APB, whose tangent PTK we know how to draw, at any given point P. Then, producing to M the line QP perpendicular to AQ, let the relation of MP to the arch PA be expressed by any equation, to find the tangent MT of the curve CMA, described from the point M. Draw qm infinitely near to QM, and MR parallel to PT; and supposing the rectification of the arch AP; make $AP = x$, $PM = y$, and it will be $Pp = \dot{x}$, $Rm = \dot{y}$, and the two triangles mRM, MPT, will be similar, and therefore $mR \,.\, RM :: MP \,.\, PT$, that is, $\dot{y} \,.\, \dot{x} :: y \,.\, PT = \frac{y\dot{x}}{\dot{y}}$, the formula for the subtangent of the curve CMA, by taking it on the tangent of the curve APB. From the given equation of the curve AMC is found the value of $\dot{x}$ or $\dot{y}$, to be substituted in the formula. All the rest is to be done as usual.

EXAMPLE.

Fig. 35.



45. While the circle DPC revolves uniformly upon the right line AB, beginning at the point A; the point C of it's periphery, which at the beginning of the motion fell upon A, leaves an impression in the plane of it's motion, which it continues till the point C arrives again at the right line AB. It will describe a curve ACB, which, from it's generation, is called a *Cycloid*. It will be the ordinary cycloid, when the circle so moves upon the right line AB, as that it shall measure out the whole exactly by it's periphery, after that the point C shall have passed from A to B, so that AB may be equal to the periphery of the same circle. It will be a prolonged cycloid when the motion is such, that the right line AB is longer than the periphery of the circle; and a contracted cycloid when the same AB is shorter than the periphery.

From the description of this curve it plainly follows, that, drawing from any point the right line MQ parallel to AB, the intercepted line MP, between the curve and the circle CPD, will have to the arch CP the same ratio as the line AB has to the whole circle.

N. B. The chord ME is omitted in Fig. 35.

Suppose

Suppose the generating circle to be in the two positions EMF, DPC; draw the chords ME, PD. Now, because the arches EM, DP, are equal, the chords EM, DP, will be equal and parallel, and therefore MP $=$ ED. But, by the nature of the curve, it is AE . EM :: AD . EMF :: AB . EMFE. And in the same ratio is also ED . MF. And MF $=$ PC, ED $=$ MP; therefore it will be MP . PC :: AD . EMF :: AB . EMFE. Therefore, if we call the right line AB $= a$, the periphery of the generating circle EMFE $= b$, and any arch or abscissa CP $= x$, the ordinate PM $= y$; the equation of the curve of the cycloid will be $x = \frac{by}{a}$.

Having therefore the equation of the curve, in order to find the subtangent, it's fluxion will be $\dot{x} = \frac{b\dot{y}}{a}$; and, instead of $\dot{x}$, substituting this value in the formula $\frac{y\dot{x}}{\dot{y}}$, it will be PT $= \frac{by}{a} = x$. Therefore, taking, on the tangent of the circle, PK, (Fig. 34.) which is supposed to be drawn, a portion PT equal to the arch of the circle AP, and drawing the right line TM to the point M, it shall be a tangent to the cycloid in the point M.

Now, besides, if the cycloid be the ordinary one; because, in this case, we shall have $b = a$, and therefore $y = x$, it will be PM $=$ PT, and the angle PTM $=$ PMT. But the external angle TPQ is double to the angle TMP, and the angles TPA, APQ, are equal, by *Euclid*, iii. 29 and 32, therefore the angle APQ will be equal to the angle TMP, and therefore the tangent MT is parallel to the chord PA.

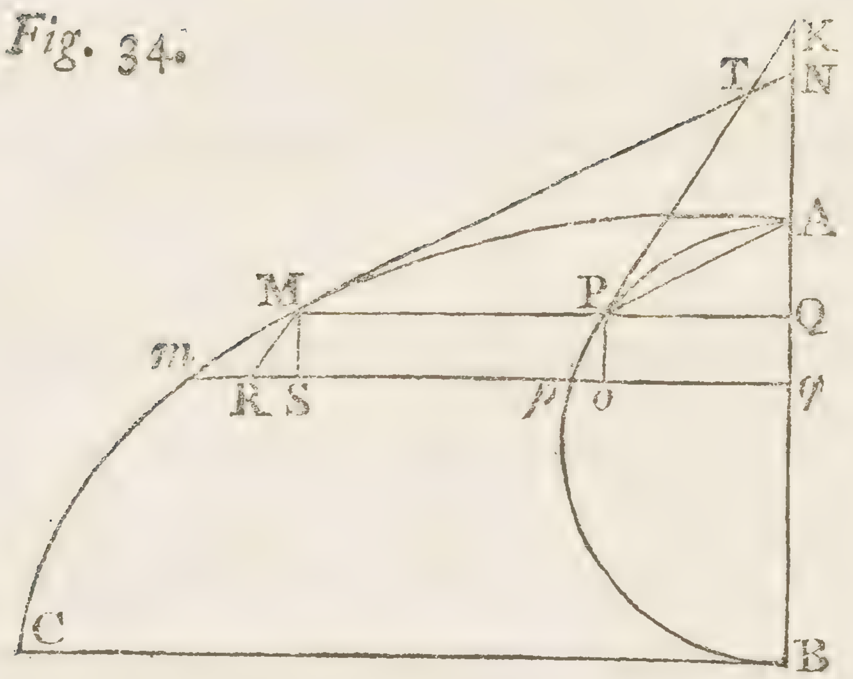
46. Without the assistance of the tangent of the curve APB, (Fig. 34.) we may have the subtangent of the curve AM, taking it in the axis KAB. Make AQ $= x$, QP $= y$, the arch AP $= s$, QM $= z$, and let the relation of the arch AP to the ordinate QM be expressed by any equation whatever. Let qm be infinitely near to QM, and MS parallel to AB. It will be MS $= \dot{x}$, S$m = \dot{z}$, and the similar triangles mSM, MQN, will give us $\dot{z} . \dot{x} :: z .$ QN $= \frac{z\dot{x}}{\dot{z}}$, a formula for the subtangent.

Instead of taking for the ordinate QM $= z$, if we take PM $= u$; drawing MR parallel to the little arch Pp, it will be mR $= \dot{u}$, RS $= po = \dot{y}$, and therefore mS $= \dot{u} + \dot{y}$. And the similar triangles mSM, MQN, will give us $\dot{u} + \dot{y} . \dot{x} :: u + y .$ QN $= \frac{\overline{u + y} \times \dot{x}}{\dot{u} + \dot{y}}$, another formula for the subtangent.

EX-

EXAMPLE I.

Fig. 34.



47. Let the curve APB be a circle whose diameter is $2r$, and let the ratio of PM to the arch PA be that of a to b; that is, let the curve AMC be a cycloid. Make $AQ = x$, $QP = y$, $QM = z$, the arch $AP = s$; then drawing mq infinitely near to MQ, MR, parallel to Pp; MS, Po, parallel to AB; it will be $mS = \dot{z}$, $RS = po = \dot{y}$, $Pp = \dot{s}$; and mR, the difference or fluxion of MP, will be $\dot{z} - \dot{y}$. But, becauſe, by the property of the curve, we have MP, to the arch PA, as a to b; in the ſame ratio, alſo, will be their differentials mR, pP; and therefore it will be $\dot{z} - \dot{y} . \dot{s} :: a . b$; that is, $\dot{z} - \dot{y} = \frac{a\dot{s}}{b}$. But $\dot{s} = \sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}$, and, by the property of the circle, $y = \sqrt{2rx - xx}$. Therefore $\dot{y} = \frac{r\dot{x} - x\dot{x}}{\sqrt{2rx - xx}}$, and $\dot{y}\dot{y} = \frac{r^2\dot{x}^2 - 2rx\dot{x}^2 + x^2\dot{x}^2}{2rx - xx}$; whence $\dot{s} = \frac{r\dot{x}}{\sqrt{2rx - xx}}$.

Wherefore, theſe values being ſubſtituted inſtead of $\dot{s}$ and $\dot{y}$ in the equation $\frac{a\dot{s}}{b} = \dot{z} - \dot{y}$, we ſhall have $\dot{z} = \frac{ar\dot{x} + br\dot{x} - bx\dot{x}}{b\sqrt{2rx - xx}}$, the differential equation of the cycloid.

Therefore, the value of $\dot{z}$, given from the equation, being ſubſtituted in the formula for the ſubtangent $\frac{z\dot{x}}{\dot{z}}$, we ſhall have $QN = \frac{bz\sqrt{2rx - xx}}{ar + br - bx}$.

Now, if the cycloid be the ordinary one, it will be $a = b$, and therefore $QN = \frac{z\sqrt{2rx - xx}}{2r - x}$; that is, $2r - x . \sqrt{2rx - xx} :: z . QN$; or $2r - x . y :: z . QN$. But, by the property of the circle, it is $2r - x . y :: y . x$. Therefore it will be $y . x :: z . QN$; that is, $QP . QA :: QM . QN$. Therefore MN will be parallel to PA.

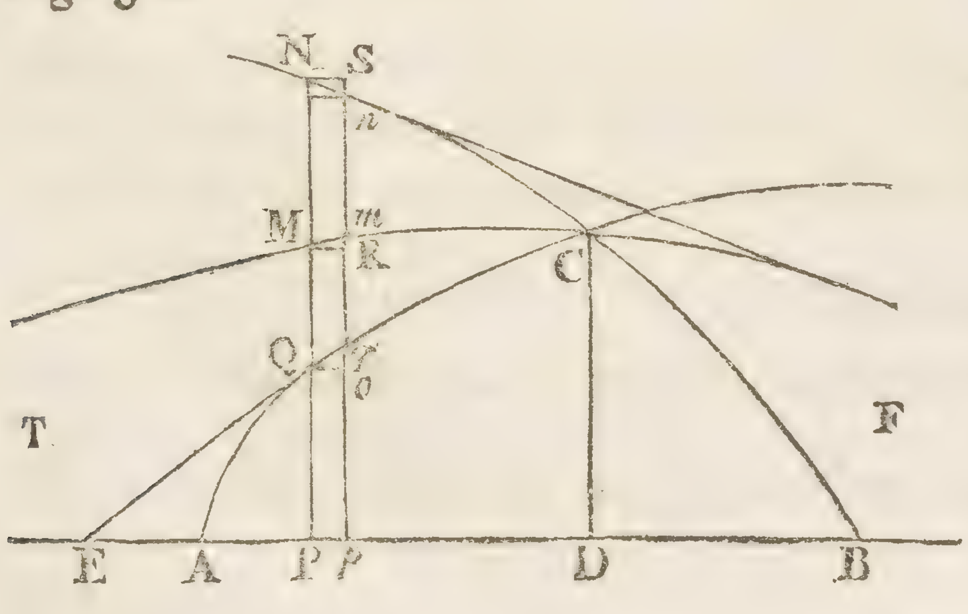
E X-

EXAMPLE II.

48. Let the curve APB be a parabola, the equation of which is $px = yy$. Make AQ $= x$, QP $= y$, and let the arch AP $= s$, PM $= u$, and the ratio of MP to the arch PA be that of a to b. Therefore it will be mR . Pp :: $a . b$. That is, $\dot{u} . \dot{s}$ :: $a . b$, and therefore $\frac{a\dot{s}}{b} = \dot{u}$. But, in the parabola, it is $y = \sqrt{px}$, and $\dot{y} = \frac{p\dot{x}}{2\sqrt{px}}$. Therefore $\dot{s} = \frac{\dot{x}\sqrt{4px + pp}}{2\sqrt{px}}$. And this value being substituted instead of $\dot{s}$ in the equation $\frac{a\dot{s}}{b} = \dot{u}$, the equation to the curve AMC will be $\frac{a\dot{x}\sqrt{4px + pp}}{2b\sqrt{px}} = \dot{u}$. Wherefore, taking the formula of the subtangent $\frac{\overline{u + y} \times \dot{x}}{\dot{u} + \dot{y}}$, which is proper to this case, and making the substitutions instead of $\dot{u}$ and $\dot{y}$, it will be QN $= \frac{\overline{u + y} \times 2b\sqrt{px}}{a\sqrt{4px + pp} + bp}$. But $y = \sqrt{px}$, by the property of the curve APB, and $\frac{as}{b} = u$, by the property of the curve AMC; wherefore QN $= \frac{2as\sqrt{px} + 2bpx}{a\sqrt{pp + 4px} + bp}$.

49. From the different manners by which many curves may be generated, arise different formulas of their subtangents, though the method of finding them is alike. It will be enough to show it in one, to give an idea of the manner, and of the artifice, which is to be used on all other occasions. Wherefore, two curves AQC, BCN, being given, having a common diameter TF, whose tangents can be drawn; let there be another curve MC such, as that the relation of the ordinates PQ, PM, PN, in respect of any point at pleasure, M, may be expressed by any equation whatever; and let the tangent MT be required, at any point M. Let pS be drawn infinitely near to PN, and the lines NS, MR, QO, parallel to AB, and make

Fig. 36.



make PE $= s$, PF $= t$, known by fuppofition, PQ $= x$, PM $= y$, PN $= z$. Becaufe of fimilar triangles QPE, qOQ, it will be QO $= \frac{s\dot{x}}{x} =$ MR $=$ NS; and, becaufe of the fimilar triangles mRM, MPT, it will be PT $= \frac{sy\dot{x}}{x\dot{y}}$, a formula for the fubtangent. Now, by differencing the equation of the curve MC, in order to have the value of $\dot{x}$, to be fubftituted in this formula, it will be given by $\dot{y}$ and $\dot{z}$; but the fubtangent itfelf is not to be had in finite terms. It is to be confidered, then, that the fimilar triangles NSn, NPF, will give us NP . PF :: nS . SN, that is, $z \,.\, t :: \pm \dot{z} \,.\,$ SN $= \pm \frac{t\dot{z}}{z}$. (That is, $\dot{z}$ muft have a pofitive fign, if, when x and y increafe, z will increafe alfo; and a negative fign, if, when x and y increafe, z will decreafe.) But it is alfo SN $= \frac{s\dot{x}}{x}$; then $\pm \frac{t\dot{z}}{z} = \frac{s\dot{x}}{x}$, and therefore $\dot{z} = \pm \frac{sz\dot{x}}{tx}$. Therefore, inftead of $\dot{z}$, putting this value in the fluxional equation of the curve MC, we fhall have the value of $\dot{x}$ expreffed by $\dot{y}$, which, being fubftituted in the formula for the fubtangent $\frac{sy\dot{x}}{x\dot{y}}$, will make the fluxions to vanifh, and the fubtangent will be expreffed in finite terms.

EXAMPLE I.

50. Let $xz = yy$ be the equation of the curve MC, the fluxion of which will be $z\dot{x} + x\dot{z} = 2y\dot{y}$; and, inftead of $\dot{z}$, fubftituting it's value $\pm \frac{sz\dot{x}}{tx}$, it will become $z\dot{x} \pm \frac{sz\dot{x}}{t} = 2y\dot{y}$, and therefore $\dot{x} = \frac{2ty\dot{y}}{tz \pm sz}$. Wherefore, inftead of $\dot{x}$, fubftituting this value in the formula for the fubtangent, it will be PT $= \frac{2styy}{tzx \pm szx} = \frac{2st}{t \pm s}$, when, inftead of yy, we put it's value xz. Now let the curve AQC be a parabola whofe parameter is b; the curve BCN a circle whofe diameter is AB $= 2a$. If, therefore, the point N falls in the periphery of the firft quadrant beginning at A, in which $\dot{z}$ is pofitive; the formula of the fubtangent PT will be $\frac{2st}{t+s}$, and the fubtangent of the circle will be $\frac{2aq-qq}{a-q} = t$, (making AP $= q$,) and that of the parabola will be $2q = s$. Therefore, thefe values of t and s being put in the expreffion $\frac{2st}{t+s}$, we fhall have PT $= \frac{8aq - 4qq}{4a - 3q}$.

51. But

51. But if the point N falls in the periphery of the other quadrant, $\dot z$ will be negative, and the formula of the fubtangent will be PT $= \frac{2st}{t - s}$. In this cafe, the fubtangent of the circle is $\frac{2aq - qq}{q - a} = t$, and that of the parabola continues to be $2q = s$. Therefore, making the fubftitution of the values of t and s in the expreffion $\frac{2st}{t - s}$, we fhall have PT $= \frac{8aq - 4qq}{4a - 3q}$; the fame as before.

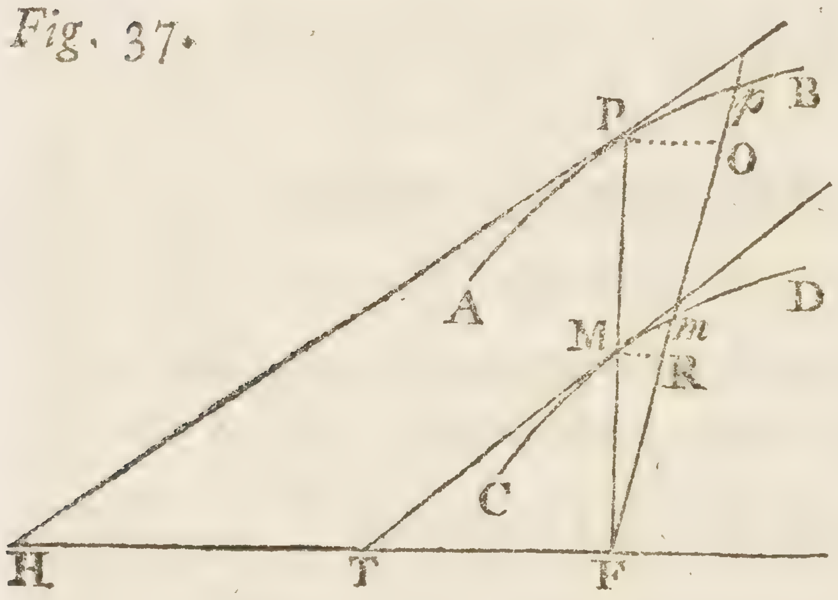
52. Let AP be denominated as before, AQ being a parabola; it will be PQ $= x = \sqrt{bq}$. And BCN being a circle, it will be PN $= z = \sqrt{2aq - qq}$. Then the equation $yy = zx$ of the curve MC will be $yy = q\sqrt{2ab - bq}$. And thus, the equation being given by the two co-ordinates AP, PM, the fubtangent PT may be found by the ufual and ordinary formulas $\frac{y\dot q}{\dot y}$. Therefore, differencing the equation $yy = q\sqrt{2ab - bq}$, it will be $y\dot y = \frac{4ab\dot q - 3bq\dot q}{4\sqrt{2ab - bq}}$. Now, multiplying the numerator and denominator of the formula $\frac{y\dot q}{\dot y}$ by y, it will be $\frac{yy\dot q}{y\dot y}$, and fubftituting the refpective values inftead of yy and $y\dot y$, it will be $\frac{yy\dot q}{y\dot y} = \frac{8aq - 4qq}{4a - 3q} =$ PT, as before.

53. Let the equation of the curve MC be more general, thus, $x^m z^n = y^{m+n}$, the fluxion of which is $mz^n \dot x x^{m-1} + nx^m \dot z z^{n-1} = \overline{m + n} \times \dot y y^{m+n-1}$. And, inftead of $\dot z$, putting it's value $\pm \frac{sz\dot x}{tx}$, it will be $\frac{tmz^n \dot x x^{m-1} \pm snz^n \dot x x^{m-1}}{t} = \overline{m + n} \times \dot y y^{m+n-1}$; and therefore $\dot x = \frac{\overline{mt + nt} \times \dot y y^{m+n-1}}{\overline{mt \pm ns} \times z^n x^{m-1}}$. Whence PT $= \frac{sy\dot x}{x\dot y} = \frac{\overline{m + n} \times sty^{m+n}}{\overline{mt \pm ns} \times z^n x^m} = \frac{m + n}{mt \pm ns} st$, if we put it's value $x^m z^n$ inftead of y^{m+n}.

54. If the two curves AC, BCN, become right lines, in the cafe of the fimple equation $xz = yy$ of the curve MC, it will be one of the Conic Sections of *Apollonius*, as is to be feen in Sect. III. of Vol. I. § 135. It will be an ellipfis, when the ordinate CD falls between the points A and B: an hyperbola, when it falls either on one fide or the other: and laftly, a parabola, when the points A, B, are infinitely diftant one from the other, that is, when one of the

right lines AC, BC, is parallel to the diameter. Hence it is manifeſt, that, in the ſame circumſtances, the ſame curves will be conic ſections, but of a ſuperior degree *in infinitum*, when the equation to the curve MC ſhall be this general one, $x^m z^n = y^{m+n}$.

Fig. 37.



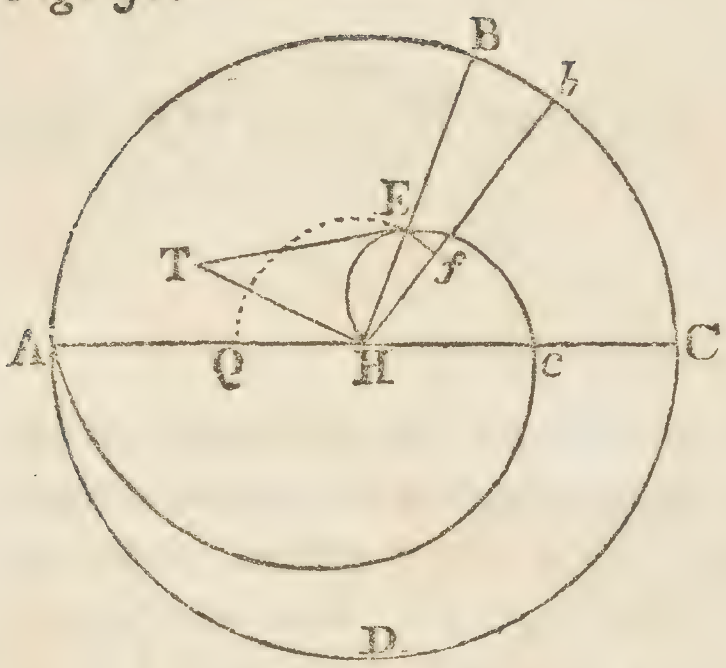
55. If the curve AP be given, having it's origin in A, of which we know how to draw the tangent; let there be another curve CMD ſuch, that, from a given point F drawing the right line FMP any how, the relation of FM to the portion AP may be expreſſed by any equation: we are to find the tangent of the curve CMD.

Let PH be a tangent to the curve APB in the point P, and let FH be drawn perpendicular to FP, and Fp infinitely near; and with centre F let the infinitely little arches MR, PO, be deſcribed; and let MT be the tangent required of the curve CMD. Make PH $= t$, FH $= s$, FM $= y$, FP $= z$, and the arch AP $= x$. Becauſe, inſtead of the infiniteſimal arches, their right ſines may be aſſumed, the triangle MRm will be right-angled at R; and, becauſe the angle MmR is not different from the angle TMF, but only by the infiniteſimal angle MFm, the two triangles MRm, TFM, may be conſidered as ſimilar; and, for the ſame reaſon, the two triangles POp, HFP, are ſimilar. Therefore it will be mR . RM :: MF . FT; that is, $\dot{y}$. MR :: y . FT, and FT $= \frac{\text{MR} \times y}{\dot{y}}$. So that, to have the value of FT, it is neceſſary to have that of MR firſt, which we might have if PO were known. Now, by the ſimilar triangles PFH, POp, it will be PH . FH :: Pp . PO; that is, $t \,.\, s :: \dot{x} \,.\, \text{OP} = \frac{s\dot{x}}{t}$. And, by the ſimilar ſectors FPO, FMR, it will be FP . PO :: FM . MR; that is, $z \,.\, \frac{s\dot{x}}{t} :: y \,.\, \text{MR} = \frac{sy\dot{x}}{zt}$. Whence FT $= \frac{syy\dot{x}}{tz\dot{y}}$, the formula for the ſubtangent. Now if, inſtead of $\dot{x}$, we ſubſtitute it's value, which may be obtained from the fluxional equation of the curve CMD, we ſhall have the ſubtangent expreſſed in finite terms.

EX-

EXAMPLE I.

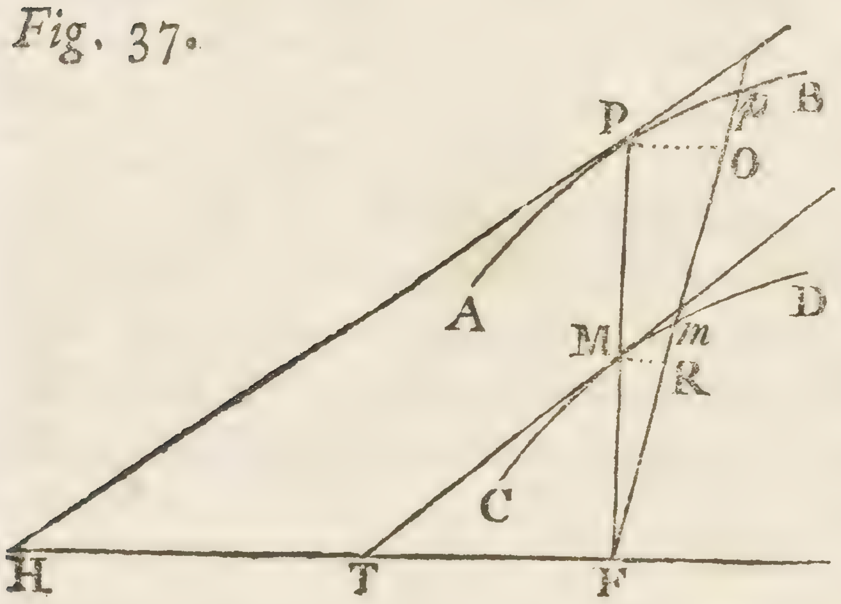
Fig. 38.



56. Let there be a circle ABCD described with centre H, and radius HA; and whilst the radius HA, with one end fixed in the centre, moves uniformly round, and with the other extremity A describes the periphery ABCD; let the point H move uniformly upon the radius HA, so that when the radius returns to it's first situation HA, the point H, in the mean time, shall pass through the radius, and shall then be found at A. The point H will then describe the curve HE*c*A, which is called the *Spiral* of *Archimedes*. From the generation of this curve, it is easy to perceive that any arch of the circle whatever, as AB, shall be to the corresponding portion of the radius HE, as the whole circle is to the whole radius. Therefore, making the radius $= r$, the periphery of the circle $= c$, the arch AB $= x$, and the ordinate HE $= y$; it will be $x \,.\, y :: c \,.\, r$; and therefore $y = \frac{rx}{c}$, an equation to the spiral, in which the ordinates proceed from the fixed point H. This being premised, if we would find ET, the tangent of the spiral; because, in this case, FP (Fig. 37.) is the radius HB of the circle, it will be $z = r$, and the two lines, PH the tangent, and FH the subtangent, (in the same Fig. 37,) are in this both perpendicular to the radius HB, (by the nature of the circle,) and consequently parallel to each other, and also equal; whence it will be $\dot{s} = t$, and therefore the general formula, in this case, will be $\frac{yy\dot{x}}{r\dot{y}}$. Then, differencing the equation $y = \frac{rx}{c}$, it will be $\dot{y} = \frac{r\dot{x}}{c}$; and the value of $\dot{x}$ being substituted in the formula, it will be $\frac{cyy}{rr} =$ HT. Or else, putting, instead of y, it's value $\frac{rx}{c}$, it will be $\frac{xy}{r} =$ HT. Therefore, with centre H, and radius HE $= y$, describing the arch EQ, and taking HT equal to the arch EQ, it shall be the subtangent. For, by similar sectors HEQ, HBA, it will be HA . AB :: HQ . QE. That is, $r \,.\, x :: y \,.\,$ QE $= \frac{xy}{r}$.

If, inſtead of making the equation $y = \frac{rx}{c}$, it were, in general, $y^m = \frac{r^m x}{c}$; that is, the periphery to the arch AB, as any power integral or fractional of the radius, to a like power of the ordinate: Then taking the fluxion of the equation, it would give us $\dot{x} = \frac{mc\dot{y}y^{m-1}}{r^m}$, and $y\dot{x} = \frac{mc\dot{y}y^m}{r^m}$. Then ſubſtituting this in the formula of the ſubtangent $\frac{yy\dot{x}}{r\dot{y}}$, it would be $\frac{mcy^{m+1}}{r^{m+1}} = \mathrm{HT}$. But $y^m = \frac{r^m x}{c}$; therefore $\frac{mxy}{r} = \mathrm{HT} = m \times \mathrm{EQ}$.

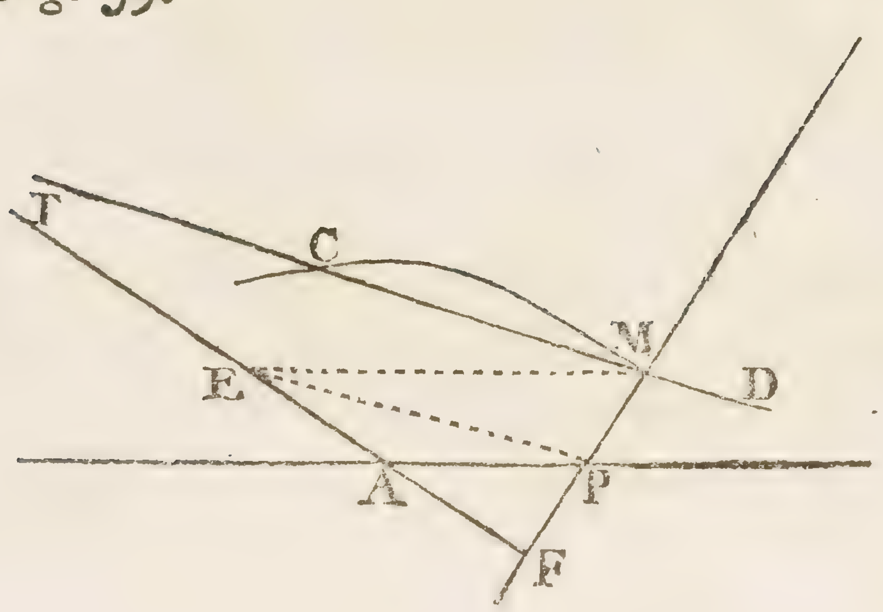
Fig. 37.



57. We ſhall have the formula of the ſubtangent more ſimple, if the equation of the curve APB were given from the relation of TM to FP. For the ſimilar triangles *p*OP, PFH, will give us PO $= \frac{s\dot{z}}{z}$, and the ſimilar ſectors FPO, FMR, will give us MR $= \frac{sy\dot{z}}{zz}$; and laſtly, the ſimilar triangles MR*m*, TFM, will give us FT $= \frac{syy\dot{z}}{zz\dot{y}}$.

EXAMPLE II.

Fig. 39.

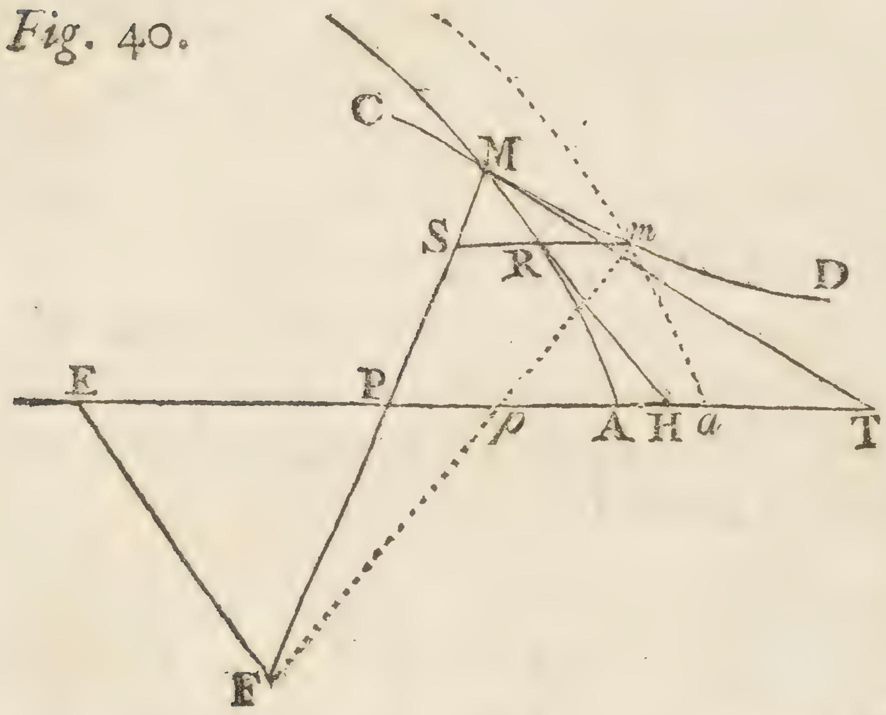


58. Let the curve CMD be above the line APB, which makes no alteration, and let APB be a right line, and let FM, FP, always differ from each other by the ſame quantity, that is, make the conſtant line PM $= a$. Then will $y - z = a$ be the equation of the curve, which is the Conchoid of *Nicomedes*, whoſe pole is the point F, and aſymptote AB. Taking the fluxions

fluxions of the equation, it will be $\dot{y} = \dot{z}$, and thence the ſubtangent FT $= \frac{sy}{zz}$.

Drawing, then, ME parallel to PA, and MT parallel to PE, MT will be a tangent to the curve in M. For it will be FA $= s$, FE $= \frac{sy}{z}$, and FT $= \frac{syy}{zz}$.

Fig. 40.



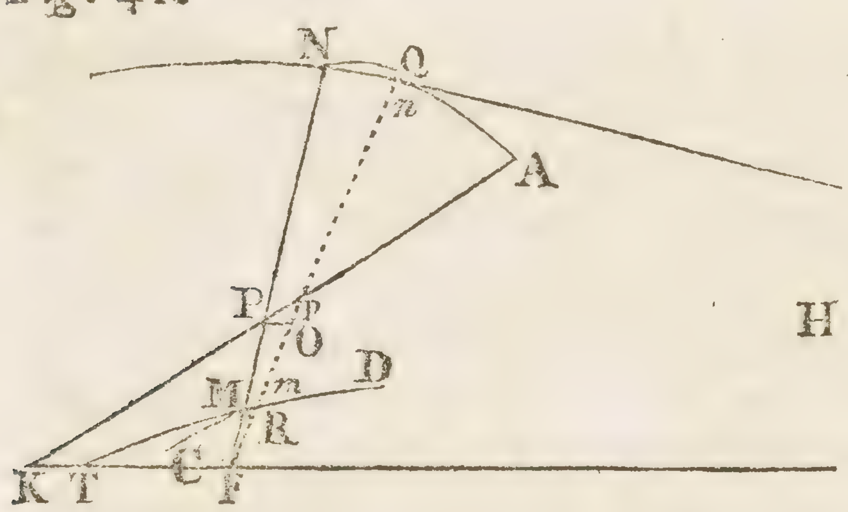
59. Any curve AM being given, to the axis EAT of which curve we know how to draw the tangent MH, at any point M; and a point F being given out of the curve, from which let be drawn the right line FPM; if we conceive the right line FPM to revolve about the immoveable point F, making the plane PAM to move upon the right line ET, always parallel to itſelf, the intercepted line PA always continuing the ſame: Then the point M, which is the common interſection of the two lines FM, AM, by this motion will deſcribe a curve CMD, the tangent of which is required. Let the plane PAM move, and, in the firſt inſtant, let it arrive at an infinitely near poſition *pam*, and let SR*m* be drawn parallel to ET. The ſimilar triangles MR*m*, MHT, would give the right line HT, which determines the tangent required, if the ſides MR, R*m*, were known. Therefore, to obtain them, let us make FP, or F*p* $= x$, FM, or F*m* $= y$, P*p* $= \dot{z}$, and the known lines PA $= a$, HM $= t$, PH $= s$. It is plain, by the conſtruction, that it will be P*p* $=$ A*a* $=$ R*m* $= \dot{z}$; and, by the ſimilar triangles FP*p*, FS*m*, it will be F*p* . P*p* :: F*m* . S*m*. That is, $x . \dot{z} :: y . Sm = \frac{y\dot{z}}{x}$. Then SR $= \frac{y\dot{z} - x\dot{z}}{x}$. And, by ſimilar triangles MPH, MSR, it will be HP . HM :: RS . RM. That is, $s . t :: \frac{y\dot{z} - x\dot{z}}{x} . MR = \frac{ty\dot{z} - tx\dot{z}}{sx}$. Laſtly, by the ſimilar triangles MR*m*, MHT, it will be MR . R*m* :: MH . HT. That is, $\frac{ty\dot{z} - tx\dot{z}}{sx} . \dot{z} :: t . HT = \frac{sx}{y - x}$.

From the point F draw FE parallel to the tangent MH, and taking HT $=$ PE, draw TM, which ſhall be a tangent to the curve at the point M. For, becauſe of ſimilar triangles PMH, PFE, it will be PM . PH :: PF . PE; that is, $y - x . s :: x . \frac{sx}{y - x} =$ PE $=$ HT.

60. It

60. It has been already demonſtrated, Vol. I. Sect. III. § 136, that, if the line AM were a right line, the curve CMD would be an hyperbola, which would have ET for one of it's two aſymptotes. If AM were a circle with centre P, the curve CMD would be the conchoid of *Nicomedes*, the pole of which is F, and it's aſymptote ET. And laſtly, if AM were a parabola, the curve CMD would be the companion of the paraboloid of *Carteſius*, that is, one of the two parabolical conchoids.

Fig. 41.



61. To the diameter AP let there be any curve AN, whoſe tangent we know how to draw, and a fixed point F out of it; and let there be another curve CMD ſuch, that, drawing, as we pleaſe, the right line FMPN from the point F, the relation between FN, FP, FM, may be expreſſed by any equation whatever. It is required to find the tangent MT, at any given point M.

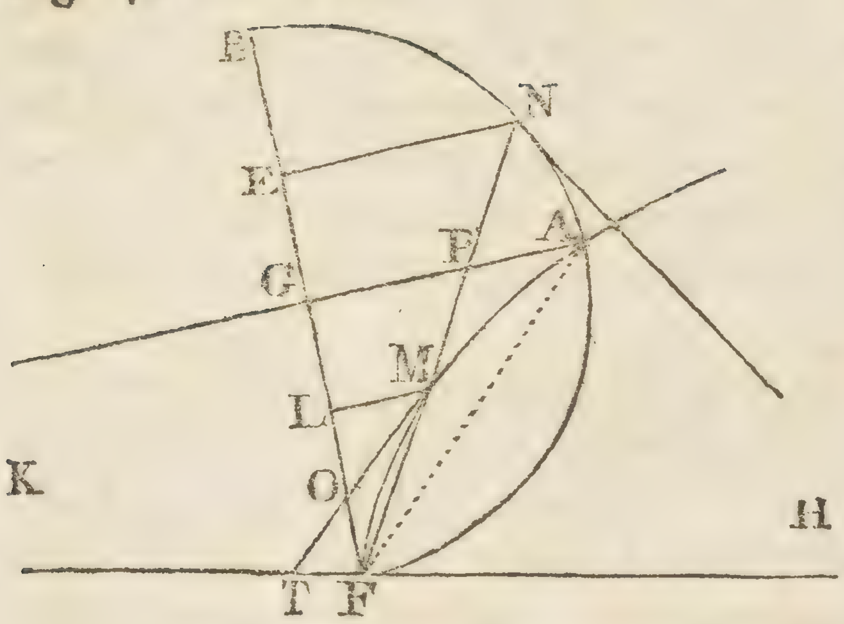
Through the point F draw HK perpendicular to FN, which meets the diameter AP produced in K, and the given tangent NH in H. Let FQ be infinitely near FN, and with centre F let the arches MR, P*o*, NQ, be deſcribed. Make FK $= s$, FH $= t$, FP $= x$, FM $= y$, FN $= z$; then it will be mR $= \dot{y}$, $po = \dot{x}$, Q$n = -\dot{z}$. And, becauſe of like triangles NQn, NFH, it will be NQ $= -\frac{t\dot{z}}{z}$. Alſo, becauſe of like ſectors FNQ, FMR, it will be MR $= -\frac{ty\dot{z}}{zz}$. Laſtly, becauſe of like triangles MRm, MFT, it will be FT $= -\frac{yyt\dot{z}}{zz\dot{y}}$, the formula required for the ſubtangent. But here it might be ſuſpected, that, taking the fluxion of the equation of the curve, the value of $\dot{y}$ to be ſubſtituted in the formula will be given by $\dot{x}$ and $\dot{z}$, by which means the fluxions would not vaniſh. Yet, however, the ſimilar ſectors FNQ, FPo, will give us Po $= -\frac{tx\dot{z}}{zz}$; and the ſimilar triangles Pop, PFK, will give us the analogy, $\dot{x} \,.\, -\frac{tx\dot{z}}{zz} :: x \,.\, s$. Whence the equation $szz\dot{x} = -txx\dot{z}$, and therefore $-\dot{z} = \frac{szz\dot{x}}{txx}$. Therefore, ſubſtitute the value of $\dot{y}$ in the formula for the ſubtangent, which value is to be obtained from the fluxional equation of the curve, and then this value inſtead of $\dot{z}$; by which the fluxions will vaniſh, and we ſhall have the ſubtangent in finite terms.

If

If the line AP were a curve inſtead of a right line, drawing the tangent PK, by the ſame way of argumentation we ſhould find the ſame value of the ſubtangent FT.

EXAMPLE.

Fig. 42.



62. Let the curve AN be a circle which paſſes through the point F, and is ſo poſited, that, from the point F drawing the perpendicular FB (produced) to AP, it may paſs through the centre G of the ſame circle; and let PN be always equal to PM: the curve CMD of the foregoing figure, that is, FMA in this, will be the ciſſoid of *Diocles*, the equation of which will be $z + y = 2x$. Then we ſhall have, by taking the fluxion, $\dot{z} + \dot{y} = 2\dot{x}$, or $\dot{y} = 2\dot{x} - \dot{z}$; and ſubſtituting this value of $\dot{y}$ in the formula $-\frac{yyt\dot{z}}{zz\dot{y}}$ of the ſubtangent, it will be $-\frac{yyt\dot{z}}{2zz\dot{x} - zz\dot{z}}$; and laſtly, putting, inſtead of $-\dot{z}$, it's value $\frac{szz\dot{x}}{txx}$, we ſhall have $\frac{styy}{2txx + szz} = \text{FT}$, the ſubtangent required.

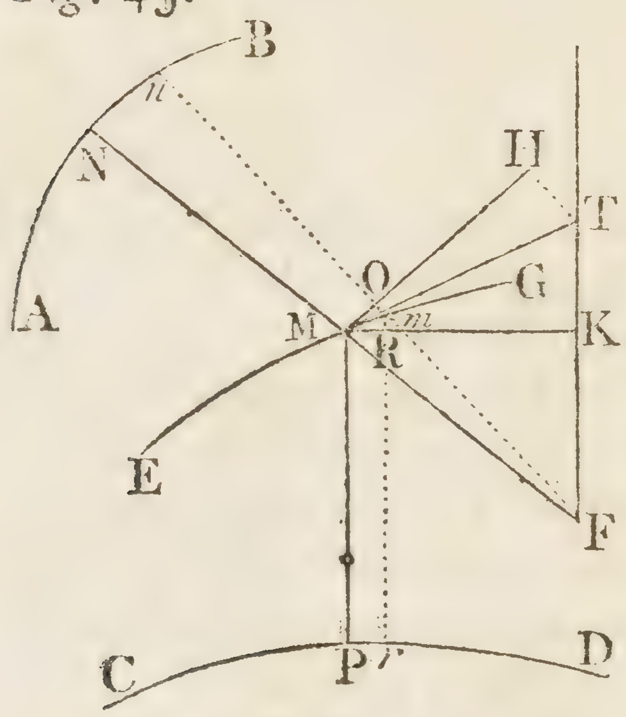
Here it is plain, that if the point M, at which the tangent is required, ſhould fall upon the point A; in this caſe, KH being perpendicular to FA, it would be FN = FP = FM = FA = FK = FH; and therefore FT $= \frac{1}{3}x = \frac{1}{3}$AF.

63. Perhaps we might find the ſubtangent of the ciſſoid more ſpeedily, by means of the uſual formula, at § 30. For, drawing NE, ML, perpendicular to FB, and making FB $= 2a$, FL $= x$, LM $= y$; by the property of the curve FMA, it will be BE = FL $= x$; and, by the property of the circle, it will be EN $= \sqrt{2ax-xx}$; and the ſimilar triangles FLM, FEN, will give FL . LM :: FE . FN, and therefore FL . LM :: EN . EB; that is, $x . y :: \sqrt{2ax - xx} . x$, whence $y = \frac{xx}{\sqrt{2ax - xx}}$, or $yy = \frac{x^3}{2a - x}$, the equation of the curve FMA. Therefore, by taking the fluxions, we ſhall have $2y\dot{y} = \frac{6axx\dot{x} - 2x^3\dot{x}}{\overline{2a - x}|^2}$; and taking the uſual formula $\frac{y\dot{x}}{\dot{y}}$, and making all the neceſſary

 ſubſtitutions,

fubftitutions, it will be $\frac{y\dot{x}}{\dot{y}} = yy \times \frac{\overline{2a - x}|^2}{3ax^2 - x^3} = LO = \frac{2ax - xx}{3a - x}$, by putting, inftead of yy, it's value $\frac{x^3}{2a - x}$.

Fig. 43.



64. Let there be two curves ANB, CPD, and a right line FK, in which are three fixed points A, C, F. Further, let the curve EMG be fuch, that, drawing through any of it's points, M, the right line FMN from the given point F, and from the point M the right line MP parallel to FK; the relation of the arch AN to the arch CP fhall be expreffed by any equation at pleafure. It is required to find the tangent of the curve EG at the point M.

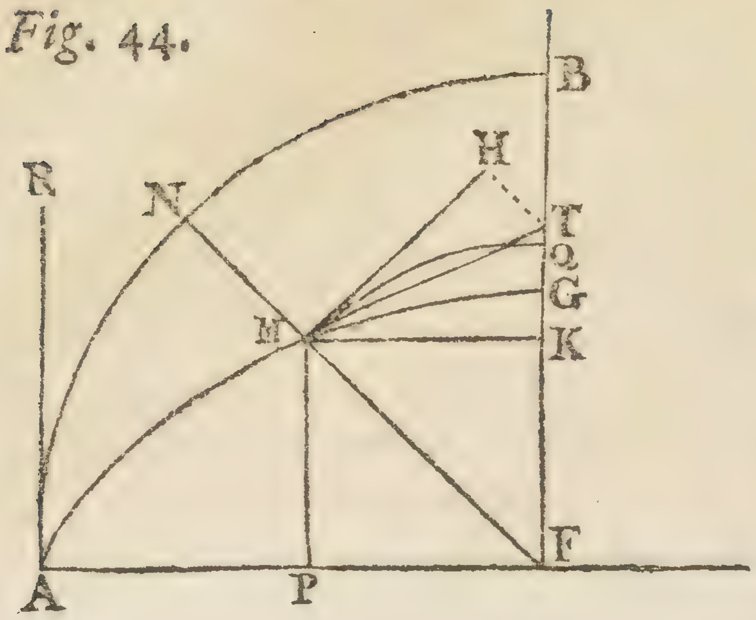
Let MT be the tangent required, which meets in T the right line FK, produced if need be, and from the point T let there be drawn TH parallel to FM, and through the point M let be drawn MRK parallel to the tangent in P, and MOH parallel to the tangent in N, and let FmOn be infinitely near to FN. Make FM $= s$, FN $= t$, MK $= u$, and the arches AN $= y$, CP $= x$; and therefore Nn $= \dot{y}$, Pp $= \dot{x}$. By the fimilar triangles FNn, FMO, it will be FN . Nn :: FM . MO; that is, $t \,.\, \dot{y} :: s \,.\, \text{MO} = \frac{s\dot{y}}{t}$. And, by the fimilar triangles MmR, MTK, and MOm, MHT, it will be MR . mM :: MK . MT, and Mm . MO :: MT . MH; and it will be alfo MR . MO :: MK . MH. That is, $\dot{x} \,.\, \frac{s\dot{y}}{t} :: u \,.\, \text{MH} = \frac{us\dot{y}}{t\dot{x}}$. Wherefore, by taking the fluxion of the given equation, we fhall have the value of $\dot{y}$ given by $\dot{x}$; and, by making the neceffary fubftitutions, we fhall have MH expreffed in finite terms. Taking, therefore, MH equal to the value now found, and parallel to the tangent in N of the curve ANB, and drawing HT parallel to MF; if from the point M be drawn the right line TM to the point T, it will be a tangent to the curve EMG in the point M.

N. B. The letter r has been put, by miftake, for the letter p, in Fig. 43.

E X-

EXAMPLE.

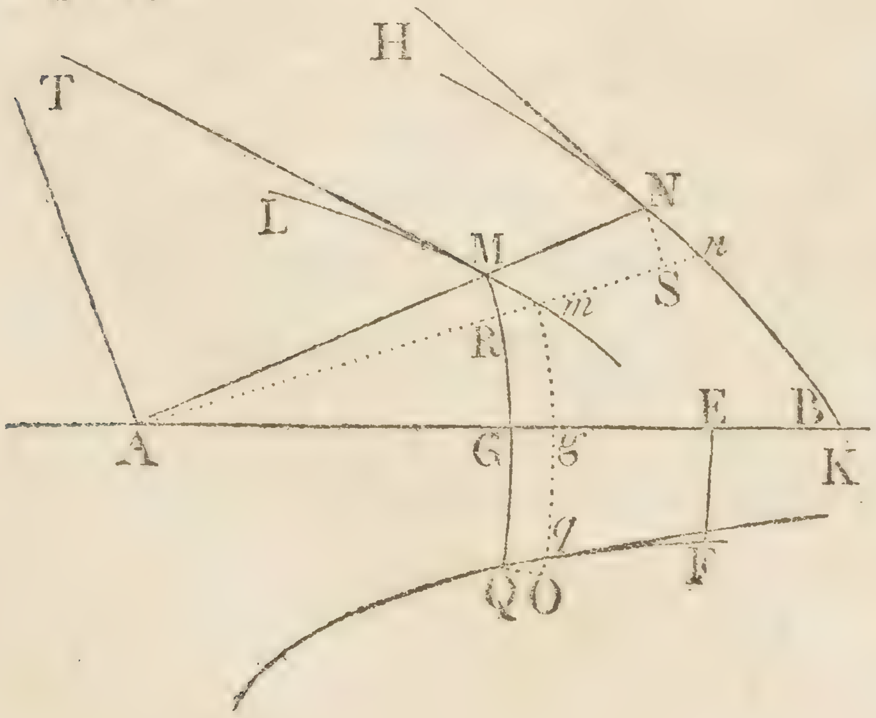
Fig. 44.



65. Let the curve ANB be a quadrant of a circle, whose centre is F; and let CPD of Fig. 43 be the radius APF of Fig. 44, which is perpendicular to the right line FKTB, and let the tangent AR be drawn. Let the radius FA be conceived to revolve equably about the centre F, and, at the same time, the tangent AR to move equably upon AF towards FB, always parallel to itself; so that, when the radius FA falls upon FB, the tangent AR may coincide with FB. By this motion, the point M, which is the intersection of the radius and the tangent, will describe the curve AMG, called the *Quadratrix* of *Dinostratus*.

It is plain, from the generation of this curve, that the arch AN will be to the intercepted line AP, as the quadrantal arch AB is to the radius AF. Therefore, making AN $= y$, AP $= x$, AB $= a$, AF $= r$, it will be $ry = ax$, and $\dot{y} = \frac{a\dot{x}}{r}$; then, substituting this value of $\dot{y}$ in the formula $\frac{us\dot{y}}{t\dot{x}}$, it will be MH $= \frac{asu}{rt}$; but, in this case, FN is the radius of the circle, and MK $=$ AF $-$ AP; then $t = r$, $u = r - x$; whence MH $= \frac{asr - asx}{rr} = \frac{as - sy}{r}$, putting, instead of ax, it's value ry from the given equation. From the point M raise MH perpendicular to FM, and equal to the arch MQ described with centre F, radius FM, and let HT be drawn parallel to FM; then MT will be a tangent to the quadratrix in the point M. For, because of similar sectors FNB, FMQ, it will be FN . NB :: FM . MQ. That is, $r \,.\, a - y :: s \,.\,$ MQ $= \frac{as - sy}{r} =$ MH.

Fig. 45.



66. Let there be two curves BN, FQ, of which it is known how to draw the tangents, and which have the right line BA for a common axis, in which are two fixed points A, E. And let there be another curve LM, such, that, drawing the right line AMN through any of it's points M, and with centre A and radius AM describing the arch MG; and from the point G letting fall GQ perpendicular to AG; the relation of the spaces

 ANB,

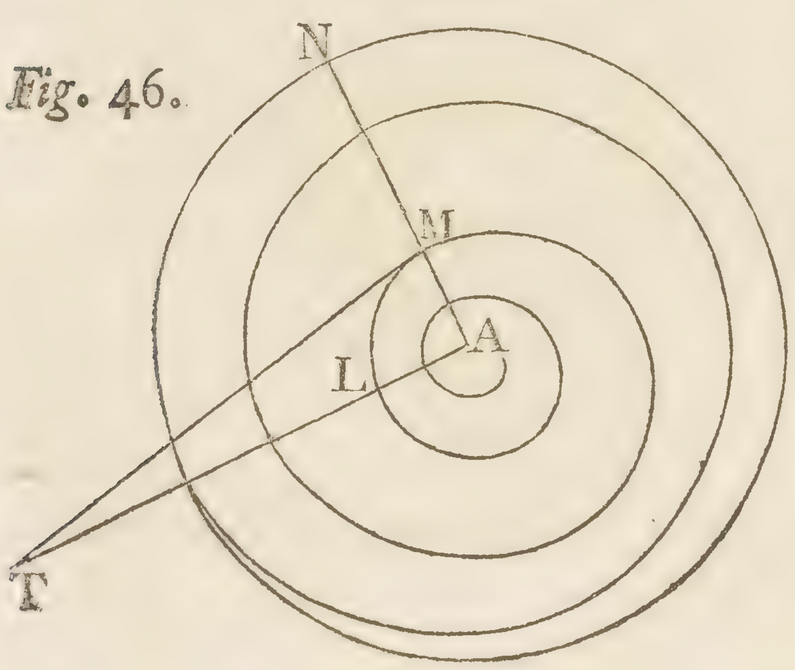
ANB, EFQG, and of the lines AM, AN, QG, may be given by the means of any equation. The tangent of the curve LM is required at the point M.

Drawing the right line ATH perpendicular to AMN, let there be another Amn infinitely near to AMN, and the arch mq, and the perpendicular gq. Then, with centre A defcribing the little arch NS, making the given fubtangents HA $= a$, GK $= b$, and make AM $= y$, AN $= z$, QG $= u$, and the fpaces EGQF $= s$, ANB $= t$, it will be Rm $=$ Gg $= \dot{y}$, Sn $= \dot{z}$. And, becaufe of the fimilar triangles KGQ, QOq, it will be Oq $= - \dot{u} = \frac{u\dot{y}}{b}$. And, by the fimilar triangles HAN, NSn, it will be SN $= \frac{a\dot{z}}{z}$. The fpace GQqg may be taken for the fpace GQOg, becaufe their difference QOg is an infinitefimal of the fecond order. Whence it will be GQqg $= u\dot{y} = - \dot{s}$. Thus, therefore, it will be ANn $= \frac{1}{2}$AN $\times$ NS $= \frac{1}{2}a\dot{z} = - \dot{t}$. Wherefore, thefe values being fubftituted, inftead of $\dot{u}$, $\dot{s}$, $\dot{t}$, in the fluxion of the propofed equation, we fhall have an equation from whence may be deduced the value of $\dot{z}$ given by $\dot{y}$. Now, becaufe of fimilar fectors ARM, ANS, it will be MR $= \frac{ay\dot{z}}{zz}$; and, by the fimilar triangles mRM, MAT, it will be AT $= \frac{ayy\dot{z}}{zz\dot{y}}$, the formula for the fubtangent; in which, inftead of $\dot{z}$, if we fubftitute it's value given by $\dot{y}$ from the equation of the curve, the fluxions will difappear, and the fubtangent will be given in finite terms.

EXAMPLE.

67. Let the fpace EGQF be double to ABN, that is, $s = 2t$; then $\dot{s} = 2\dot{t}$. But $\dot{s} = - u\dot{y}$, and $\dot{t} = - \frac{1}{2}a\dot{z}$; therefore it will be $u\dot{y} = a\dot{z}$, and $\dot{z} = \frac{u\dot{y}}{a}$. Then the fubtangent is AT $= \frac{uyy}{zz}$.

Fig. 46.



Let the curve BN be a circle with centre A, radius AN $= c$; whence $z = c$; and let the curve FQ be an hyperbola with the equation $uy = ff$; the fubtangent will be AT $= \frac{ffy}{cc}$; that is, the ratio of AM to AT will be conftant. The curve LM (Fig. 46.) will be called, in this cafe, the *Logarithmic Spiral*.

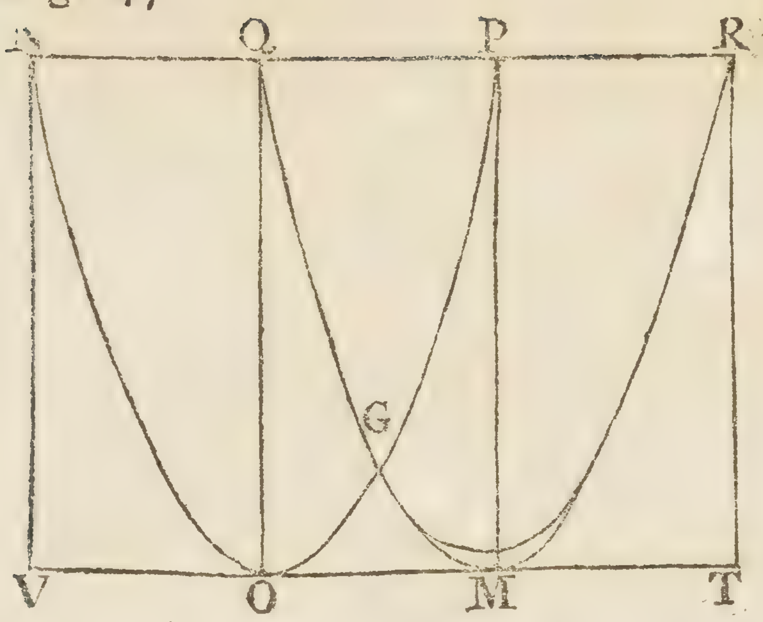
7 Here

Here it is manifeſt, that the curve LM will make an infinite number of circumvolutions before it arrives at the point A; foraſmuch as, when the point G (Fig. 45.) coincides with A, the ſpace s will be infinite, as may be ſeen from the Inverſe Method of Fluxions. For then, alſo, the ſpace t muſt be infinite, which cannot be but after infinite revolutions of the radius AM.

68. It remains, laſtly, to conſider a particular caſe belonging to Tangents. It has been ſeen that, the co-ordinates of any curve being x and y, the general formula of the ſubtangent will be $\frac{y\dot{x}}{\dot{y}}$, or $\frac{x\dot{y}}{\dot{x}}$, according as y or x ſupplies the place of the ordinate. Wherefore, the fluxion of the equation of the curve being taken, if from thence we deduce the value of $\dot{x}$ or $\dot{y}$, this value, being ſubſtituted in the general formula, will give us a fraction in finite terms, which is the expreſſion or value of the ſubtangent for any point of the propoſed curve. Now, if we deſire the ſubtangent for any determinate point of the curve, nothing elſe is required to be done, but to ſubſtitute in this fraction, inſtead of x and y, their values which they have at the point given. But it may ſometimes happen, that, by ſubſtituting, inſtead of x or y, a determinate value in the fraction which expreſſes the ſubtangent, or otherwiſe, in the ratio of $\dot{x}$ to $\dot{y}$ deduced from the fluxional equation of the curve, all the terms in the numerator and denominator may vaniſh of themſelves, and that there will only ariſe $\frac{\dot{x}}{\dot{y}} = \frac{0}{0}$, and thence, alſo, the ſubtangent will be $\frac{0}{0}$, from whence, however, we are not to infer that the ſubtangent is nothing in this point.

For an example, let us take the curve belonging to this equation $y^4 - 8ay^3 - 12axyy + 16aayy + 48aaxy + 4aaxx - 64a^3x = 0$, and let y be the abſciſs, and x the ordinate. Therefore $\frac{x\dot{y}}{\dot{x}}$ will be the formula for the ſubtangent. Therefore, by taking the fluxion of this equation, we ſhall have $\frac{\dot{y}}{\dot{x}} = \frac{3ayy - 12aay - 2aax + 16a^3}{y^3 - 6ayy - 6axy + 8aay + 12aax}$, and the ſubtangent will be $\frac{x\dot{y}}{\dot{x}} = \frac{3axyy - 12aaxy - 2aaxx + 16a^3x}{y^3 - 6ayy - 6axy + 8aay + 12aax}$. Now, if we would have the ſubtangent to that point of the curve, which correſponds to the abſciſs $y = 2a$, it being alſo in this caſe $x = 2a$, by the given equation; make the ſubſtitutions in the fraction which expreſſes the ratio of $\dot{x}$ to $\dot{y}$, and we ſhall find it to be $\frac{12a^3 - 24a^3 - 4a^3 + 16a^3}{8a^3 - 24a^3 - 24a^3 + 16a^3 + 24a^3}$, that is, $\frac{0}{0}$, becauſe all the terms deſtroy one another; and therefore the ſubtangent alſo, at this point, is $\frac{0}{0}$, which informs us of nothing, although one or more ſubtangents may belong to that point.

Fig. 47.



69. This case will always happen, whenever the curve has several branches which intersect one another, and when we would have a tangent at the point of concourse. And, indeed, the curve NOPQMR (Fig. 47.) of the proposed equation has two such branches, which cut one another in the point G, to which exactly corresponds $y = 2a$, OT being the axis of the y's, and it's beginning at O. Also, $x = 2a$, taking the x's in the axis OQ.

To give a reason for this case, it is enough to take notice of two things. The first is, that, at the point of concourse of the different branches of the curve, several roots of the equation become equal to one another. Thus, as to the proposed equation, in the point G the two values of x are equal, and also, two are equal of the four values of y. The second is, (what is demonstrated in *Des Cartes's* Algebra,) that if an equation which contains equal roots be multiplied, term by term, into any arithmetical progression, the product will be equal to nothing, and will contain in it fewer by one of the equal roots. And if this product be again multiplied by an arithmetical progression, the product will, in like manner, be equal to nothing, and will contain still fewer by one of the equal roots, than were contained by the first product; that is, fewer by two of the equal roots, than were contained by the first equation. And thus on successively to that product, which shall contain only one of the equal roots.

If, therefore, any equation of a curve, treating x as variable and y as constant, shall be multiplied by an arithmetical progression which terminates in nothing; in the case of equal roots the product shall be equal to nothing; and it will also be so, if the product be divided by x, which division will succeed when the last term is multiplied by nothing. The same thing will obtain also by treating y as variable and x as constant, and multiplying the equation by such an arithmetical progression as has nothing, or o, to put under the last term.

This being supposed, it is easy to perceive that such an operation as this performs the very same thing as taking the fluxion; that is, if it treats x as variable, and multiplies the equation by an arithmetical progression, the first term of which is the greatest exponent of x, and the last term is nothing, and produces a product multiplied into $\dot{x}$. Then, if it treats y as variable, and multiplies the equation by an arithmetical progression, the first term of which is the greatest exponent of y, and the last is nothing, or o, and produces a product multiplied into $\dot{y}$. But, in the case of equal roots of x, and in that of equal roots of y, as well the product multiplied by $\dot{x}$, as that by $\dot{y}$, are equal to nothing. So that the ratio $\frac{\dot{x}}{\dot{y}} = \frac{0}{0}$ ought to arise, in that point wherein two branches of the curve intersect each other.

 That

That this may be feen more fully, I here fet in order the equation of the propofed curve according to the letter y, and multiply it by an arithmetical progreffion, the laft term of which is o.

$$\left.\begin{array}{lllll} y^4 & - 8ay^3 & - 12axy^2 & + 48aaxy & + 4aaxx \\ & & + 16aay^2 & & - 64a^3x \end{array}\right\} = 0.$$

$$4, \quad 3, \quad 2, \quad 1, \quad 0,$$

The product will be

$$4y^4 - 24ay^3 - 24axy^2 + 32aay^2 + 48aaxy = 0.$$

That is, dividing by $4y$,

$$y^3 - 6ay^2 - 6axy + 8aay + 12aax = 0.$$

Then I fet the fame equation in order according to the letter x, and multiply it by the arithmetical progreffion, the laft term of which is o.

$$\left.\begin{array}{lll} 4aax^2 & + 48aayx & + y^4 \\ & - 64aaax & - 8ay^3 \\ & - 12ayyx & + 16a^2y^2 \end{array}\right\} = 0.$$

$$2, \quad 1, \quad 0,$$

The product will be

$$8aax^2 + 48aayx - 64a^3x - 12ayyx = 0.$$

That is, dividing by $4x$,

$$2aax + 12aay - 16a^3 - 3ayy = 0.$$

This being done, I take the fluxion of the propofed equation, which is $4y^3\dot{y} - 24ay^2\dot{y} - 24axy\dot{y} - 12ay^2\dot{x} + 32aay\dot{y} + 48aax\dot{y} + 48a^2y\dot{x} + 8a^2x\dot{x} - 64a^3\dot{x} = 0$; that is, dividing it by 4, and tranfpofing the terms belonging to $\dot{x}$,

$$y^3 - 6ay^2 - 6axy + 8a^2y + 12a^2x \text{ into } \dot{y}$$
$$= 3ay^2 - 12aay + 2aax + 16a^3 \text{ into } \dot{x}.$$

Now here the multiplier of $\dot{y}$ is the firft product into the arithmetical progreffion, and confequently $= 0$ in relation to the point G, in which y has two equal values. And the multiplier of $\dot{x}$ is the fecond product into it's arithmetical progreffion with it's figns changed, which does not hinder it being $= 0$, in relation to the fame point G, in which x has two equal values. Therefore it will be $\dot{y} \times 0 = \dot{x} \times 0$, that is, $\frac{\dot{y}}{\dot{x}} = \frac{0}{0}$ in the point G.

But, if to multiply any equation by an arithmetical progreffion, or to find it's fluxion, (which is the fame thing,) bring it to pafs, that, on the fuppofition of

of equal roots, that cafe will arife of which we are treating, that is, $\frac{\dot{y}}{\dot{x}} = \frac{0}{0}$; it alfo brings it to pafs, that, in the equation derived from thence, there will be one lefs of thofe equal roots. Wherefore, if the equation propofed have two equal roots, when differenced it will have but one of thofe equal roots. And, if the propofed equation have three, by differencing again that which was differenced before, (affuming as conftant the differences or fluxions $\dot{x}$, $\dot{y}$,) the equation thence arifing will have only one; and fo on. Therefore, if we affume as conftant the fluxions $\dot{x}$, $\dot{y}$, as well the terms multiplied into $\dot{x}$ as thofe multiplied into $\dot{y}$, will mutually deftroy each other, in the fuppofition of fuch a determinate value of x and y; alfo, the terms multiplied into $\ddot{x}$ and $\ddot{y}$ will deftroy one another. By proceeding in this way of operation, equations will be reduced to contain only one of the number of equal roots which they had at firft; and therefore, finally differencing the laft, to obtain the ratio of $\dot{y}$ to $\dot{x}$, there can no longer arife the cafe of $\frac{\dot{y}}{\dot{x}} = \frac{0}{0}$.

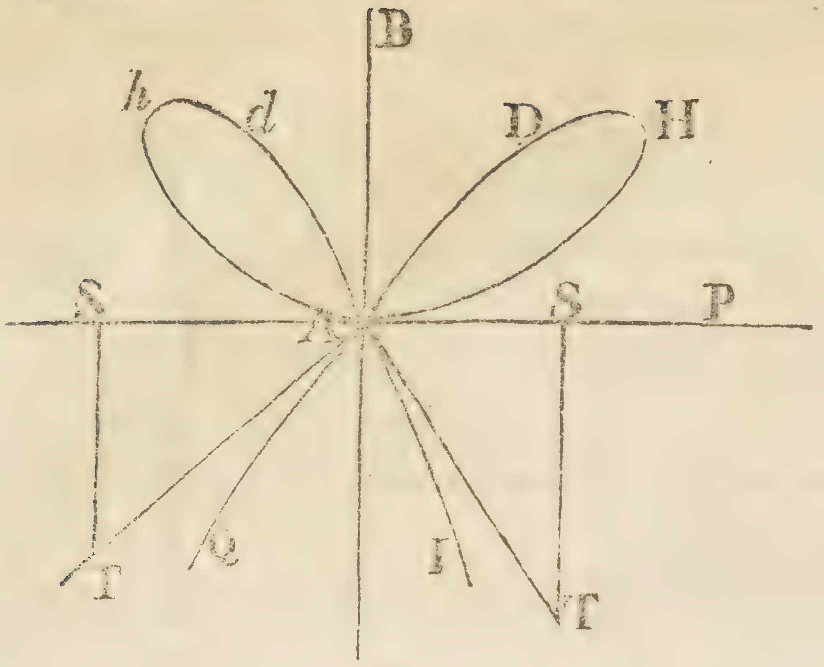
Therefore I refume the foregoing equation whofe fluxion was found to be $y^3\dot{y} - 6ay^2\dot{y} - 6axy\dot{y} - 3ay^2\dot{x} + 8aay\dot{y} + 12aax\dot{y} + 12aay\dot{x} + 2aax\dot{x} - 16a^3\dot{x} = 0$. But, becaufe, by fubftituting, inftead of y, it's value $2a$, and, inftead of x, it's correfpondent value $2a$, in order to have the tangent at the point G; I find only $\frac{\dot{y}}{\dot{x}} = \frac{0}{0}$: I go on to difference that already differenced, taking always for conftant the fluxions $\dot{x}$, $\dot{y}$, and I fhall obtain $3y^2\dot{y}^2 - 12ay\dot{y}^2 - 6ax\dot{y}^2 + 8aa\dot{y}^2 - 12ay\dot{y}\dot{x} + 24aa\dot{y}\dot{x} + 2aa\dot{x}^2 = 0$.

Inftead of y and x, I fubftitute their values $2a$, in relation to the point G; and I find $\dot{x} = \pm\dot{y}\sqrt{8}$. Then, in the general formula for the fubtangent $\frac{x\dot{y}}{\dot{x}}$, putting the values of $x = 2a$, and $\dot{x} = \pm\dot{y}\sqrt{8}$, I fhall finally have the fubtangent $= \pm\frac{a}{\sqrt{2}}$; or, to fpeak more properly, the two fubtangents correfponding to the point G, one pofitive, the other equal to it, but negative.

If the curve fhall have three equal roots at the point in which the tangent is required, that is, if the curve fhall have three branches which meet one another in that point; becaufe, after the equation has been differenced once, it will ftill have two equal roots; it muft be differenced again, that we may have the ratio of $\dot{y}$ to $\dot{x}$: It will give us, notwithftanding, by what has been already faid, the ratio $\frac{\dot{y}}{\dot{x}} = \frac{0}{0}$; and therefore it will be neceffary to take the difference or fluxion a third time. And, in general, the equation muft be fo often differenced as is the number of equal roots, or the branches of the curve; and from the laft difference muft be obtained the ratio of $\dot{y}$ to $\dot{x}$. And fo many will be the tangents as are the branches of the curve, which cut one another in that point.

Let

Fig. 48.



Let the curve be QADHAhdAI, whose equation is $x^4 - ayxx + by^3 = 0$, and which has three branches QAD, IAd, hAH, which cut one another in A. And let AP be the axis belonging to x, and AB perpendicular to AP, the axis belonging to y, and the point A their common origin. By differencing the equation, it will be $4x^3\dot{x} - 2ayx\dot{x} - axx\dot{y} + 3byy\dot{y} = 0$; that is, $\frac{\dot{x}}{\dot{y}} = \frac{axx - 3byy}{4x^3 - 2ayx}$. But, if we would have the tangent at the point A, because there it is $x = 0$, $y = 0$; it will be $\frac{\dot{y}}{\dot{x}} = \frac{0}{0}$. We must therefore go on to second fluxions, and the equation will be $12xx\dot{x}\dot{x} - 2ay\dot{x}\dot{x} - 4ax\dot{x}\dot{y} + 6by\dot{y}\dot{y} = 0$. But from this we shall only obtain $\frac{\dot{x}}{\dot{y}} = \frac{0}{0}$, every term being multiplied by $x = 0$, by supposition, or by $y = 0$. Therefore, differencing for the third time, it will be $24x\dot{x}^3 - 6a\dot{y}\dot{x}^2 + 6b\dot{y}^3 = 0$. Here, making $x = 0$, the first term vanishes, and therefore it is $a\dot{y}\dot{x}^2 = b\dot{y}^3$, from whence we have three values of $\dot{y}$; that is, $\dot{y} = 0$, and $\dot{y} = \pm\frac{\dot{x}\sqrt{a}}{\sqrt{b}}$, which give us three ratios of $\dot{x}$ to $\dot{y}$; that is to say, three tangents at the point A. One of them will be infinite, which coincides with the axis AP, and serves for the branch hAH. The other, taking any line AS, and drawing ST perpendicularly in such a manner, as that it may be ST . SA :: $\sqrt{a}$. $\sqrt{b}$; the lines TA will be tangents in the point A, one of the branch QAD, the other of the branch IAd.

70. The truth of these conclusions may also be demonstrated after another manner, and, as they say, *à posteriori*. The differentials of finite equations, which are found by the foregoing rules of differencing, are not really the complete differentials, the rules giving us only those terms which contain the first differences, or of one dimension only; and omitting, for brevity-sake, and for greater convenience, the differences of other degrees, or of greater dimensions: which, by the principles of the calculus, would make those terms in which they are found to be relatively nothing.

Resuming the equation $\left.\begin{matrix} y^4 - 8ay^3 - 12axy^2 + 48a^2yx + 4a^2x^2 \\ + 16a^2y^2 \qquad\qquad - 64a^3x \end{matrix}\right\} = 0$,

it's fluxion or difference will be $4y^3\dot{y} - 24ay^2\dot{y} - 12ayy\dot{x} - 24axy\dot{y} + 32a^2y\dot{y} + 48aax\dot{y} + 48aay\dot{x} + 8aax\dot{x} - 64a^3\dot{x} = 0$. But here, if y be considered as increased by it's fluxion or difference, and likewise x; and that in the proposed equation, instead of y and it's powers, we should write $y + \dot{y}$ and it's corresponding powers; and should do the same by writing $x + \dot{x}$ and it's powers instead of those

those of x; we should then have the terms as they are set in order in the following Table.

I.	II.	III.	IV.	V.	
$+ y^4$	$+ 4y^3\dot{y}$	$+ 6yy\dot{y}\dot{y}$	$+ 4y\dot{y}^3$	$+ \dot{y}^4$	
$- 8ay^3$	$- 24ay^2\dot{y}$	$- 24ay\dot{y}\dot{y}$	$- 8a\dot{y}^3$		
$- 12axy^2$	$- 24axy\dot{y}$	$- 12ax\dot{y}\dot{y}$	$- 12a\dot{x}\dot{y}^2$		
$+ 16aay^2$	$- 12ayy\dot{x}$	$- 24ay\dot{x}\dot{y}$			
$+ 48aaxy$	$+ 32a^2y\dot{y}$	$+ 16aa\dot{y}\dot{y}$			$= 0.$
$+ 4aaxx$	$+ 48a^2y\dot{x}$	$+ 48aa\dot{x}\dot{y}$			
$- 64a^3x$	$+ 48a^2x\dot{y}$	$+ 4aa\dot{x}\dot{x}$			
	$+ 8a^2x\dot{x}$				
	$- 64a^3\dot{x}$				

Now the sum of all these columns, excepting the first, which is the proposed equation itself, will be their complete and entire fluxion. But, because the last or fifth column is infinitely little in respect of the fourth, and the fourth in respect of the third, and the third in respect of the second; we assume the second column alone for the fluxion of the proposed equation, which compendium proceeds from the common rule of differencing. But it can be so only when the columns after the second are absolutely nothing. If, therefore, a case shall arise, in which the second column is absolutely nothing, the third may not be nothing in respect of it, and therefore ought not to be omitted, but will itself be the differential of the first. And the same may be said of the fourth, when the second and third are nothing; and so of the rest. But this case precisely happens, when we seek the relation of $\dot{x}$ to $\dot{y}$ in the proposed equation, in that point in which it is $y = 2a$, and $x = 2a$; because, making the necessary substitutions, we find the second column itself to be nothing; and therefore we go on to make use of the third. And this is exactly the same thing as to difference the equation twice, as appears from hence.

71. By the same principles, and after the same manner, a like case may be resolved, which arises in the construction of curves, when the ordinate is expressed by a fraction, the denominator and numerator of which become each equal to nothing, when a determinate value is assigned to the absciss.

Now, to remove this difficulty, it is enough to consider the fraction as if it expressed the ordinates of two curves, which meet in some point of their common axis. And because, in this point, their ratio cannot be expressed otherwise than by $\frac{0}{0}$, it is necessary to find what may be their ratio in a point infinitely near it, that is, when they are increased by an infinitesimal. That is to say, we must proceed to differencing the numerator, and then the denominator of the said fraction, and that once, twice, or oftener, till at last, putting the determinate value of the absciss in the fraction, it may no longer be $\frac{0}{0}$, for the same reason mentioned before, concerning the columns of differentials.

Let

Let the equation be $y = \frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{aax}}{a - \sqrt[4]{ax^3}}$. Taking $x = a$, and making the ſubſtitution, it will be $y = \frac{0}{0}$, from whence we cannot therefore infer, that when the abſciſs $x = a$, the correſponding ordinate will be $y = 0$. For, by differencing the numerator, and then the denominator of the fraction, it will be $y = \frac{\overline{a^3\dot{x} - 2x^3\dot{x}} \times \overline{2a^3x - x^4}|^{-\frac{1}{2}} - \frac{1}{3}a^3\dot{x} \times a^{-\frac{4}{3}}x^{-\frac{2}{3}}}{-\frac{3}{4}axx\dot{x} \times a^{-\frac{3}{4}}x^{-\frac{9}{4}}}$. Then, dividing both above and below by $\dot{x}$, and making $x = a$, it will be $y = \frac{16}{9}a$.

Let the equation be $y = \frac{a\sqrt[3]{4a^3 + 4x^3} - ax - aa}{\sqrt{2aa + 2xx} - x - a}$, in which, if we put $x = a$, it will become $y = \frac{0}{0}$. Wherefore, differencing, firſt, the numerator, and then the denominator of the fraction, it will be $y = \frac{4axx \times \overline{4a^3 + 4x^3}|^{-\frac{2}{3}} - a}{2x \times \overline{2aa + 2xx}|^{-\frac{1}{2}} - 1}$, omitting $\dot{x}$, which ſhould be in both the numerator and the denominator. But now, in this fraction, if we put $x = a$, it will be ſtill $y = \frac{0}{0}$. Therefore, proceeding to difference this ſecond fraction alſo, we ſhall have $y = \frac{32a^4x \times \overline{4a^3 + 4x^3}|^{-\frac{5}{3}}}{4aa \times \overline{2aa + 2xx}|^{-\frac{3}{2}}}$, omitting the $\dot{x}$. And now, making $x = a$, it will be $y = 2a$.

SECT. III.

The Method of the Maxima *and* Minima *of Quantities.*

Fig. 49.

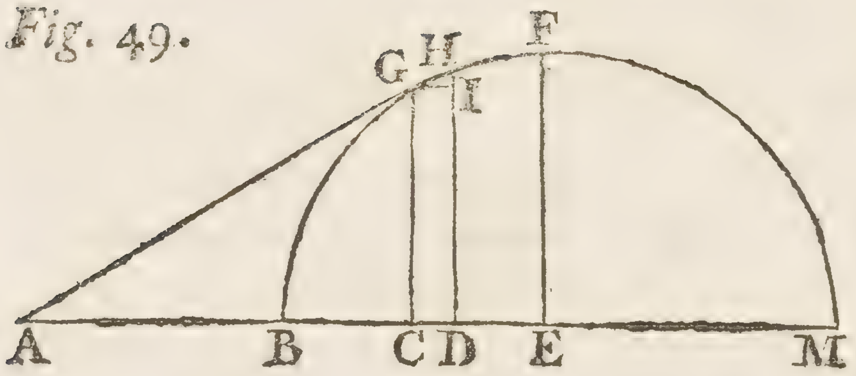


Fig. 50.

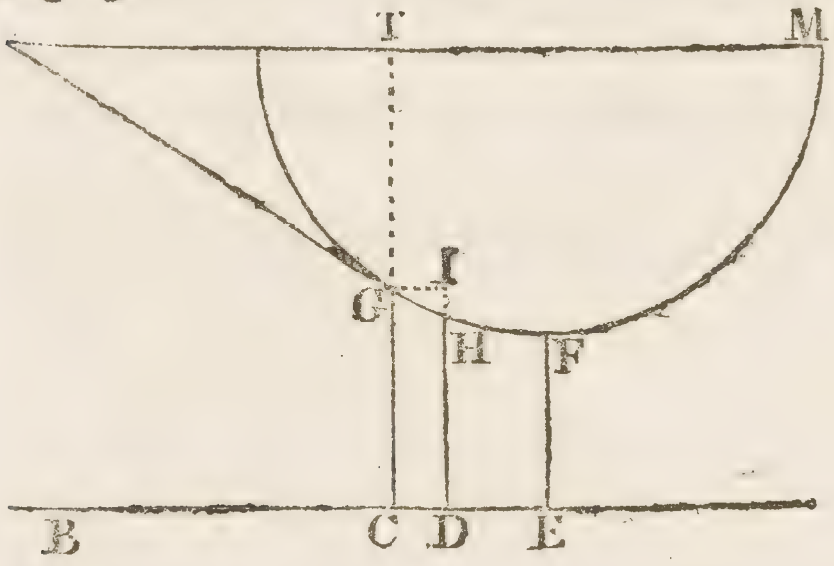
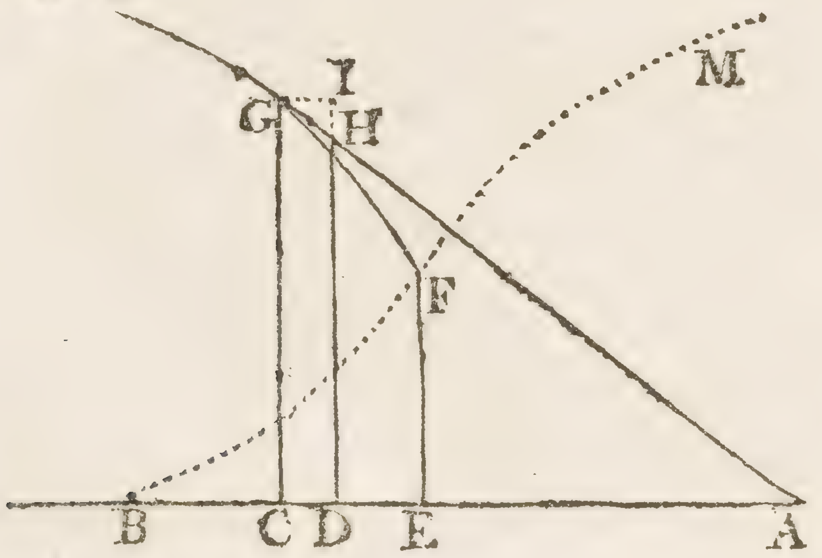


Fig. 51.

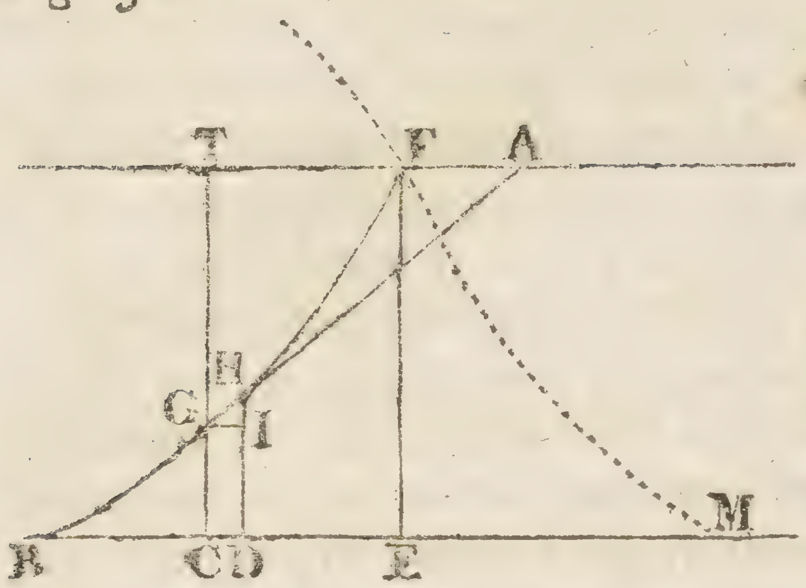


72. In any curve whatever, whose ordinates are parallel, if, the abscifs BC (Fig. 49, 50, 51, 52,) continually increafing, the ordinate CG increafes alfo to a certain point E, after which it decreafes, or is no longer an ordinate of any kind; or, on the contrary, the abscifs increafing, the ordinate CG goes on continually decreafing to a certain point E, after which it either increafes, or elfe is no more: In this cafe, the ordinate EF is called a *Maximum* or a *Minimum*.

In the curve GHF, let EF be the greateft of the ordinates, (Fig. 49.) or the leaft, (Fig. 50.) taking any abscifs BC, and drawing the ordinate CG; let GA be fuppofed to be a tangent at the point G, and DH to be infinitely near to CG. Make BC $= x$, CG $= y$, and drawing GI parallel to BC, it will be GI $=$ CD $= \dot{x}$, and IH $= \dot{y}$. Now, becaufe the triangles ACG, GHI, are fimilar, in Fig. 49, it will be AC . CG :: GI . IH. And, becaufe the triangles ATG, GHI, are fimilar, in Fig. 50, it will be AT . TG :: GI . IH. This being fuppofed, let the ordinate GC, being always parallel to itfelf,

N. B. The letter A is omitted in Fig. 50.

Fig. 52.



itſelf, be conceived to approach to the greateſt or leaſt ordinate EF. It is plain, that, as CG approaches to EF, the ſubtangent AC, or AT, will always become greater and greater; ſo that, when CG falls upon EF, the tangent will become parallel to BC, and conſequently the ſubtangent will be infinite. Therefore, in this caſe, we ſhall have AC to CG, or AT to TG, an infinite ratio, CG ſtill remaining a finite quantity. But, ſince it is always AC to CG, or AT . TG :: GI . IH, GI to IH will alſo have an infinite ratio. Therefore it will be as nothing in reſpect of $\dot{x}$, that is, $\dot{y} = 0$ in the point of the greateſt or leaſt ordinate.

Let the curve be GHF, (Fig. 51, 52.) EF the leaſt of the ordinates, (Fig. 51.) or the greateſt (Fig. 52.); therefore, taking any abſciſs BC, and drawing the ordinate CG, the tangent GA, DH infinitely near to CG, and GI parallel to BC; and making $BC = x$, $CG = y$, it will be $GI = CD = \dot{x}$, $IH = \dot{y}$. Now, becauſe of the ſimilar triangles ACG, GIH, it will be (Fig. 51.) AC . CG :: GI . IH; and, becauſe of the ſimilar triangles ATG, GIH, it will be (Fig. 52.) AT . TG :: GI . IH. Now, the ordinate CG always remaining parallel to itſelf, and continually approaching towards the greateſt or leaſt ordinate, the ſubtangent AC or AT will always become leſs and leſs; ſo that, when CG falls upon EF, the tangent will become perpendicular to BC, and conſequently the ſubtangent will be nothing. Therefore, in this caſe, we ſhall have AC to CG, or AT to TG, in the ratio of nothing to a finite quantity; and therefore, GI to IH being in the ſame ratio, $\dot{x}$ will be nothing in reſpect of $\dot{y}$, that is, $\dot{y} = \infty$, in the point of the greateſt or leaſt ordinate. Wherefore the general formula for the greateſt and leaſt ordinate will be $\dot{y} = 0$, or elſe $\dot{y} = \infty$.

73. Therefore, the equation of the curve being given, of which we would find the greateſt or leaſt ordinate, we muſt difference it to find the value of the fraction or ratio $\frac{\dot{y}}{\dot{x}}$; then making the ſuppoſition of $\dot{y} = 0$, or elſe of $\dot{x} = 0$, that is, $\dot{y} = \infty$, we ſhall have the value of the abſciſs x, to which belongs the greateſt or leaſt, y; and this value, being ſubſtituted in the propoſed equation, will give us the greateſt or leaſt ordinate, as required. Only here we muſt obſerve, that, in the caſe of the ſuppoſition of $\dot{y} = \infty$, that is, of $\dot{x} = 0$, x will ſupply the place of the ordinate; if in the other ſuppoſition, it is y that does the ſame. That, if neither the firſt ſuppoſition of $\dot{y} = 0$, nor the ſecond of $\dot{y} = \infty$, will ſupply us with any real value of y, it is then to be concluded, that the propoſed curve has no greateſt or leaſt ordinate.

74. This method will help us to acquire a complete and exact idea of curve-lines; to find in what points the tangents are parallel to their conjugate axes, &c. Besides which, it may be applied to an infinite number of questions, which we may want to have resolved, whether geometrical or physical. Such it would be to inquire, among the infinite parallelopipeds of a given solidity, which is that which has the least surface: as it would be to inquire, among the infinite different ways along which a moving body may pass, to go from one point to another not in the same vertical line, which is that which may be described in the shortest time, according to some given law of motion: and many others of a like kind. In such questions must be found an analytical expression of what we would have to be a *maximum*, or a *minimum*, which may be put equal to y. Then taking the fluxion, we must proceed according to the rules here given.

EXAMPLE I.

75. Let there be a curve with this equation $2ax - xx = yy$, and let it be required to know, to what point of the axis, or of the abscis x, the greatest ordinate y corresponds, and what that ordinate is.

The fluxional equation of this will be $2a\dot{x} - 2x\dot{x} = 2y\dot{y}$, that is, $\frac{\dot{y}}{\dot{x}} = \frac{a-x}{y}$. Making the supposition of $\dot{y} = 0$, the numerator of the fraction ought to be nothing, or $a - x = 0$, whence $x = a$. Therefore the greatest ordinate belongs to that abscis which is equal to a. This value being substituted instead of x in the proposed equation, it will be $2aa - aa = yy$, that is, $y = \pm a$. Therefore the greatest ordinate, positive and negative, will be equal to a. Making the supposition of $\dot{y} = \infty$, the denominator of the fraction ought to be nothing, and therefore it will be $y = 0$. Wherefore, substituting this value instead of y in the proposed equation, we shall have $x = 0$, and $x = 2a$; which is as much as to say, that $x = 0$ will be the least, and $x = 2a$ the greatest: Or, more properly, that, when $x = 0$, and $x = 2a$, then $\dot{y}$ being infinite in respect of $\dot{x}$, the subtangent will be nothing, or the tangent will be parallel to the ordinate y.

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EXAMPLE II.

76. Let it be the curve of this equation $xx - ax = yy$. By taking the fluxions, it will be $\frac{\dot{y}}{\dot{x}} = \frac{2x - a}{2y}$. The supposition of $\dot{y} = 0$ gives here $x = \frac{1}{2}a$. But this value being substituted instead of x in the proposed equation, y will be found imaginary; so that the curve has no ordinate corresponding to such an abscis, and therefore much less will it have a greatest or a least. The supposition of $\dot{y} = \infty$, that is, of $\dot{x} = 0$, will here give $y = 0$: which declares that the tangent will be perpendicular to the axis of the abscis x in the point in which $y = 0$; which corresponds to the two abscisses $x = 0$, and $x = a$. For, instead of y, substituting 0 in the proposed equation, it will be $xx - ax = 0$, and therefore $x = 0$, and $x = a$.

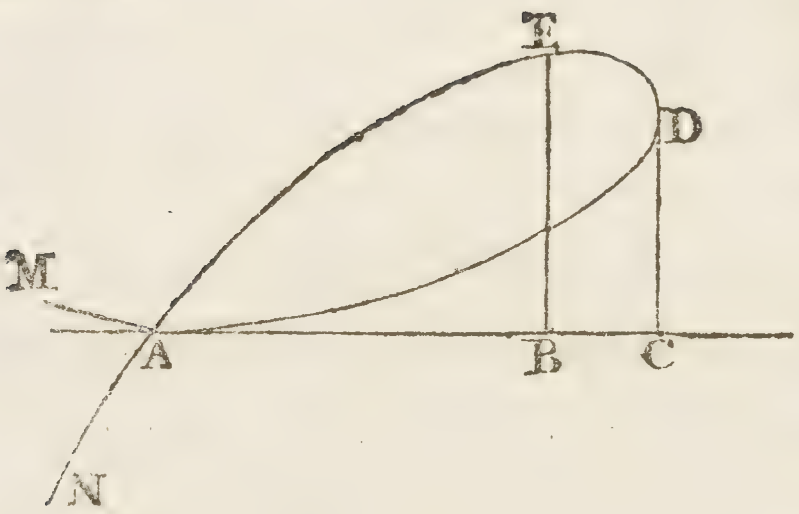
EXAMPLE III.

77. Let the curve belong to this equation $2axy = a^3 + axx - bxx$, in which x is the abscis, and y the ordinate. By taking the fluxions, it will be $2ax\dot{y} + 2ay\dot{x} = 2ax\dot{x} - 2bx\dot{x}$, and therefore $\frac{\dot{y}}{\dot{x}} = \frac{ax - bx - ay}{ax}$. The supposition of $\dot{y} = 0$ gives $x = \frac{ay}{a - b}$; and this value being substituted in the proposed equation, it will be $\frac{2aayy}{a - b} = a^3 + \frac{a^3y^2 - a^2by^2}{\overline{a - b}|^2}$, that is, $yy = a \times \overline{a - b}$, and $y = \pm \sqrt{aa - ab}$, the greatest or least ordinate. And, since we have $x = \frac{ay}{a - b}$, substituting this in the value of y, it will be $x = \pm \frac{a\sqrt{a}}{\sqrt{a - b}}$, the abscis, to which belongs the greatest or least ordinate now found. The supposition of $\dot{y} = \infty$, or $\dot{x} = 0$, gives us $ax = 0$, that is, $x = 0$. And making the substitution in the proposed equation, it will be $a^3 = 0$; which implies that a given finite quantity is as nothing: so that the curve will have no other *maxima* or *minima* but those found in the first supposition, which, because of the ambiguity of the signs, are two, and those equal; one of which is positive, and corresponds to the positive abscisses, the other negative, and belongs to the negative abscis.

78. This method, indeed, gives us the *maxima* and *minima*, but ambiguoufly and indifcriminately; nor by this can we diftinguifh one from the other. But they become known when the progrefs of the curve is known. But, without fuch knowledge, we may proceed after this manner. Let there be affigned a value to the abfcifs in the given equation, which is either a little greater or a little lefs than that which anfwers to the greateft or leaft ordinate with which we are concerned, and the value of the ordinate which arifes from thence will determine the queftion. For, if it fhall be greater than that which the method difcovers, the queftion is about a *minimum*; but, being lefs than that, the queftion is about a *maximum*. Therefore the curve of this Example will have two leaft ordinates.

EXAMPLE IV.

Fig. 53.



79. Let the curve MADEAN belong to this equation $x^3 + y^3 = axy$; make $AB = x$, and $BE = y$. By differencing, we fhall have $\frac{\dot{y}}{\dot{x}} = \frac{ay - 3xx}{3yy - ax}$; and therefore, making the fuppofition of $\dot{y} = 0$, it will be $y = \frac{3xx}{a}$. Then fubftituting this value in the equation, we fhall find $x = \frac{1}{3}a\sqrt[3]{2}$. Wherefore, fince $y = \frac{3xx}{a}$, it will be $y = \frac{1}{3}a\sqrt[3]{4} = BE$, the greateft ordinate in the curve, which correfponds to the abfcifs $x = \frac{1}{3}a\sqrt[3]{2} = AB$. The fuppofition of $\dot{x} = 0$ will give us $x = \frac{3yy}{a}$, and making the fubftitution in the given equation, it will be $y = \frac{1}{3}a\sqrt[3]{2}$, whence $x = \frac{1}{3}a\sqrt[3]{4}$, the greateft AC, to which correfponds $y = CD = \frac{1}{3}a\sqrt[3]{2}$, which is the tangent in the point D.

80. But, before we proceed to more Examples, it will be convenient to provide for a cafe, which fometimes is wont to happen; and that is, that as well the fuppofition of $\dot{y} = 0$, as that of $\dot{y} = \infty$, will furnifh the fame value of the ordinate, or of the abfcifs; in which cafe, no *maximum* or *minimum* will be determined, but only a point of interfection or the meeting of two branches of

of the curve. And the reafon of this is plain; forafmuch as, $\frac{\dot{y}}{\dot{x}}$ being equal to a fraction, if from the numerator we derive the fame value of x, for example, as from the denominator, this value or root being fubftituted, will make each of them equal to nothing, and therefore in fuch a point of the curve it will be $\frac{\dot{y}}{\dot{x}} = \frac{0}{0}$. But it has been already fhown before, at § 69, that when $\frac{\dot{y}}{\dot{x}} = \frac{0}{0}$, it always indicates the meeting of two branches of the curve. Therefore, &c.

EXAMPLE V.

81. Let the curve GFM (Fig. 51.) be the cubic parabola with the equation $y - a = \sqrt[3]{a^3 - 2aax + axx}$, BE $=$ EF $= a$, BC $= x$, CG $= y$. Taking the fluxions, it will be $\frac{\dot{y}}{\dot{x}} = \frac{2ax - 2aa}{3 \times \overline{a^3 - 2aax + axx}|^{\frac{2}{3}}}$. The fuppofition of $\dot{y} = 0$ will give us $x = a$, and the fuppofition of $\dot{y} = \infty$ will give, in like manner, $x = a$. Therefore the curve has a point of interfection F, which correfponds to the abfcifs $x = a$, and to the leaft ordinate $y = a$; which is derived from the propofed equation, by fubftituting it's value in the place of x.

Let us take the fame equation, but freed from radicals, that is, $y^3 - 3ay^2 + 3aay - a^3 = a^3 - 2aax + axx$. By taking the fluxions, it will be $\frac{\dot{y}}{\dot{x}} = \frac{2ax - 2aa}{3yy - 6ay + 3aa}$. The fuppofition of $\dot{y} = 0$ will give $x = a$, and putting this value in the propofed equation, we have $y = a$. The fuppofition of $\dot{y} = \infty$ will alfo give $y = a$, and therefore $x = a$; and $y = a$ gives us the point F, which is a point of meeting or contact of the two branches GF, FM, and, at the fame time, the leaft ordinate y.

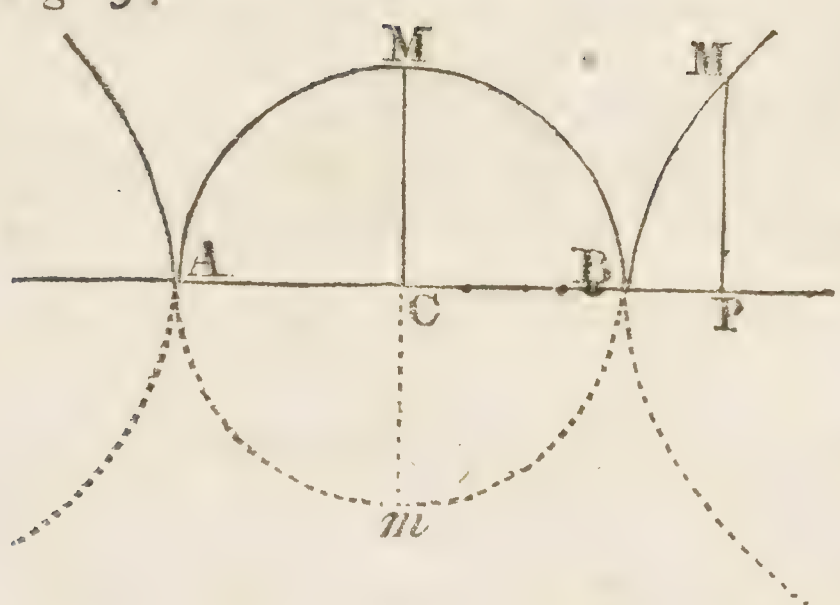
But, if we fhould operate upon the equation $y - a = a^{\frac{1}{3}} \times \overline{a - x}|^{\frac{2}{3}}$, which expreffes the branch GF alone, (the other branch FM would be expreffed by $y - a = a^{\frac{1}{3}} \times \overline{x - a}|^{\frac{2}{3}}$,) we fhould have $\frac{\dot{y}}{\dot{x}} = \frac{-2a^{\frac{1}{3}}}{3 \times \overline{a - x}|^{\frac{1}{3}}}$. The fuppofition of $\dot{y} = 0$, informs us of nothing. The fuppofition of $\dot{y} = \infty$ gives us $x = a$, and therefore $y = a$. And the point F, in this cafe, fupplies us with a *maximum* in refpect of x, and a *minimum* in refpect of y.

82. I faid that the fuppofition of $\dot{y} = 0$, which here gives $2a^{\frac{1}{3}} = 0$, informs us of nothing, meaning in refpect of finite *maxima*; for, taking in the infinite alfo, it fupplies us with two of them. If $2a^{\frac{1}{3}} = 0$, it will be then $x = 0$; and fubftituting this value in the propofed equation, it will be $\frac{y}{0} = \sqrt[3]{xx}$, that is, $x = \pm \sqrt{\frac{y^3}{0}}$; and therefore x and y are infinite. The *maxima* are two, one belonging to the branch FG, the other to the branch FM; for, putting $a = 0$, the equations exprefs them both.

This cafe will generally arife, as often as the fuppofition of $\dot{y} = 0$, or of $\dot{y} = \infty$, exhibits a conftant finite expreffion, or a conftant divifor, to be equal to nothing; which value, being fubftituted in the propofed equation, does not bring us to an imaginary quantity, or to a contradiction. And the reafon of it is this, becaufe a finite quantity cannot be taken for nothing, but only in refpect of an infinite quantity.

EXAMPLE VI.

Fig. 54.



83. Let the curve belong to the equation $x^4 - 2ax^3 + aaxx = y^4$. Make AB $= a$, AC or AP $= x$, CM or PM $= y$. Taking the fluxions, it will be $\frac{\dot{y}}{\dot{x}} = \frac{4x^3 - 6ax^2 + 2aax}{4y^3}$. The fuppofition of $\dot{y} = 0$ will give us three values of x, that is, $x = 0$, $x = a$, $x = \frac{1}{2}a$. The value $x = 0$, being fubftituted in the propofed equation, makes $y = 0$. The value $x = a$, makes $y = 0$. The value $x = \frac{1}{2}a$, makes $y = \pm \frac{1}{2}a$. The fuppofition of $\dot{y} = \infty$ gives us $y = 0$; fo that y has the fame value in both the fuppofitions, when $x = 0$ and $x = y$. Whence the points A, B, will be points of meeting of the branches of the curve, and $x = \frac{1}{2}a =$ AC will give the greateft ordinate $y = \pm \frac{1}{2}a =$ CM, or Cm. The *locus* of the foregoing Example may be called a double *locus*, which arifes from one or other of the two fimple formulas, ($ax - xx = yy$ to the circle, and $xx - ax = yy$ to the hyperbola,) being raifed to it's fquare. Whence it would not be fufficient to reduce the equation to a fimple circle, or to a fimple hyperbola; but it will be neceffary to have a view to the complication of the two *loci* or curves with each other.

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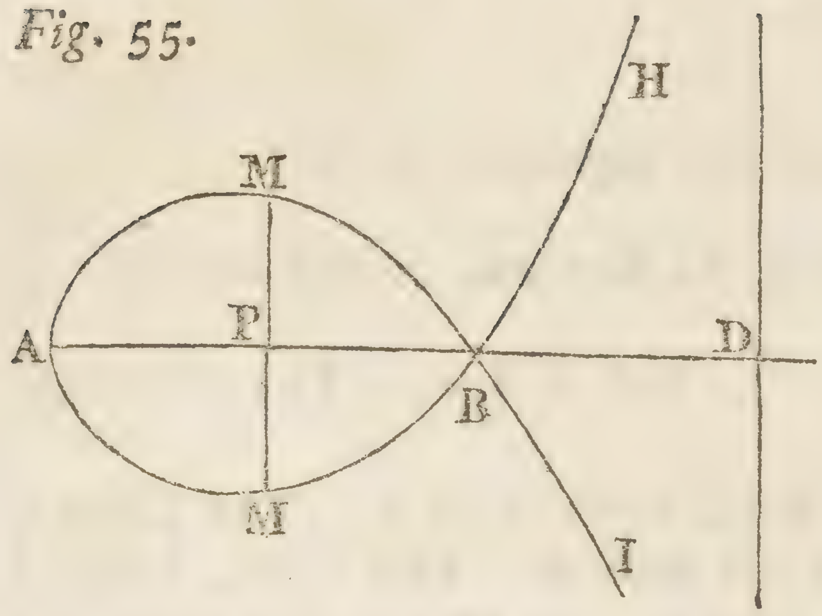
EXAMPLE VII.

Fig. 55.



84. Let it be the curve of Fig. 55, the equation of which is $yy = \frac{aax - 2axx + x^3}{2a - x}$. Make AP $= x$, PM $= y$, AD $= 2a$. The fluxions will be $\frac{\dot{y}}{\dot{x}} = \frac{a^3 - 4a^2x + 4ax^2 - x^3}{y \times \overline{2a - x}|^2}$; that is, $\frac{\dot{y}}{\dot{x}} = \frac{a^3 - 4aax + 4axx - x^3}{a - x \times \sqrt{x} \times \overline{2a - x}|^{\frac{3}{2}}}$. Before I proceed, I shall here observe that both the numerator and the denominator of the fraction are divisible by $a - x$; therefore, in the supposition of $\dot{y} = 0$, and in that of $\dot{y} = \infty$, we shall have $a - x = 0$, or $x = a$. And this, being substituted, will give $y = 0$, and therefore the curve will have a node in the axis at the point B, making AB $= a$. Therefore, making the division, it will be $\frac{\dot{y}}{\dot{x}} = \frac{aa - 3ax + xx}{\overline{2a - x} \times \sqrt{2ax - xx}}$. The supposition of $\dot{y} = 0$ will give $x = \frac{3a \pm a\sqrt{5}}{2}$. But the value $x = \frac{3a + a\sqrt{5}}{2}$ cannot be of use, because, being substituted in the proposed equation, it makes the ordinate imaginary; and this, in general, is imaginary, when x is assumed greater than $2a$, as may be plainly seen. Wherefore, substituting the other value, $x = \frac{3a - a\sqrt{5}}{2}$, it gives $y = \pm a\sqrt{\frac{7a - 3a\sqrt{5}}{a + a\sqrt{5}}}$. Making, then, AP $= \frac{3a - a\sqrt{5}}{2}$, PM, P*m*, will be the greatest ordinates, one positive, the other negative; as above.

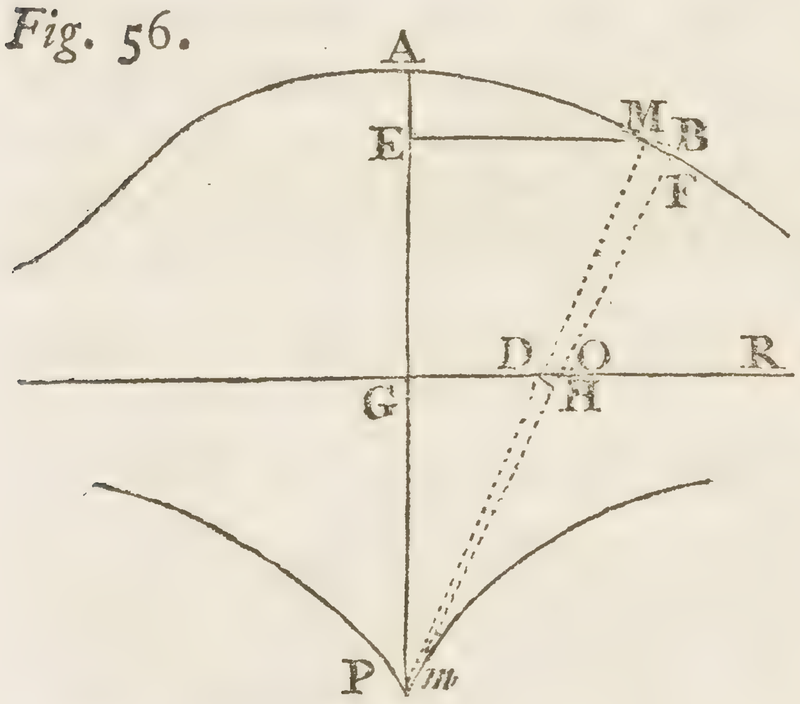
The supposition of $\dot{y} = \infty$ will give $x = 0$, and $x = 2a$. These values being substituted in the proposed equation, we shall have $y = 0$, and $y = \infty$; that is, taking $x = 0$, or in the point A, the tangent will be parallel to the ordinate PM. And taking $x = 2a =$ AD, the ordinate will be infinite, that is, will become an asymptote to the curve, in respect of the branches BH, BI.

N. B. By mistake of the Wood cutter, a Roman M has been put in the lower part of Fig. 55, instead of an Italic *m*.

EXAMPLE VIII.

85. Let the curve be the conchoid with the equation $yy = \frac{aaxx - x^4 + 2aabx - 2bx^3 - bbxx + aabb}{xx}$. Taking the fluxions, it will be $\frac{\dot{y}}{\dot{x}} = \frac{-x^4 - bx^3 - aabx - aabb}{\pm xx\sqrt{aaxx - x^4 + 2aabx - 2bx^3 - b^2x^2 + a^2b^2}}$. In Vol. I. § 239, I have already confidered three cafes of this curve. The firft is, when $a = b$. The fecond, when b is lefs than a. The third, when b is greater than a. As to the firft cafe, the curve will be that of Fig. 56, and the equation $yy = \frac{a^4 + 2a^3x - 2ax^3 - x^4}{xx}$. Making GA $=$ GP $= a$, GE $= x$, EM $= y$; and, taking the fluxions, it is $\frac{\dot{y}}{\dot{x}} = \frac{-x^4 - ax^3 - a^3x - a^4}{\pm xx\sqrt{a^4 + 2a^3x - 2ax^3 - x^4}}$. The fuppofition of $\dot{y} = 0$ will give the numerator equal to nothing, that is, $\overline{x + a} \times \overline{x^3 + a^3} = 0$; and therefore $x = -a$, which value, fubftituted in the equation of the curve, gives $y = 0$. The fuppofition of $\dot{y} = \infty$

Fig. 56.



gives the denominator equal to nothing, that is, $xx\sqrt{\overline{x + a} \times \overline{aa - xx}} = 0$, and therefore $x = 0$, $x = -a$, and $x = a$. But the value $x = -a$ was alfo found in the fuppofition of $\dot{y} = 0$. Therefore, when it is $x = -a$, that is, taking GP $= a$, the curve will have a point P, where two branches meet each other.

The value $x = a$, being fubftituted in the equation, will give us $y = 0$; and therefore the fame x will be $= a =$ GA, to which correfponds $y = 0$. The value $x = 0$, being fubftituted, will give $y = \infty$. Therefore, through the point G, where $x = 0$, if a line be drawn parallel to the ordinates, it will touch the curve at an infinite diftance, that is, it will be an afymptote.

As

Fig. 57.

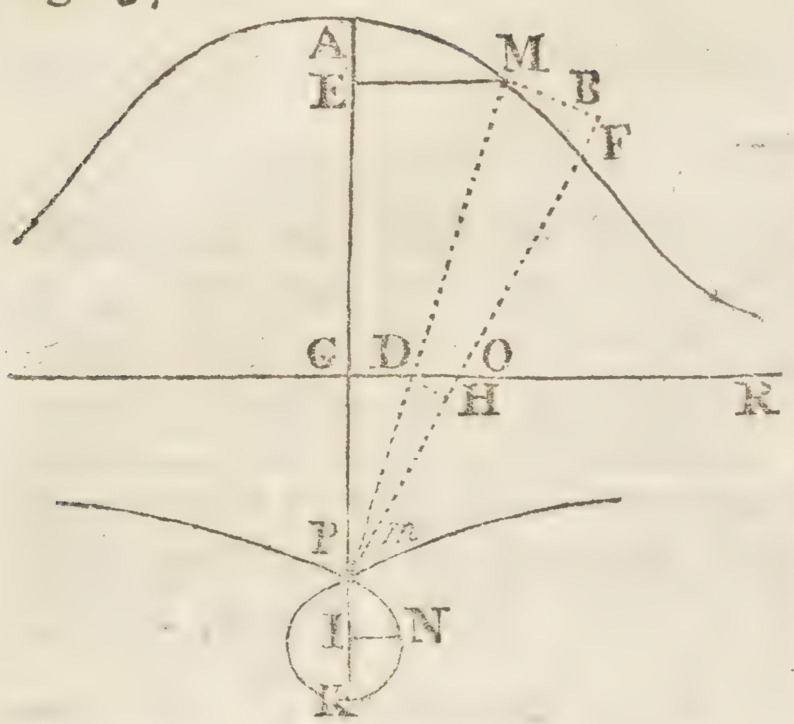
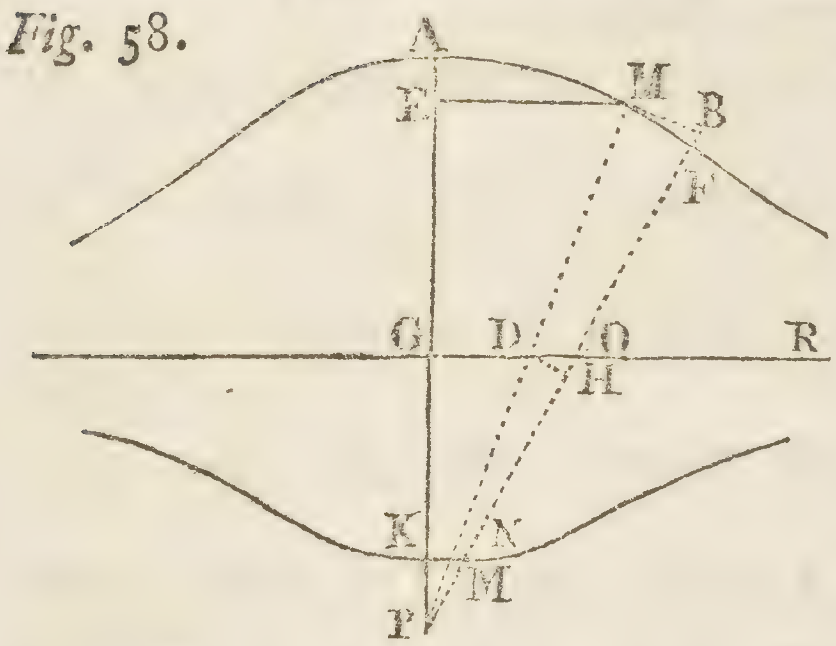


Fig. 58.



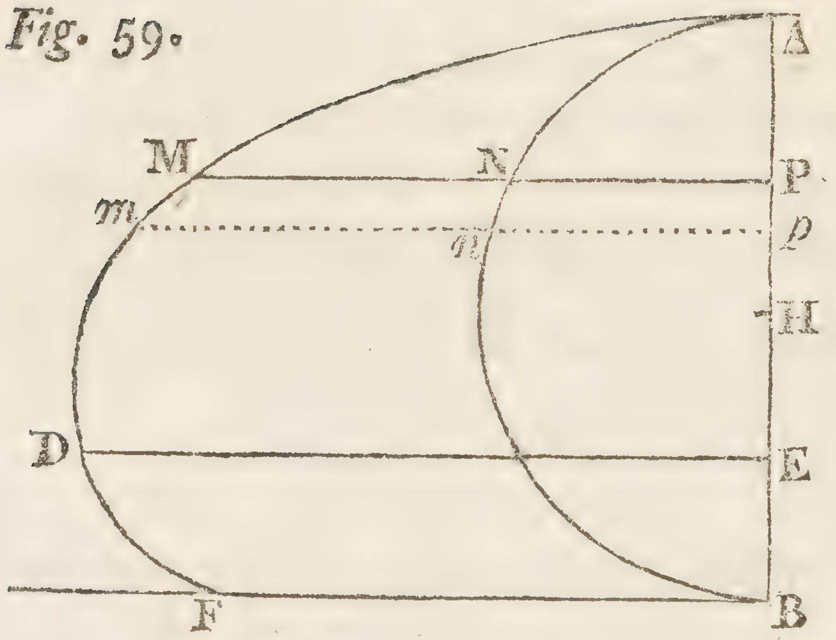
As to the other two cafes, Fig. 57, 58. Let GA = GK = a, GP = b, and the reft as above. The fuppofition of $\dot{y} = 0$ will give $-x^4 - bx^3 - aabx - aabb = 0$; that is, $\overline{x+b} \times \overline{-x^3 - aab} = 0$, and therefore $x = -b$, $x = \sqrt[3]{-aab}$. The fuppofition of $\dot{y} = \infty$, will give $xx\sqrt{a^2x^2 - x^4 + 2a^2bx - 2bx^3 - b^2x^2 + a^2b^2} = 0$, that is, $xx\sqrt{\overline{x+b}^2 \times \overline{aa-xx}} = 0$, and thence $x = 0$, $x = -b$, $x = a$, $x = -a$.

The value $x = -b$, which is the fecond cafe, being fubftituted in the equation, makes $y = 0$, and is exhibited by both the fuppofitions. Therefore (Fig. 57.) taking GP on the negative fide, and equal to $-b$, the point P fhall be a meeting or an interfection of two branches of the curve. The fame value $x = -b$, being fubftituted in the equation of the curve $\pm y = \frac{b+x}{x}\sqrt{aa-xx}$, in the third cafe, gives the radical negative, becaufe of b greater than a, and therefore the curve is imaginary, and of no ufe.

The value $x = \sqrt[3]{-aab}$, fubftituted in the equation of the curve, gives us $y = \pm\sqrt{\frac{\overline{aa-bb} \times \sqrt[3]{abb} + 3ab\sqrt[3]{-aab} + 3abb}{\sqrt[3]{abb}}}$, which is therefore imaginary when b is greater than a, (Fig. 58.) and therefore, in like manner, ferves to no purpofe in this third cafe. But it gives y real when b is lefs than a; and therefore, (Fig. 57.) making GI $= \sqrt[3]{-aab}$, IN will be the greateft ordinate, or y, as above. The value $x = 0$ here gives $y = \infty$, that is, an afymptote. The value $x = \pm a$ gives $y = 0$; that is, the tangent in the points A, K, is parallel to the ordinate.

EXAMPLE IX.

Fig. 59.

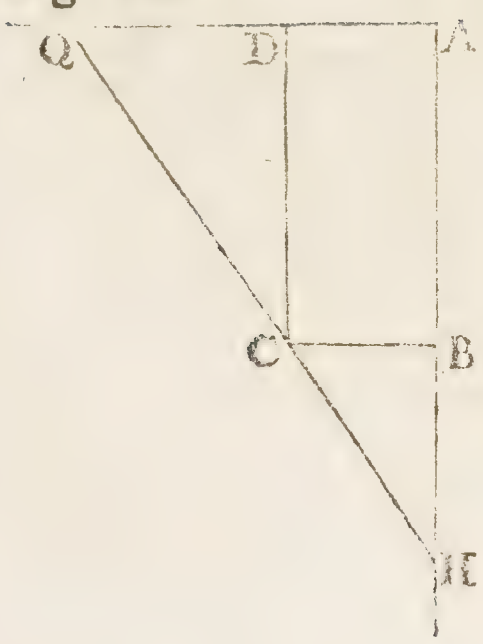


86. Let AMF be half the contracted cycloid. Make $AB = 2a$, $BF = b$, $AP = x$, $PM = z$, the semiperiphery $ANB = c$, the arch $AN = q$; it will be $PN = \sqrt{2ax - xx}$, $NM = z - \sqrt{2ax - xx}$; and, by the property of the curve, it is $ANB . BF :: AN . NM$; that is, $c . b :: q . NM = \frac{bq}{c}$. Therefore $\frac{bq}{c} = z - \sqrt{2ax - xx}$. By differencing, it is $\frac{b\dot{q}}{c} = \dot{z} - \frac{a\dot{x} - x\dot{x}}{\sqrt{2ax - xx}}$. Now, drawing mp infinitely near to MP, it will be $Nn = \dot{q} = \frac{a\dot{x}}{\sqrt{2ax - xx}}$. Whence, making the substitution in the equation, we shall have $\frac{\dot{z}}{\dot{x}} = \frac{ab + ac - cx}{c\sqrt{2ax - xx}}$. The supposition of $\dot{z} = 0$ will give here $x = \frac{ab}{c} + a$. Therefore, if H be the centre of a circle, taking HE equal to the fourth proportional of the semiperiphery ANB, of the right line BF, and of the radius; the corresponding ordinate will be the greatest, as was required.

The supposition of $\dot{z} = \infty$ gives us $x = 0$, and $x = 2a$; which is as much as to say, that, in the points A, F, the tangent will be parallel to the ordinates.

PROBLEM I.

Fig. 60.



87. A rectangle ADCB being given, the least right line QH is required, which can be drawn through the point C in the angle QAH.

Make $AB = a$, $BC = b$, $BH = x$; it will be $CH = \sqrt{bb + xx}$; and, because of the similar triangles HBC, HAQ, we shall have $HB . HC :: HA . HQ$; that is, $x . \sqrt{bb + xx} :: x + a . HQ = \frac{x + a}{x}\sqrt{bb + xx}$.

Wherefore,

Wherefore, ſuppoſing HQ $= y$, as if it were the ordinate of a curve, we ſhall have $y = \frac{x + a}{x}\sqrt{bb + xx}$, and, by differencing, it will be $\frac{\dot{y}}{\dot{x}} = \frac{x^3 - abb}{xx\sqrt{bb + xx}}$. The ſuppoſition of $\dot{y} = 0$ will give $x = \sqrt[3]{abb}$; and therefore, making BH $= \sqrt[3]{abb}$, and drawing HCQ, it will be the leaſt line, as required. The ſuppoſition of $\dot{y} = \infty$ will give $x = \sqrt{-bb}$, and $x = 0$, which anſwers no purpoſe; it not being meant that the right line drawn through the point C, which, in this caſe, would be BC infinitely produced, ſhould be a *maximum*, for that reaſon becauſe infinite. Wherefore, in ſuch caſes as theſe, it will be ſufficient to difference that expreſſion, which we would have to be a *maximum* or *minimum*, and afterwards to ſuppoſe the numerator equal to nothing, and then the denominator.

PROBLEM II.

Fig. 61.

88. The right line AB being divided into three given parts, AC, CF, FB, the point E is required, in which the middle portion CF is to be divided, ſo that the rectangle AE $\times$ EB to the rectangle CE $\times$ EF, may have the leaſt poſſible ratio.

Make AC $= a$, CF $= b$, CB $= c$, and CE $= x$; then AE $= a + x$, EB $= c - x$, EF $= b - x$; and therefore the ratio will be $\frac{\text{AE} \times \text{EB}}{\text{CE} \times \text{EF}} = \frac{ac + cx - ax - xx}{bx - xx}$, which muſt be a *minimum*. The fluxion, therefore, will be $\frac{cxx - axx - bxx + 2acx - abc}{(bx - xx)^2} \times \dot{x}$; and making the numerator equal to nothing, we ſhall have $x = \frac{-ac \pm \sqrt{abcc - abbc - aabc + aacc}}{c - b - a}$. One of the values is poſitive, which gives the point required, E, from C towards B. The other is negative, which would give us the point E, from C towards A. Making the denominator equal to nothing, we ſhall have $x = 0$, and $x = b$, in which two caſes the ratio of the rectangles will be a *maximum*; for, taking $x = 0$, the point E falls in C; and taking $x = b$, the point E falls in F; and therefore, in each caſe, the rectangle CE $\times$ EF is nothing.

PRO-

PROBLEM III.

89. The given right line AB is to be fo cut in the point C, as that the product ACq $\times$ CB fhall be the greateft of all fuch products.

Make AB $= a$, AC $= x$, then CB $= a - x$. Therefore ACq $\times$ CB $= axx - x^3$. The differential will be $2ax\dot{x} - 3xx\dot{x}$, which, compared to nothing, will give $x = \frac{2}{3}a$, and $x = 0$. Wherefore, taking AC $= x = \frac{2}{3}a$, the product will be the greateft poffible; and taking $x = 0$, the product will be a kind of *minimum*, becaufe it will be nothing, the point C falling in A. The differential not being a fraction, the other ufual fuppofition cannot take place, of the denominator being made equal to nothing. But if we will confider the expreffion of the product $axx - x^3$ as an ordinate of a curve, by the laws of homogeneity that product may be divided by a conftant plane, and thus the differential will be a fraction with a conftant denominator. But that conftant quantity can never be nothing, but only relatively in refpect of x being affumed infinite; and furely then the product muft be a *maximum*, when it is AC $= x = \infty$.

I faid that the product ACq $\times$ CB is a *maximum*, when it is AC $= \frac{2}{3}a$; which will be plainly feen by defcribing the curve of the equation $\frac{axx - x^3}{aa} = y$. For all the ordinates between A and B are lefs than that which correfponds to the abfcifs $x = \frac{2}{3}a$. The other value, $x = 0$, being fubftituted, it will be $y = 0$, from whence it may be concluded, that this value will be of no ufe.

90. In the foregoing Problem, and in all others of a like nature, this method may be made ufe of to difcover, whether the queftions propofed are concerning a *maximum* or a *minimum*.

PROBLEM IV.

91. Among all the parallelopipeds that are equal to a given cube, and of which one fide is given; it is required to find that which has the leaft furface.

Let the given cube be a^3, and the known fide of the parallelopiped $= b$. Let one of the fides fought be x, and then the third will be $\frac{a^3}{bx}$, becaufe the

product of the three makes the given cube a^3. The products of the fides, taken two and two, that is, bx, $\frac{a^3}{x}$, and $\frac{a^3}{b}$, form the three planes which are half the fuperficies of the parallelopiped, and therefore the fum of thefe, that is, $bx + \frac{a^3}{x} + \frac{a^3}{b}$, muft be the *minimum* required. Therefore, taking the fluxions, we fhall have $b\dot{x} - \frac{a^3\dot{x}}{xx}$, or $\frac{bxx - a^3}{xx}\dot{x}$. The fuppofition of the numerator equal to nothing gives $x = \sqrt{\frac{a^3}{b}}$. Therefore the three fides of the required parallelopiped will be b, $\sqrt{\frac{a^3}{b}}$, and $\frac{a^3}{b\sqrt{\frac{a^3}{b}}}$, or $\sqrt{\frac{a^3}{b}}$. Therefore the two fides required will be equal. The fuppofition of the denominator, being equal to nothing, ferves to no purpofe; for then $x = 0$, which contradicts the Problem.

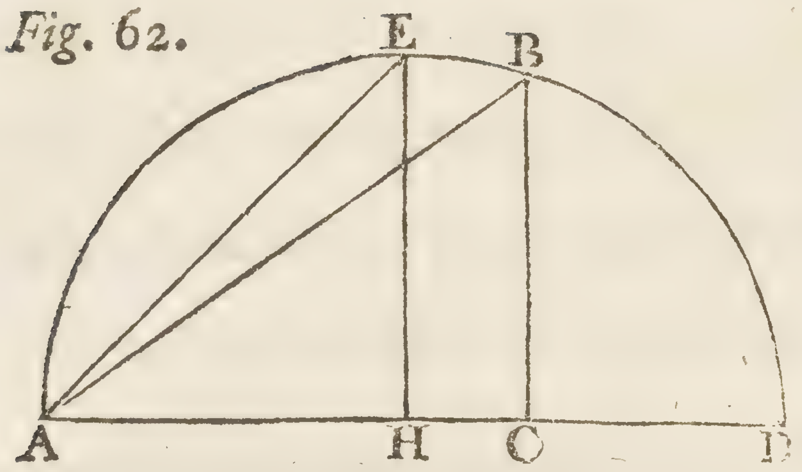
If we would have a parallelopiped with the conditions affigned, but without affuming any fide as given; making one fide $= x$, the two others will be equal, and each $= \sqrt{\frac{a^3}{x}}$. The fum of the three fides or planes, which is to be a *minimum*, will be $2x\sqrt{\frac{a^3}{x}} + \frac{a^3}{x}$, which, by differencing, is $\frac{a^3\dot{x}}{x\sqrt{\frac{a^3}{x}}} - \frac{a^3\dot{x}}{xx}$; or thus, $\frac{a^3x\dot{x} - a^3\dot{x}\sqrt{\frac{a^3}{x}}}{xx\sqrt{\frac{a^3}{x}}}$. Here, making the numerator equal to nothing, we fhall have $x = a$, and, in like manner, the other two fides will be $= a$; fo that the cube itfelf will be the parallelopiped required.

PROBLEM V.

92. Among the infinite cones that may be infcribed in a fphere, to determine that whofe convex fuperficies is the greateft; the bafe being excluded.

In

Fig. 62.

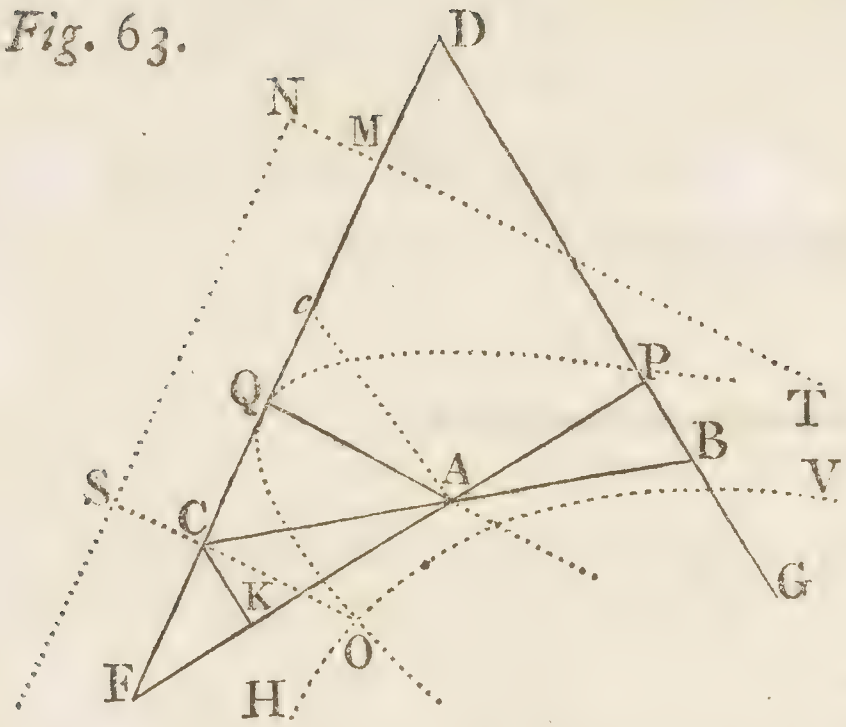


In the femicircle ABD let there be the triangles ABC, AEH, and let a femicircle revolve about it's diameter AD. At the fame time that it defcribes a fphere, the triangles will defcribe fo many cones. But, as it is demonftrated by *Archimedes*, that the fuperficies of the infcribed cones will be to each other as the rectangles AE $\times$ EH, AB $\times$ BC; the queftion is reduced to this, to determine fuch a point C in the diameter AD, that the product AB $\times$ BC may be a *maximum*.

Therefore make AC $= x$, AD $= a$; by the property of the circle, it will be CB $= \sqrt{ax - xx}$, AB $= \sqrt{ax}$, and AB $\times$ BC $= \sqrt{ax} \times \sqrt{ax - xx}$ $= \sqrt{aaxx - ax^3}$. Therefore, taking the fluxions, we fhall have $\frac{2aax\dot{x} - 3axx\dot{x}}{2\sqrt{aaxx - ax^3}}$. And making the numerator equal to nothing, it will be $x = \frac{2}{3}a$, and $x = 0$. Making the denominator $= 0$, it will be $x = a$, and $x = 0$. Taking, therefore, AC $= \frac{2}{3}$AD, the fuperficies of the cone defcribed by the triangle ABC will be the greateft, as required. The other two values $x = 0$, and $x = a$, can be of no ufe in this Problem, as is evident.

PROBLEM VI.

Fig. 63.



93. The angle FDG being given, and the point A being given in pofition, to find the leaft right line, which, in the given angle, can pafs through the point A.

Let CB be the line required, and let AQ be drawn perpendicular to FD, FAP perpendicular to DG, and CK perpendicular to FP. Becaufe the angle FDG is given, and the angle FPD is a right one, the angle AFQ will be known. But the point A is alfo given in pofition; then the lines QA, QF, FA, QD, will alfo be known. Therefore make QF $= a$, QA $= c$, QD $= b$, and QC $= x$. Therefore it will be FA $= \sqrt{aa + cc}$, CA $= \sqrt{cc + xx}$, FD $= b + a$, and FC $= a - x$. But, becaufe of fimilar triangles FAQ, FDP, it will be FA

FA . FQ :: FD . FP. Wherefore $FP = \frac{aa + ab}{\sqrt{aa + cc}}$, and $AP = \frac{ab - cc}{\sqrt{aa + cc}}$. Now, becauſe of ſimilar triangles ACK, ABP, it will be AK . CA :: AP . AB. Therefore $AB = \frac{\overline{ab - cc} \times \sqrt{cc + xx}}{cc + ax}$, and thence $CB = \sqrt{cc + xx} + \frac{ab - cc}{cc + ax}\sqrt{cc + xx}$, which is to be a *minimum*. Therefore, taking the fluxions, it will be $\frac{x\dot{x}}{\sqrt{cc + xx}} + \frac{x\dot{x} \times \overline{ab - cc} \times \overline{cc + ax} - a\dot{x} \times \overline{ab - cc} \times \overline{cc + xx}}{\overline{cc + ax}|^2 \times \overline{cc + xx}|^{\frac{1}{2}}}$. And, putting the numerator $= 0$, (firſt reducing to a common denominator,) it will be $x^3 + \frac{2c^2x^2}{a} + \frac{bc^2x}{a} + \frac{c^4}{a} - bc^2 = 0$, which is a ſolid equation.

To conſtruct it, I take the equation to the parabola $xx = ay$; making the ſubſtitution, it will be $xy + \frac{2ccy}{a} + \frac{bccx}{aa} + \frac{c^4}{aa} - \frac{bcc}{a} = 0$, a *locus* to the hyperbola between it's aſymptotes.

This ſuppoſed, on the right line QD is taken $QM = \frac{2cc}{a}$, and drawing the right line $MN = \frac{bcc}{aa}$ from the point M, and parallel to AQ, NS is drawn parallel to QD, and between the aſymptotes NS, NT, the hyperbola HOV is deſcribed with the conſtant rectangle $\frac{2bc^4 + aabcc - ac^4}{a^3}$. And, on the right line QF, from the point Q let the x's be taken, and the y's perpendicular to them. Then, with the axis AQ, vertex Q, and parameter $= a$, let the parabola QO of the equation $xx = ay$ be deſcribed. From the point O, in which the parabola cuts the hyperbola, let OC be drawn parallel to AQ; and from the point C let the right line CAB be drawn through the point A. This ſhall be the *minimum* required.

And, indeed, by the conſtruction, it is $NS = x + \frac{2cc}{a}$, $SO = y + \frac{bcc}{aa}$. And, by the property of the hyperbola, it ought to be NS $\times$ SO, equal to the conſtant rectangle. Therefore $xy + \frac{2ccy}{a} + \frac{bccx}{aa} + \frac{2bc^4}{a^3} = \frac{2bc^4 + aabcc - ac^4}{a^3}$. But $CO = y = \frac{xx}{a}$, by the property of the parabola. Therefore, inſtead of y, ſubſtituting this value, we ſhall have $\frac{x^3}{a} + \frac{2ccxx}{aa} + \frac{bccx}{aa} = \frac{bcc}{a} - \frac{c^4}{aa}$; that is, $x^3 + \frac{2ccxx}{a} + \frac{bccx}{a} + \frac{c^4}{a} - bcc = 0$, which is the very equation from whence the value of x was to be derived. Therefore, &c.

I have here made the suppofition, that the numerator of the fraction, which expreffes the *minimum*, is to be nothing. The other fuppofition, that the denominator muft be nothing, will give $\overline{cc + ax}|^2 \times \sqrt{cc + xx} = o$, that is, $\sqrt{cc + xx} = o$, $cc + ax = o$. But $\sqrt{cc + xx} = o$ gives us $x = \sqrt{-cc}$, which is imaginary, and therefore of no ufe. $cc + ax = o$ gives us $x = -\frac{cc}{a}$. But, taking $Qc = x = -\frac{cc}{a}$, and drawing Ac, the triangle QAc will be fimilar to the triangle QFA, or PFD, and therefore the angle QcA will be equal to the angle FDP. Whence cA will be parallel to DP; which is as much as to fay, that a line drawn from the point c, and through the point A in the given angle FDG, will be infinite, which is a kind of *maximum*.

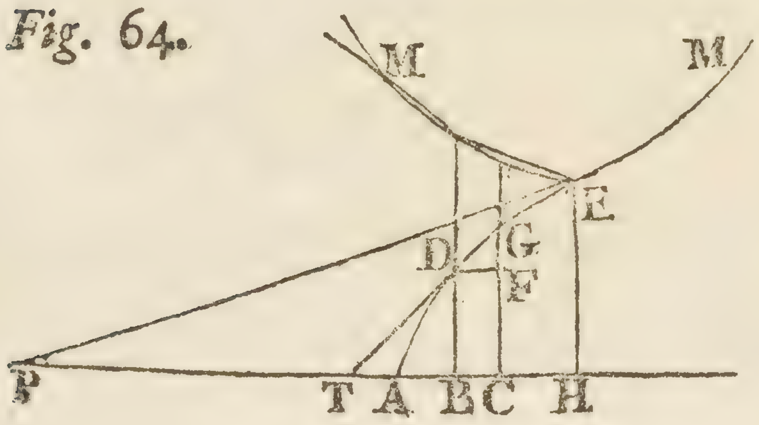
It may be fhown ftill in a fhorter manner, that the right line here fought will be infinite. For, in the expreffion $\sqrt{cc + xx} + \frac{ab - cc}{cc + ax}\sqrt{cc + xx} = CB$, inftead of x, if we fubftitute it's value $-\frac{cc}{a}$, the denominator becomes nothing, and therefore the line is infinite.

SECT. IV.

Of Points of Contrary Flexure, and of Regreffion.

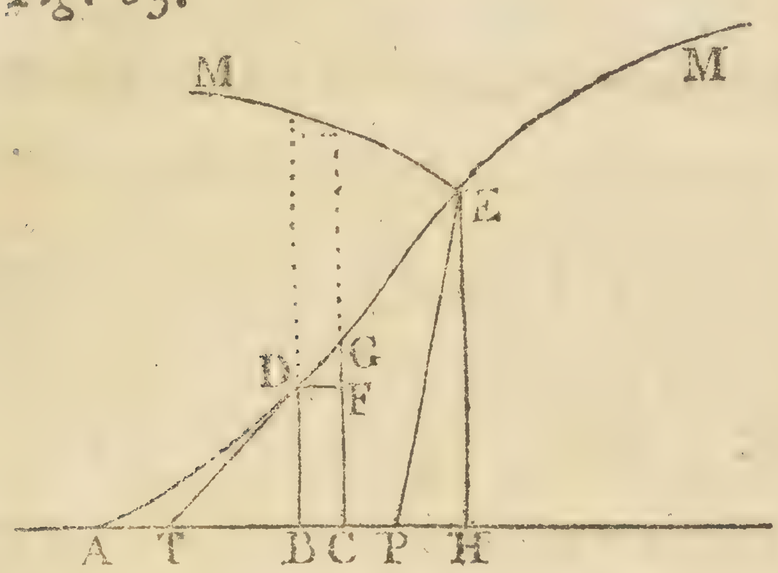
94. In Sect. VI. Vol. I. it has been faid already, what are Contrary Flexures and Regreffions of Curves. Suppofing, therefore, that to be already known, let ADEM be a curve whofe ordinates are parallel, and which in E has a contrary flexure or regreffion. Taking any abfcifs, AB $= x$, and it's ordinate BD $= y$, and drawing CF parallel and indefinitely near to BD; it is plain, that, affuming $\dot{x} =$ BC as conftant, that, as the abfcifs AB $= x$ continually increafes, the fluxion GF of the ordinate

Fig. 64.



nate BD, that is, $\dot{y}$, will always become lefs and lefs, till the ordinate becomes HE, which correfponds to the point of contrary flexure or of regreffion: after which point, in both cafes, the fluxion $\dot{y}$ will go on continually increafing. Therefore, in the point of contrary flexure or regreffion, $\dot{y}$ will be a *minimum.* Whence, by the Method of *Maxima* and *Minima*, $\ddot{y} = 0$, or elfe $\ddot{y} = \infty$, will be the formula of contrary flexure or regreffion.

Fig. 65.



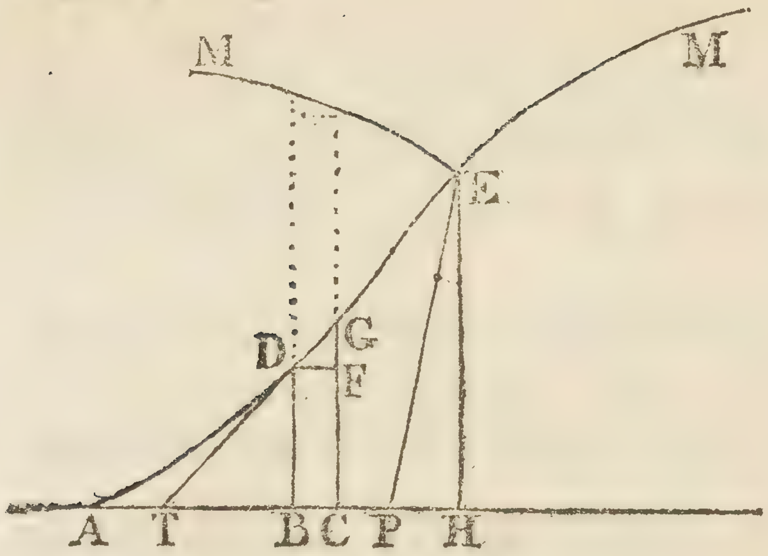
If the curve fhall be firft convex, and afterwards concave to the axis AH; the abfcifs increafing continually, the fluxion or difference of the ordinate will increafe to the point E of contrary flexure or regreffion, after which it will go on decreafing. Therefore, in this point, $\dot{y}$ is a *maximum*, and, for that reafon, we may put $\ddot{y} = 0$, or elfe $\ddot{y} = \infty$.

The fame thing may alfo be inferred from this confideration, that, in a curve firft concave towards it's axis, the fecond fluxion of the ordinate y, that is, $\ddot{y}$, is negative to the point E of regreffion or contrary flexure, after which it becomes pofitive. And, in curves that are firft convex, that fecond fluxion is pofitive as far as the point E, after which it becomes negative. But no quantity from pofitive can become negative, or from negative can become pofitive, but it muft pafs through either nothing or infinite. Therefore, in the point E of regreffion or contrary flexure, it ought to be $\ddot{y} = 0$, or elfe $\ddot{y} = \infty$.

Let the right line DT (Fig. 64.) be a tangent in the point D to the curve AEM, which is firft concave towards the axis; and alfo, the right line EP at the point E. As the abfcifs AB increafes, the line AT, intercepted between the tangent and the origin of the abfcifs will always increafe fo far till the point B falls in H, after which, in the cafe of contrary flexure, the abfcifs ftill increafing, that intercepted line will decreafe. Therefore, in the point E of contrary flexure, that intercepted line AP $= \frac{y\dot{x}}{\dot{y}} - x$ ought to be a *maximum.* Wherefore, by differencing, taking $\dot{x}$ for conftant, it will be $\frac{\dot{y}\dot{y}\dot{x} - y\dot{x}\ddot{y} - \dot{y}\dot{y}\dot{x}}{\dot{y}\dot{y}}$, equal to nothing, or to infinite; that is, by reducing, and dividing by $-y\dot{x}$, and multiplying by $\dot{y}\dot{y}$, it will be, finally, $\ddot{y} = 0$, or $\ddot{y} = \infty$. In cafe that the point E be a point of regreffion, if the intercepted line AT increafe, the abfcifs AB will alfo increafe, till the point T falls in P, and the abfcifs fhall be AH; beyond which point T the abfcifs will go on decreafing. Therefore AH will be a *maximum*, and it's difference will be equal to nothing, or infinite. Therefore, relatively to fuch a difference, the difference of AP will be infinite, or nothing. Therefore $\ddot{y} = \infty$, or $\ddot{y} = 0$, as before.

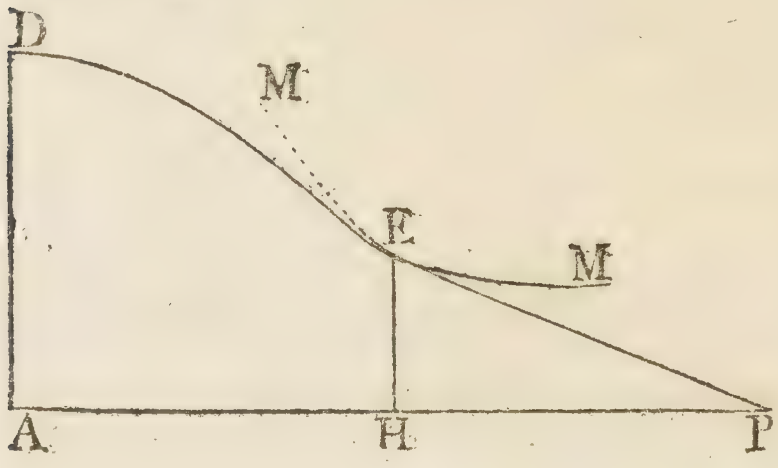
L 2 If

Fig. 65.



If the curve be firſt convex to the axis, the intercepted line AT will be $= x - \frac{y\dot{x}}{\dot{y}}$, and the difference $\frac{\dot{x}\dot{y}\dot{y} - \dot{x}\dot{y}\dot{y} + y\dot{x}\ddot{y}}{\dot{y}\dot{y}}$, that is, $\frac{y\dot{x}\ddot{y}}{\dot{y}\dot{y}}$; and therefore, dividing by $y\dot{x}$, and multiplying by $\dot{y}\dot{y}$, we ſhall have neither more nor leſs than $\ddot{y} = 0$, or elſe $\ddot{y} = \infty$.

Fig. 66.



In the curve DEM, the origin of the abſciſſes x being A, and E the point of contrary flexure, the intercepted line AP will be equal to AH + HP. But, in this caſe, the ſubtangent HP is negative, that is, $-\frac{y\dot{x}}{\dot{y}}$. Therefore it will be AP $= x - \frac{y\dot{x}}{\dot{y}}$. Hence we ſee, that in no caſe the intercepted line AP can be $x + \frac{y\dot{x}}{\dot{y}}$.

95. The formula here found will ſerve for curves which have parallel ordinates, or ſuch as are referred to an axis or diameter. But it is different in curves that are referred to a focus.

Fig. 67.

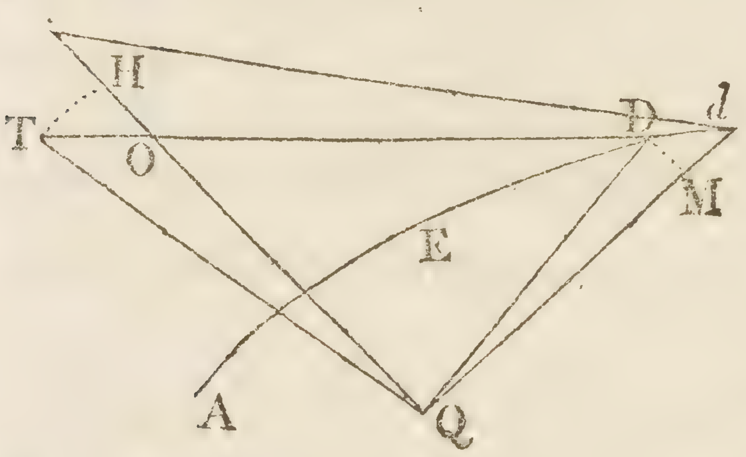
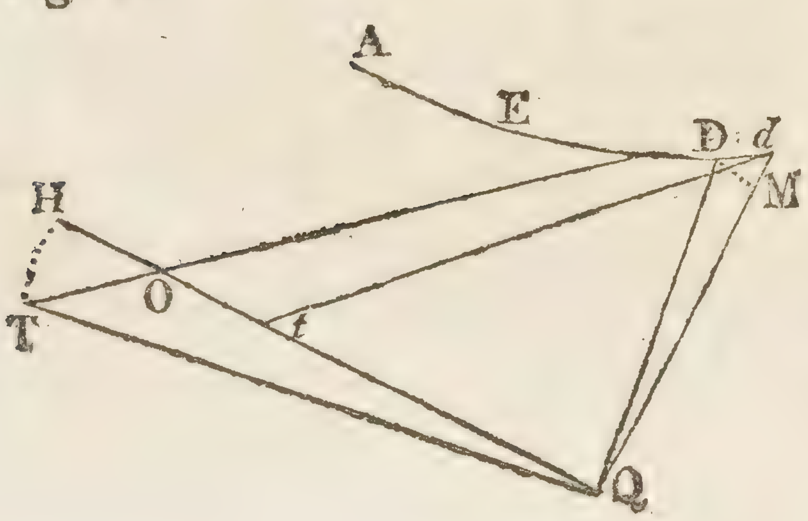


Fig. 68.



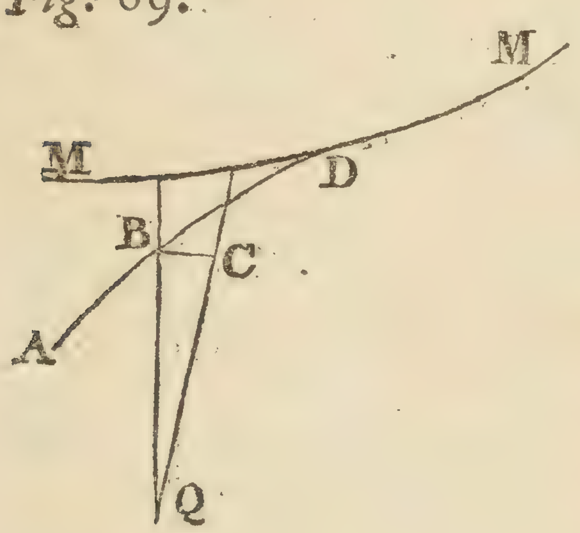
Let the curve be ADE, (Fig. 67, 68.) it's focus Q, from whence the ordinates QD proceed; and let Q*d* be infinitely near to QD. Draw QT perpendicular to QD, and and Q*t* perpendicular to Q*d*. Draw DT a tangent to the curve in the point D, and *dt* a tangent in the point *d*. Let Q*t* (produced if need be,) meet DT in the point *o*. Now it is plain, that, as the ordinates increaſe, if the curve be concave towards the focus Q, (Fig. 67.) Q*t* will be greater than QT. But, if the curve be convex towards the focus Q, (Fig. 68.) Q*t* will be leſs than QT. Therefore, as the curve changes from being concave to convex, or *vice verſâ*, that is, in the point of contrary flexure or regreſſion, the line or quantity *ot*, from

from being positive, ought to become negative, or the contrary, and therefore must pass through nothing or infinite.

Wherefore, make $QD = y$, $DM = \dot{x}$, and with centre Q let the infinitesimal arches DM, TH, be described. The two triangles *d*MD, *d*QT, will be similar, as also, *d*Q*o*, TH*o*, and therefore it will be *d*M . MD :: *d*Q (or DQ) . QT. That is, $\dot{y} . \dot{x} :: y . QT = \frac{y\dot{x}}{\dot{y}}$. But the two sectors DQM, TQH, are also similar; whence QD . DM :: QT . TH. That is, $y . \dot{x} :: \frac{y\dot{x}}{\dot{y}} . TH = \frac{\dot{x}\dot{x}}{\dot{y}}$. And, because of the similar triangles *d*Q*o*, TH*o*, it will be *d*Q (or DQ) . Q*o* (or QT) :: TH . H*o*. That is, $y . \frac{y\dot{x}}{\dot{y}} :: \frac{\dot{x}\dot{x}}{\dot{y}} . HO = \frac{\dot{x}^3}{\dot{y}^2}$. But H*t* (Fig. 67.) is the difference of QT, that is, $Ht = \frac{\dot{x}\dot{y}\dot{y} - y\dot{x}\ddot{y}}{\dot{y}\dot{y}}$, taking $\dot{x}$ for constant. Therefore $to = tH + Ho = \frac{\dot{x}\dot{y}\dot{y} - y\dot{x}\ddot{y} + \dot{x}^3}{\dot{y}\dot{y}}$, which must be equal to o, or to ∞. And therefore, also, multiplying by $\dot{y}\dot{y}$, and dividing by $\dot{x}$, it will be $\dot{y}\dot{y} - y\ddot{y} + \dot{x}\dot{x}$, equal to nothing, or infinite.

In Fig. 68, the line *ot* becomes negative, and therefore $= \frac{-\dot{x}\dot{y}\dot{y} + y\dot{x}\ddot{y} - \dot{x}^3}{\dot{y}\dot{y}}$. Therefore, dividing by $-\dot{x}$, and multiplying by $\dot{y}\dot{y}$, it will be $\dot{x}\dot{x} + \dot{y}\dot{y} - y\ddot{y}$ equal to o, or to ∞.

Fig. 69.



Wherefore, if any curve be referred to a focus Q, whose ordinates are $QB = y$, and the little arches $BC = \dot{x}$, and shall have a contrary flexure or regression; the general formula to determine it will be $\dot{y}\dot{y} + \dot{x}\dot{x} - y\ddot{y} = o$, or $= \infty$.

Here, if we suppose y infinite, the two first terms of the formula will be nothing in respect of the third, and therefore it will be $-y\ddot{y}$, equal to nothing, or infinity; and dividing by $-y$, we shall have $\ddot{y} = o$, or $\ddot{y} = \infty$; which is the formula of the first case of curves referred to a diameter, as it ought to be. For, supposing y infinite, the ordinates become parallel to one another.

96. The nature of a curve being given by means of an equation, and $\dot{x}$ being supposed constant; by differencing twice, if the curve be algebraical, or once, if it be a differential of the first degree, that we may have the value of $\ddot{y}$ expressed by $\dot{x}$; this, compared to o or ∞, will give those values of the absciss x, to which will correspond that ordinate y, which meets the curve in the points of contrary flexure or regression. Wherefore, if those values be substituted in the

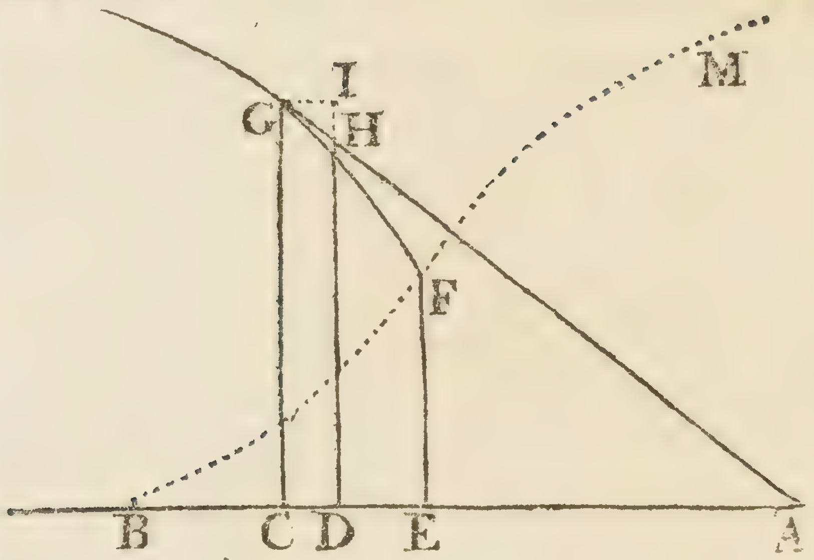
the equation of the curve inſtead of x, we ſhall have y either real or imaginary. If y be imaginary, or ſhall involve a contradiction, then the curve will have no ſuch points.

97. To diſtinguiſh the points of contrary flexure from thoſe of regreſſion, becauſe this method gives us each of them indiſcriminately, it will be ſufficient to ſee the progreſs of the curve, by taking an ordinate very near. And this will afford light enough to remove any doubt about it.

98. Curves may have another kind of regreſſion, different from this which has been conſidered. And that is, when the curve returns backwards towards it's origin, turning it's cavity the ſame way as it did before it's regreſſion. After I have firſt treated on the Radii of Curvature, I ſhall give a general formula, alſo, for regreſſions of this ſecond ſort, at the end of the following Section.

EXAMPLE I.

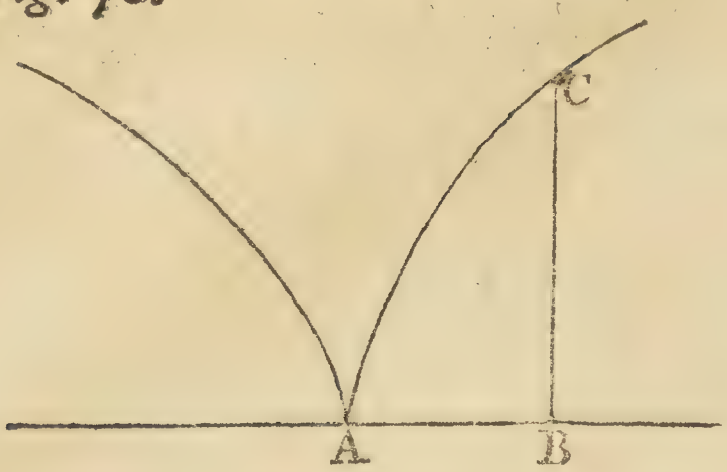
Fig. 51.



99. Let there be a cubic parabola with the equation $y = a + \sqrt[3]{a^3 - 2aax + axx}$, which, in § 81, has been found to have a point of interſection. Now, by differencing, it will be $\dot{y} = \frac{-2aa\dot{x} + 2ax\dot{x}}{3 \times \overline{a^3 - 2aax + axx}|^{\frac{2}{3}}}$, and differencing again, taking $\dot{x}$ conſtant, it will be $\ddot{y} = -\frac{2a\dot{x}\dot{x}}{9 \times \overline{a^3 - 2aax + axx}|^{\frac{2}{3}}}$. The ſuppoſition of $\ddot{y} = 0$ will give us $-2a\dot{x}\dot{x} = 0$, which is of no uſe; making, therefore, the ſuppoſition of $\ddot{y} = \infty$, it will be $9 \times \overline{a^3 - 2aax + axx}|^{\frac{2}{3}} = 0$, that is, $aa - 2ax + xx = 0$, and therefore $x = a$. This value being ſubſtituted inſtead of x in the propoſed equation, it will be $y = a$, and therefore the curve has a contrary flexure, or regreſſion, which correſponds to the abſciſs $x = a$, to which belongs the ordinate $y = a$. And, becauſe we know otherwiſe, that this is alſo a point of interſection; it cannot therefore be a point of contrary flexure, but muſt be a regreſſion.

In

Fig. 70.

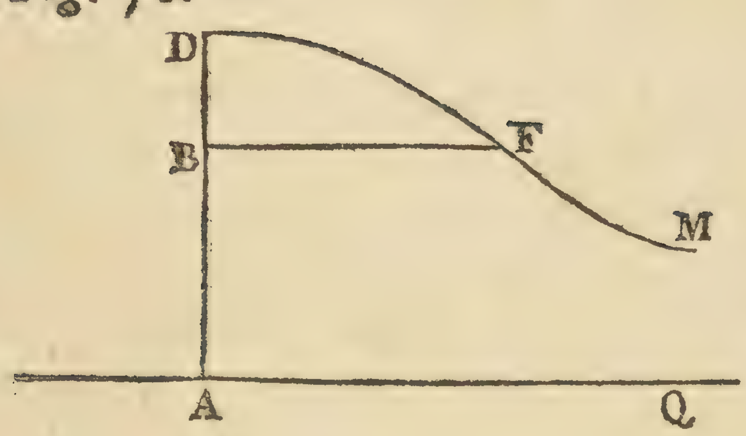


In the fame cubic parabola, taking the abfcifs AB $= x$ from the vertex A, and the ordinate BC $= y$; the equation is $axx = y^3$, the fluxion of which is $2ax\dot{x} = 3yy\dot{y}$. And taking the fluxions again, making $\dot{x}$ conftant, it will be $\ddot{y} = \frac{-6y\dot{y}\dot{y} + 2a\dot{x}\dot{x}}{3yy}$. But, by the equation, it is $3yy = 3x\sqrt[3]{aax}$, and, by the firft differencing, $\dot{y} = \frac{2ax\dot{x}}{3x\sqrt[3]{aax}}$. Therefore, making the fubftitutions, it will be $\ddot{y} = \frac{-2a\dot{x}\dot{x}}{9x\sqrt[3]{aax}}$.

The fuppofition of $\ddot{y} = 0$ has no ufe. The fuppofition of $\ddot{y} = \infty$ will give $9x\sqrt[3]{aax} = 0$, that is, $x = 0$; which value, being fubftituted in the equation, gives $y = 0$. Therefore the curve has a regreffion at the vertex A.

EXAMPLE II.

Fig. 71.



100. Let the curve be DFM, commonly called the Witch, the equation of which is $y = a\sqrt{\frac{a-x}{x}}$, AB $= x$, BF $= y$, AD $= a$; by differencing, $\dot{y} = -\frac{aa\dot{x}}{2x\sqrt{ax - xx}}$; and taking $\dot{x}$ conftant, and differencing again, it will be $\ddot{y} = \frac{3a^3\dot{x}\dot{x} - 4aax\dot{x}\dot{x}}{4x \times \overline{ax-xx}|^{\frac{3}{2}}}$.

The fuppofition of $\ddot{y} = 0$ will give $3a^3 - 4aax = 0$, that is, $x = \frac{3}{4}a$; which value, being fubftituted in the equation of the curve, gives $y = a\sqrt{\frac{1}{3}}$. Whence, taking AB $= \frac{3}{4}a$, the ordinate BF $= a\sqrt{\frac{1}{3}}$ will meet the curve in the point F, which will be a contrary flexure. The fuppofition of $\ddot{y} = \infty$ gives us $4x \times \overline{ax - xx}|^{\frac{3}{2}} = 0$, that is, $x = 0$, and $x = a$. The firft value fubftituted in the equation makes $y = \infty$, the fecond, $y = 0$. But neither the one nor the other cafe infer a contrary flexure, but only that the afymptote AQ, as alfo the tangent in the point D, is parallel to the ordinates.

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EXAMPLE III.

Fig. 72.

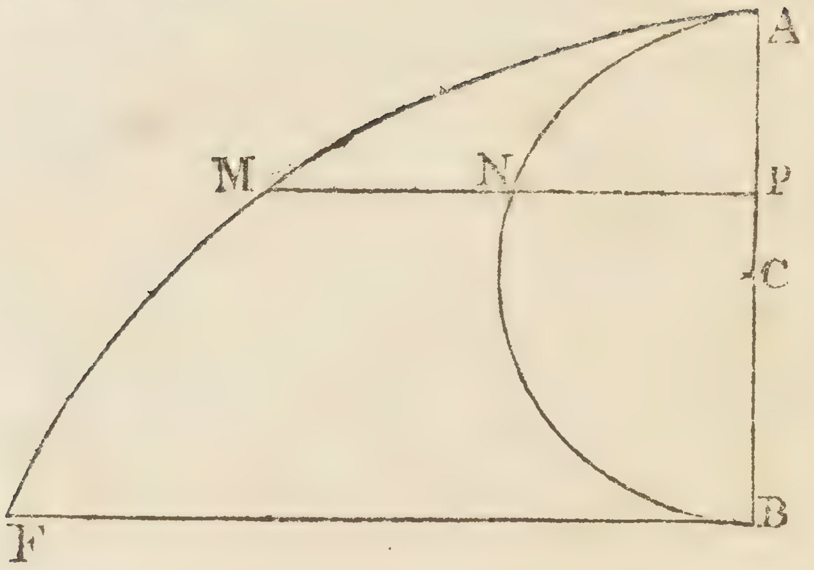


Fig. 73.

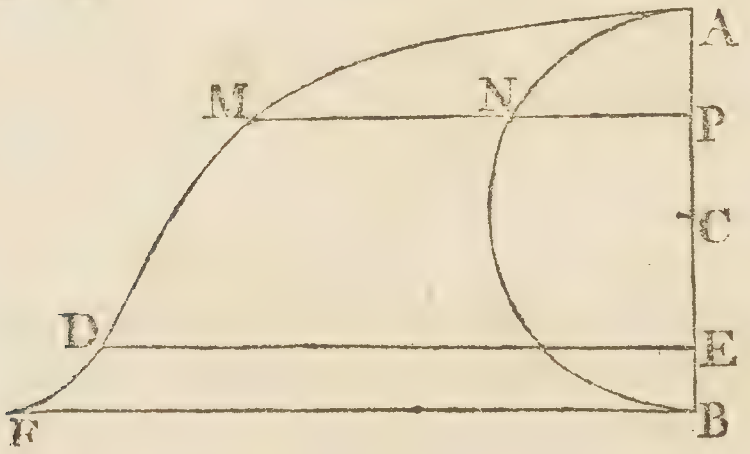
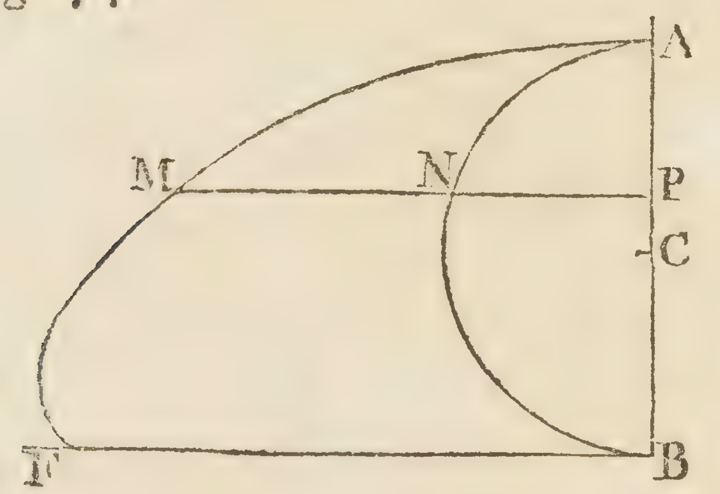


Fig. 74.



101. Let AMF (Fig. 72, 73, 74.) be a cycloid with the equation $\dot{z} = \frac{ar\dot{x} + br\dot{x} - bx\dot{x}}{b\sqrt{2rx - xx}}$, § 47. By differencing, it will be $\ddot{z} = \frac{arx - arr - brr}{b \times \overline{2rx - xx}|^{\frac{3}{2}}} \times \dot{x}\dot{x}$.

The suppofition of $\ddot{z} = 0$ will give $arx - brr - arr = 0$, that is, $x = r + \frac{br}{a}$. If a be greater than b, it will be the protracted cycloid. Whence, taking CE from the centre, and equal to the fourth proportional of BF, the femicircle, and the radius, and drawing the ordinate ED, (Fig. 73.) it will meet the curve in the point of contrary flexure D. If a be lefs than b, (Fig. 74.) the cycloid will be contracted. But when $a < b$, the line $x = r + \frac{br}{a}$ will be greater than $2r$, that is, greater than AB, in which cafe the ordinates are imaginary; becaufe there is no part of the curve under the point F. Therefore the curve has no point of contrary flexure or regreffion. If it be $a = b$, it will be the common cycloid, (Fig. 72.) and therefore $x = r + \frac{br}{a} = 2r = AB$, and $y = BF$; which gives no contrary flexure or regreffion, but only informs us that the tangent in F will be parallel to the abfcifs or diameter AB.

The fuppofition of $\ddot{z} = \infty$ gives us $b \times \overline{2rx - xx}|^{\frac{3}{2}} = 0$, that is, $x = 0$, and $x = 2r$. The value $x = 0$, in all the three cafes, gives the tangent in the point A parallel to the ordinates. The value $x = 2r$, in the firft and fecond cafe, gives the tangent in the point F, in the fame manner, parallel to the ordinates. But, in the third cafe, it gives us a contradiction. For, the equation

equation being $\dot{z} = \frac{\dot{x}\sqrt{2r - x}}{\sqrt{x}}$, inftead of x fubftituting it's value $2r$, it will be $\dot{z} = 0$. But it cannot be $\dot{z} = 0$, and at the fame time $\ddot{z} = \infty$; therefore fuch a value ferves to no purpofe in this cafe.

EXAMPLE IV.

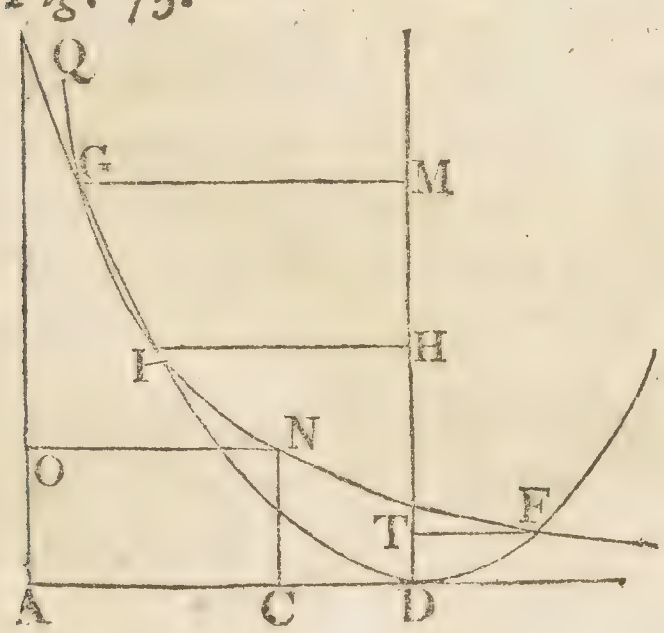
102. Let the curve be the conchoid of *Nichomedes*, confidered above at § 85, the equation of which is $yy = \frac{aaxx - x^4 + 2aabx - 2bx^3 - bbxx + aabb}{xx}$, or $y = \frac{\overline{b + x} \times \sqrt{aa - xx}}{x}$. Taking the fluxions, it will be $\dot{y} = \frac{-x^3\dot{x} - aab\dot{x}}{xx\sqrt{aa - xx}}$; and taking them again, making $\dot{x}$ conftant, $\ddot{y} = \frac{2a^4b - a^2x^3 - 3a^2bx^2}{x^3 \times \overline{aa - xx}|^{\frac{3}{2}}} \times \dot{x}\dot{x}$.

As to the three ufual cafes, which this curve may have, I begin with the firft, when $a = b$, (Fig. 56.) This fuppofed, it will be $\ddot{y} = \frac{2a^5 - aax^3 - 3a^3xx}{x^3 \times \overline{aa - xx}|^{\frac{3}{2}}}\dot{x}\dot{x}$.

The fuppofition of $\ddot{y} = 0$ will give $2a^5 - aax^3 - 3a^3xx = 0$, that is, $x^3 + 3ax^2 - 2a^3 = 0$; and, refolving the equation, it is $x = \sqrt{3aa} - a$, $x = -\sqrt{3aa} - a$, and $x = -a$. The firft value gives us the abfcifs GE $= x = \sqrt{3aa} - a$, to which belongs the ordinate EM $= y = \frac{\sqrt{3aa} \times \sqrt{2a\sqrt{3aa} - 3aa}}{\sqrt{3aa} - a}$, which meets the curve in M, the point of contrary flexure; the fecond value is of no fervice, becaufe it makes the equation of the curve imaginary; the third gives us a regreffion in the point P.

As to the other two cafes, the fuppofition of $\ddot{y} = 0$ gives $2aab - x^3 - 3bxx = 0$, or $x^3 + 3bx^2 - 2aab = 0$. Now, to have the roots of this equation, I make $xx = bz$, a *locus* to the *Apollonian* parabola; and, making the fubftitution, there arifes the fecond *locus* $xz + 3bz - 2aa = 0$, which is to the hyperbola.

Fig. 75.



Between the afymptotes AQ, AD, take AC $= 2a$, the perpendicular CN $= a$, AD $= 3b$, and taking the abfcifs x from the point D on the afymptote AD, let the hyperbola GNF be defcribed, with the conftant rectangle $= 2aa$; it will pafs through the point N. Then raifing DM perpendicular to DA, on the axis DM, with the vertex D, and parameter $= b$, let the parabola of the equation $xx = bz$ be defcribed.

 If,

If, therefore, we affume b greater than a, becaufe AD $= 3b$, AC $= 2a$, CD will be greater than b. Now, taking in the parabola the abfcifs $z = a$ $=$ CN, the ordinate will be $x = \sqrt{ab}$. But if a be lefs than b, alfo $\sqrt{ab}$ will be lefs than b, and thence alfo lefs than CD. Therefore the parabola will cut the hyperbola between N and D, fuppofe in the point I.

Now, if we affume $x = -a$, it will be in the parabola $z = \frac{aa}{b}$, and in the hyperbola $z = \frac{2aa}{-a+3b}$; but $\frac{aa}{b}$ is greater than $\frac{2aa}{-a+3b}$; therefore the parabola will cut the hyperbola in fuch a point I, as that it will be HI $= -x$ lefs than a. Therefore this abfcifs will have in the conchoid a real ordinate, which here determines the contrary flexure in the point N, for example, of the lower branch KN. The line GM, drawn from the point G, another interfection of the parabola and hyperbola, will neceffarily be greater than a, and therefore to fuch an abfcifs there can be no correfponding real ordinate in the conchoid; fo that this value is of no ufe. Laftly, the third value TF will give us an abfcifs, to which an ordinate belongs in the upper branch, which meets the curve in the point of contrary flexure M.

Fig. 58.

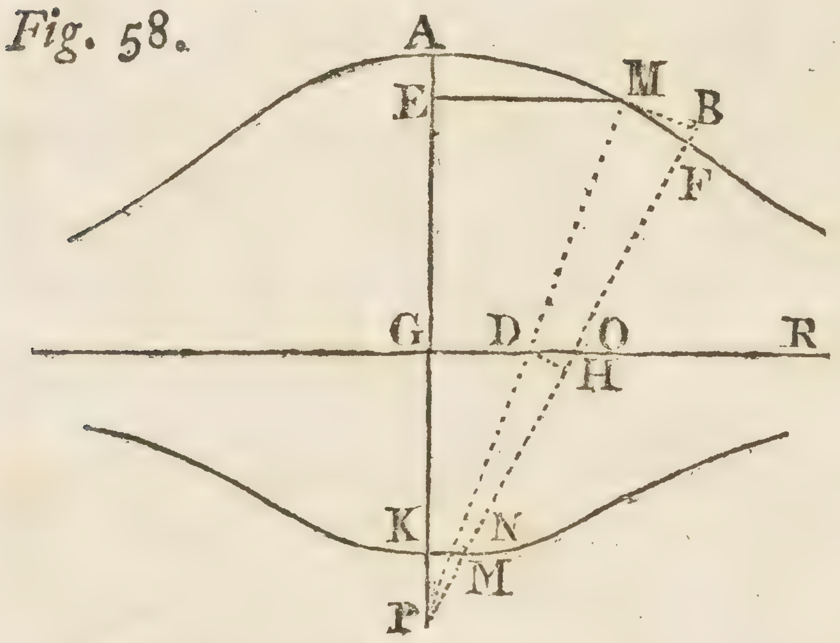
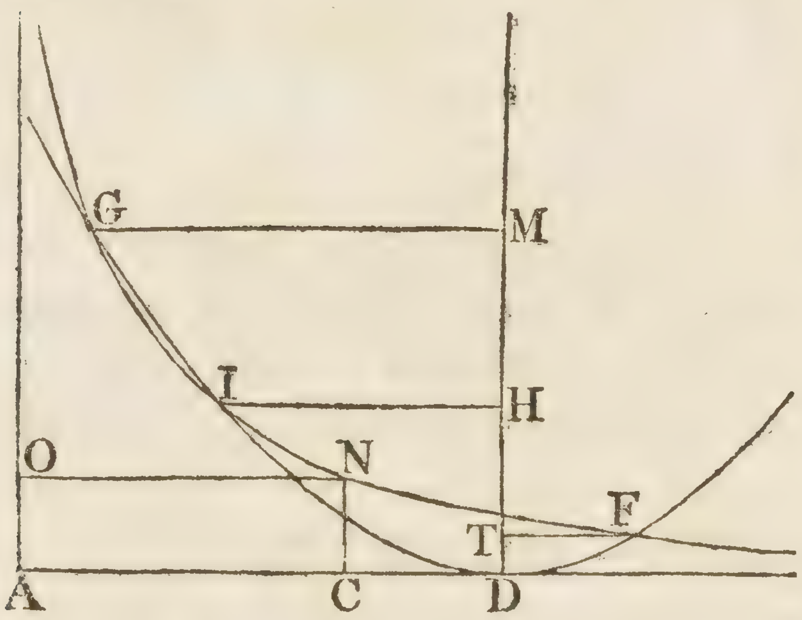


Fig. 76.



Let b be lefs than a; then CD will be lefs than b; and in the parabola, taking $z = a =$ CN, the ordinate will be $x = \sqrt{ab}$, that is, greater than b, and therefore greater than CD. Whence the parabola will pafs between N and C: fo that it will either not cut the hyperbola, and the two negative values of x in the equation $x^3 + 3bx^2 - 2aab = 0$ will be imaginary; or, if it cut it, they will always be greater than a, to which, in the conchoid, (Fig. 57.) imaginary ordinates correfpond, and therefore are of no fervice. Wherefore the parabola will certainly cut the hyperbola, on the pofitive fide, in the point F for example. Whence TF, which is lefs than a, will be the value of x, to which the ordinate correfponds in the branch AM of the conchoid, which it meets in M, the point of contrary flexure.

I faid that if the parabola cut the hyperbola between N and O, the two negative values of x would be greater than a. For, taking $x = -a$ in the parabola,

it will be $z = \frac{aa}{b}$, and in the hyperbola $z = \frac{2aa}{3b - a}$. But $\frac{aa}{b}$ is lefs than $\frac{2aa}{3b - a}$, for b is lefs than a. Now, if fo be that x negative be not greater than a, the parabola would not cut the hyperbola; fo that it will cut it in a point in which x fhall be greater than a. Taking x pofitive equal to a, it will be in the parabola $z = \frac{aa}{b}$, and in the hyperbola $z = \frac{2aa}{3b + a}$. But $\frac{aa}{b}$ is greater than $\frac{2aa}{3b + a}$; fo that the parabola will cut the hyperbola in fuch a point F, that TF will be lefs than a.

The fuppofition of $\ddot{y} = \infty$ gives us $x^3 \times \overline{aa - xx}\,^{\frac{3}{2}} = 0$, that is, $x = 0$, and $x = \pm a$; which is as much as to fay that the afymptote and tangent in A are parallel to the ordinates in all the three cafes, as likewife the tangent in K, in the fecond and third cafe: and in the firft, that in P there is a point of interfection, (as the regreffions alfo intimate,) becaufe the fame value $x = - a$ has alfo been already fupplied from the fuppofition of $\ddot{y} = 0$; which point of interfection has alfo been found before, at § 85.

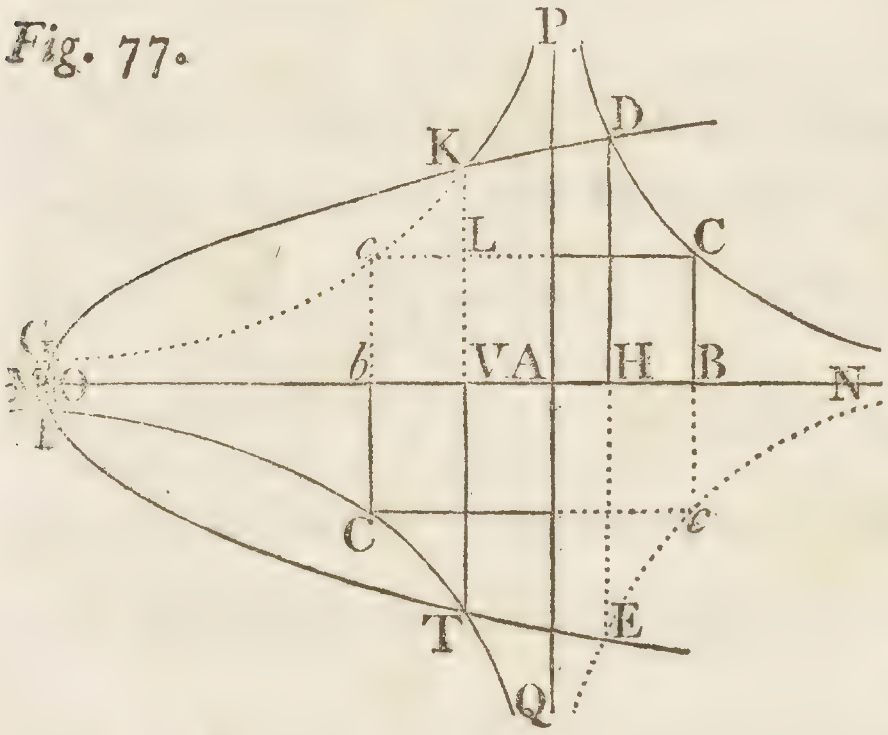
103. The fame after another manner. I take the fame conchoidal curve, but with all it's ordinates proceeding from a fixed point, or from the pole P. Therefore make $PM = y$, (Fig. 56, 57, 58.) and draw PF infinitely near to PM. Then with centre P defcribe the little arches MB, DH; make $MB = \dot{x}$, $AG = a$, $GP = b$, and make $PD = z$, $HO = \dot{z}$. By the property of the curve, the equation will be $y = z \pm a$; that is, $y = z + a$ in refpect of the curve above the afymptote GR, and $y = z - a$ in refpect to the curve below it.

Therefore, finding the fluxions, it will be in both cafes $\dot{y} = \dot{z}$. Becaufe of fimilar triangles PGD, DHO, (for the angles GDP, DOH, do not differ but by the infinitely little angle DPH, and the angles at H and G are right angles,) we fhall have PG . GD :: DH . HO; that is, $b \,.\, \sqrt{zz - bb} :: \frac{z\dot{x}}{y} \,.\, \dot{z}$; and therefore $\dot{z} = \frac{z\dot{x}\sqrt{zz - bb}}{by}$. But $\dot{z} = \dot{y}$, therefore $\dot{y} = \frac{z\dot{x}\sqrt{zz - bb}}{by}$; and taking the fluxions again, making $\dot{x}$ conftant and putting $\dot{z}$ inftead of $\dot{y}$, $\ddot{y} = \frac{2byzz - b^3y - bz^3 + b^3z}{bbyy\sqrt{zz - bb}} \times \dot{x}\dot{z}$; and then putting the value of $\dot{z}$, we fhall have $\ddot{y} = \frac{2yz^3 - bbyz - z^4 + bbzz}{bby^3} \times \dot{x}\dot{x}$; and laftly, fubftituting the value of $y = z \pm a$, it will be $\ddot{y} = \frac{z^4 \pm 2az^3 \mp abbz}{bb \times \overline{z \pm a}|^3} \times \dot{x}\dot{x}$.

The formula of curves referred to a *focus* has been found to be $\dot{x}\dot{x} + \dot{y}\dot{y} - y\ddot{y} = 0$, or else $= \infty$. Therefore, putting the values of y, of $\dot{y}$, and of $\ddot{y}$, it will be $\frac{aabb \pm 3abbz \mp 2az^3}{bb \times \overline{z \pm a}|^2} \times \dot{x}\dot{x} = 0$, or else $= \infty$. The supposition of the formula being equal to o, will give $abb \pm 3bbz \mp 2z^3 = 0$. In the first place, let it be $a = b$, and let us consider the upper branch; it will be $z^3 - \frac{3}{2}aaz - \frac{1}{2}a^3 = 0$, and the three values of z are $z = -a$, $z = \frac{a - \sqrt{3aa}}{2}$, and $z = \frac{a + \sqrt{3aa}}{2}$. But it is $y = z + a$; therefore it will be $y = 0$, $y = \frac{3a + \sqrt{3aa}}{2}$, and $y = \frac{3a - \sqrt{3aa}}{2}$. The third value is of no use, because it gives the ordinate less than $2a$, where there is no curve. The second gives the ordinate y, which meets the curve in the point of contrary flexure, for example, at M. The first is also supplied by considering the lower branch, and determines the point of regression P; and, in respect of the inferior branch, will be $z^3 - \frac{3}{2}aaz + \frac{1}{2}a^3 = 0$. Hence the three values, $z = a$, $z = \frac{-a \pm \sqrt{3aa}}{2}$. But, in this case, $y = z - a$, so that we shall have $y = 0$, $y = \frac{-3a \pm \sqrt{3aa}}{2}$. The two last values serve to no purpose, because they give y negative, where there is no curve.

As to the other two cases, (Fig. 57, 58.) it will be $z^3 - \frac{3}{2}bbz \mp \frac{1}{2}abb = 0$. To obtain the roots of this equation, I put $zz = \frac{1}{2}bp$, a *locus* to the *Apollonian* parabola; and making the substitution, there arises a second *locus* which is to the hyperbola, $pz - 3bz = \pm ab$; that is, the *homogeneum comparationis* is positive in regard to the upper branch of the curve, and negative in regard to the lower. Between the asymptotes PQ, NM, perpendicular in A, are described the opposite hyperbolas (Fig. 77.) in the angles PAN, MAQ, if the *homogeneum* be positive, and in the angles PAM, NAQ, if it be negative. And, supposing b to be greater than a, make AB $= b$, BC $= a$; the hyperbolas will pass through the point C. And taking AM $= 3b$, from the point M in the asymptote MN let the p's proceed. Then at the vertex M, with axis MN, and parameter $\frac{1}{2}b$, let there be described the parabola EMD of the equation $zz = \frac{1}{2}bp$. Then taking p = MB $= 2b$, the ordinate in the parabola is $z = b$, greater than a, that is, than bc, the parabola will pass without the points C, and will cut the hyperbolas DC, CT,

Fig. 77.



CT, in the points D, T, I, from which the right lines DH, TV, IO, being drawn parallel to the aſymptote QP, will be the three roots or values of z in the equation $z^3 - \frac{3}{2}bbz - \frac{1}{2}abb = 0$, that is, in reſpect of the upper branch of the conchoid. But $y = z + a$, then DH $+ a$ ſhall be the ordinate y, which meets the curve in the point of contrary flexure, for example in M, (Fig. 58.) The other two roots VT, OI, ſerve to no purpoſe; for, being negative, and a adjoined to VT, the difference, or y, will be negative; and a, adjoined to OI, the difference will be poſitive, but leſs than a; and, in this caſe, the curve will not correſpond to y negative, or leſs than a. As to the inferior branch of the conchoid, that is, in the equation $z^3 - \frac{3}{2}bbz + \frac{1}{2}abb = 0$, the three roots will be OG, VK, HE; but if from the firſt, and from the third, a be ſubtracted to have y, the difference will be negative, that is, y negative, to which the curve does not correſpond, and therefore they will be of no uſe. If a be ſubtracted from the ſecond, VK, the difference LK will be the ordinate y, which meets the curve in the point of contrary flexure, that is, in N.

Fig. 58.

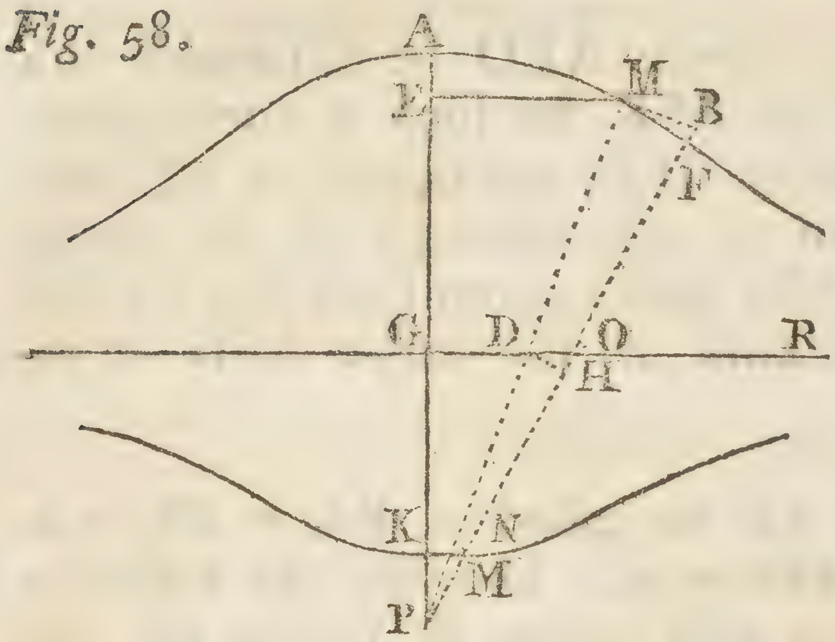
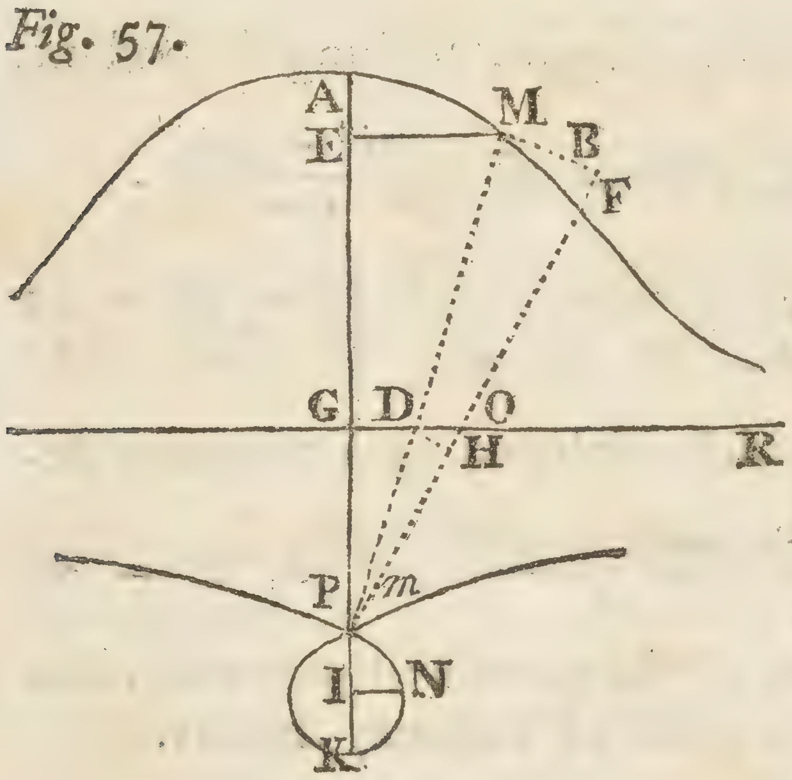


Fig. 57.



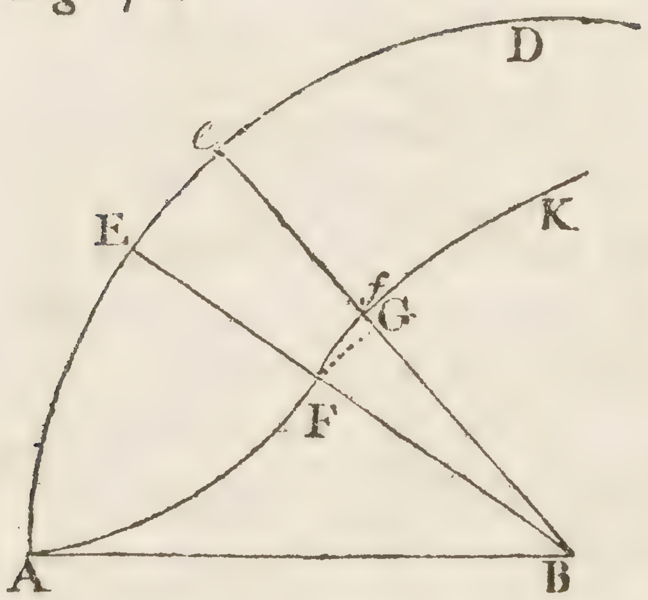
Suppoſing b leſs than a, the parabola will paſs between the points c, C, of the hyperbolas GcK, ICT; and therefore the two negative values of z in the equation $z^3 - \frac{3}{2}bbz - \frac{1}{2}abb = 0$, by adding a, will give y leſs than a, to which the curve does not correſpond. The third, by adding a, will give y, which will meet the curve in the contrary flexure, as at M, (Fig. 57.)

As to the inferior branch, that is, to the equation $z^3 - \frac{3}{2}bbz + \frac{1}{2}abb = 0$, from the two poſitive roots, which are leſs than b, ſubtract a; and alſo, being ſubtracted from the negative root, we ſhall always have negative y greater than PK, to which the curve does not correſpond. Therefore the inferior branch of the conchoid, when b is leſs than a, has neither contrary flexure nor regreſſion.

The ſuppoſition of the formula being $= \infty$, gives, in all the three caſes, $z = \mp a$, and therefore $y = 0$. In Fig. 58, the value $y = 0$ ſerves to no purpoſe, becauſe there is no curve. In Fig. 56, 57, it gives the tangent in P, which is alſo a point of regreſſion in Fig. 56, but not ſo in Fig. 57.

EXAMPLE V.

Fig. 78.



104. Let the circle AED be defcribed with centre B, and let AFK be fuch a curve, that, drawing any radius BFE, the fquare of FE may be always equal to the rectangle of the correfpondent arch AE, into a given right line b; and the contrary flexure of the curve AFK is required.

Let the arch AE be called z, BA = BE = a, BF = y, and FG = $\dot{x}$. Drawing Be infinitely near to BE, and with centre B, radius BF, defcribing the little arch FG; by the nature of the curve, it will be $bz = aa - 2ay + yy$. Then taking the fluxions, it is $b\dot{z} = -2a\dot{y} + 2y\dot{y}$, whence $\dot{z} = \frac{2y\dot{y} - 2a\dot{y}}{b} =$ Ee. But, becaufe of fimilar fectors BEe, BFG, it will be BE . BF :: Ee . FG; that is, $a . y :: \frac{2y\dot{y} - 2a\dot{y}}{b} . \dot{x}$. Whence $\dot{x} = \frac{2yy\dot{y} - 2ay\dot{y}}{ab}$. And taking the fluxions again, making $\dot{x}$ conftant, it will be $4y\dot{y}\dot{y} + 2yy\ddot{y} - 2a\dot{y}\dot{y} - 2ay\ddot{y} = 0$, whence $y\ddot{y} = \frac{a\dot{y}\dot{y} - 2y\dot{y}\dot{y}}{y - a}$.

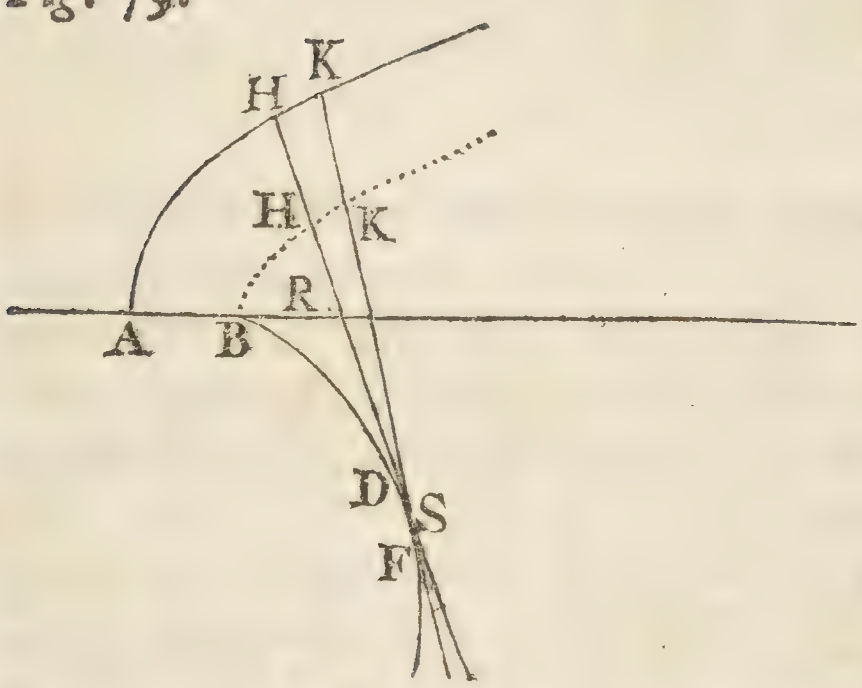
In the general formula of curves referred to a *focus*, $\dot{x}\dot{x} + \dot{y}\dot{y} - y\ddot{y} = 0$; fubftitute the values of $\dot{x}\dot{x}$ and of $y\ddot{y}$ given by $\dot{y}$, and we fhall have $\frac{4y^4\dot{y}^2 - 8ay^3\dot{y}^2 + 4a^2y^2\dot{y}^2 + a^2b^2\dot{y}^2}{aabb} - \frac{a\dot{y}^2 - 2y\dot{y}^2}{y - a}$; which, reduced to a common denominator, will be $\frac{4y^5 - 12ay^4 + 12aay^3 - 4a^3yy + 3aabby - 2a^3bb}{aabb \times \overline{y - a}} = 0$, or $= \infty$. Wherefore, this equation being conftructed, one of the roots will give the value of the ordinate y, which meets the curve in the point of contrary flexure.

SECT.

SECT. V.

Of Evolutes, and of the Rays of Curvature.

Fig. 79.



105. Let the curve be BDF, and let it be involved or wound about by the thread ABDF; that is, the thread being faftened by one of it's ends in the fixed and immoveable point F, let it be conceived to be ftretched along the curve BDF, fo that the portion AB may fall upon the tangent of the curve AR in the point B. Let the thread move or unwind by it's extremity A, continually evolving the curve, but in fuch a manner that it may always have the fame degree of tenfion. By this motion, the point A will defcribe the curve AHK.

The curve BDF is called the *Evolute* of the curve AHK, as has been already faid before, at § 16. And the curve AHK is called the *Involute* of BDF, or the curve generated by the evolution of BDF; and the portions AB, HD, KF, of the thread are called the *Rays* of the Evolute, or *Rays of Ofculation.*

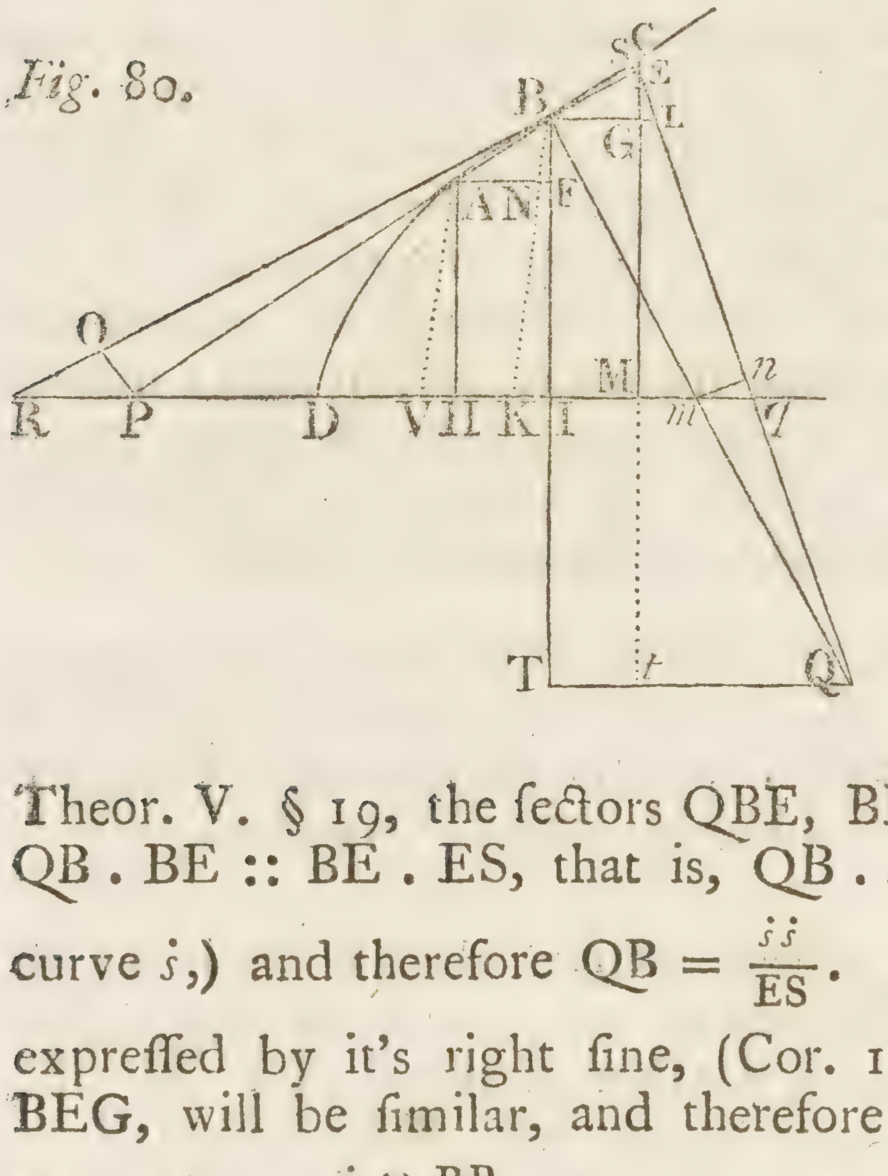
106. Now, becaufe the length of the thread ABDF always continues the fame, it follows from thence, that the difference of the rays of ofculation AB, HD, will be equal to BD, the correfponding portion of the curve. As alfo, the other portion DF is equal to the difference of the radii HD, KF, and the whole curve BDF is equal to the difference of the radii AB, KF. And if the radius AB fhould be none at all, that is, if the point A fhould fall in B, the radius HD would be equal to the portion BD, and the radius FK to the whole curve BDF.

107. From

107. From the generation of the curve AHK, by the unwinding of the thread, it may be clearly seen that every radius HD, KF, at it's extremities D, F, is a tangent to the evolute BDF.

108. Let the arch HK of the curve AHK be an infinitesimal; therefore, also, the arch DF of the evolute will be an infinitesimal; and, as it has been demonstrated in Coroll. 4. Theor. I. § 6. that any infinitely little arch of a curve has the same properties as the arch of a circle: and in Theor. IV. § 15. that the radius HD being produced, so that it may meet the radius KF in S, the lines SH, SK, differ from each other only by an infinitely little quantity of the third degree; therefore those lines SH, SK, may be assumed as equal: and therefore they are perpendicular to the curve AHK in the points H, K. But the two lines HD, HS, differ from each other by DS, an infinitesimal of the first order, and HD is finite; therefore they may be assumed as equal. Wherefore, to determine any point D in the evolute, that is, to determine the length of any ray of osculation or of curvature HD, it will suffice to have given in position the perpendicular HS of the given curve AHK, (which is done by the Method of Tangents;) the point S may be determined, in which it is cut by the infinitely near perpendicular KS. This may be done in the following manner.

Fig. 80.



109. First, let the curve DABE be referred to it's axis; let the two infinitely little arches be AB, BE, the perpendicular BQ, and the other EQ, which meets it in the point required, Q. Make, as usual, DH $= x$, HA $= y$; draw AF, BG, parallel to DM, and the chord PABC which meets ME produced in C, and draw the other chord EBR. Now, with centre B, and distances BE, BP, the little arches ES, PO, being described, it will be AF $= \dot{x}$, FB $= \dot{y}$, AB $= \dot{s}$ $= \sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}$. But, by Coroll. 2. Theor. V. § 19, the sectors QBE, BES, are similar. Therefore we shall have QB . BE :: BE . ES, that is, QB . $\dot{s}$ :: $\dot{s}$. ES, (calling the element of the curve $\dot{s}$,) and therefore QB $= \frac{\dot{s}\dot{s}}{\text{ES}}$. Now, because the little arch PO may be expressed by it's right sine, (Cor. 1. Theor. III. § 9.) the triangles RPO, BEG, will be similar, and therefore BE . EG :: RP . PO; that is, $\dot{s} . \dot{y}$ :: RP . PO $= \frac{\dot{y} \times \text{RP}}{\dot{s}}$. But the sectors BPO, BES, are also similar; and therefore it will be BP . PO :: BE . ES; that is, $\frac{y\dot{s}}{\dot{y}} . \frac{\dot{y} \times \text{RP}}{\dot{s}}$ :: $\dot{s}$. ES $= \frac{\dot{y}\dot{y} \times \text{RP}}{y\dot{s}}$.

And

And laſtly, $QB = \frac{y\dot{s}^3}{\dot{y}^2 \times RP}$, a general formula for the rays of oſculation, or the radii of curvature, in which nothing elſe remains to be done, but to ſubſtitute the value of RP, the fluxion of $DP = \frac{y\dot{x}}{\dot{y}} - x$, according to the different hypotheſis of the firſt fluxion which is to be taken for conſtant.

If no firſt fluxion be taken for conſtant, it will be $RP = \frac{y\dot{y}\ddot{x} - y\dot{x}\ddot{y}}{\dot{y}\dot{y}}$, and therefore $QB = \frac{\overline{\dot{x}\dot{x} + \dot{y}\dot{y}}|^{\frac{3}{2}}}{\dot{y}\ddot{x} - \dot{x}\ddot{y}}$.

If $\dot{x}$ be aſſumed as conſtant, it will be $RP = -\frac{y\dot{x}\ddot{y}}{\dot{y}\dot{y}}$, and therefore $QB = \frac{\overline{\dot{x}\dot{x} + \dot{y}\dot{y}}|^{\frac{3}{2}}}{-\dot{x}\ddot{y}}$.

If $\dot{y}$ be aſſumed as conſtant, it will be $RP = \frac{y\ddot{x}}{\dot{y}}$, and therefore $QB = \frac{\overline{\dot{x}\dot{x} + \dot{y}\dot{y}}|^{\frac{3}{2}}}{\dot{y}\ddot{x}}$.

If $\dot{s}$ be aſſumed as conſtant, that is, $\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}$, it will be $\dot{x}\ddot{x} + \dot{y}\ddot{y} = 0$, and $-\ddot{y} = \frac{\dot{x}\ddot{x}}{\dot{y}}$; whence $RP = \frac{y\ddot{x} \times \overline{\dot{x}\dot{x} + \dot{y}\dot{y}}}{\dot{y}^3}$, and therefore $QB = \frac{\dot{y}}{\ddot{x}}\sqrt{\dot{y}\dot{y} + \dot{x}\dot{x}}$; or elſe, ſubſtituting the value $\ddot{x}$, $QB = \frac{\dot{x}\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}}{-\ddot{y}}$. Therefore, in the expreſſion of $QB = \frac{\overline{\dot{x}\dot{x} + \dot{y}\dot{y}}|^{\frac{3}{2}}}{\dot{y}\ddot{x} - \dot{x}\ddot{y}}$, in which, as no fluxion is taken for conſtant, it will be ſufficient to expunge the term $\dot{y}\ddot{x}$, in the ſuppoſition of $\dot{x}$ conſtant; to expunge the term $\dot{x}\ddot{y}$, in the ſuppoſition of $\dot{y}$ conſtant; and to put, inſtead of $-\ddot{y}$, it's value $\frac{\dot{x}\ddot{x}}{\dot{y}}$, in the ſuppoſition of $\dot{s}$ conſtant.

110. The curve may be referred to a diameter, or the co-ordinates may be inclined to each other in an oblique angle. Make the abſciſs $DV = x$, $VK = \dot{x}$, the ordinate $VA = y$, and the reſt as above. Becauſe the angle DKB is known, the angle BNF will be known alſo. Wherefore, it being $NB = \dot{y}$, NF and FB will be given, and therefore AB, or $\dot{s}$. But the triangle RPO is ſimilar to the triangle ABF, for the angles at O and F are right ones, and the angle ORP does not differ from the angle FAB but by an infinitely little angle RBP. Wherefore there will be given RP, PO, and thence ES, and finally, QB.

111. From

Subosculatrix, or Co-radius, what.

111. From the extremity of the radius of curvature BQ is drawn QT parallel to the axis DM, which meets in T the ordinate BI produced; the right line BT is called the *Subosculatrix*, or the *Co-radius*. The radius BQ being given, the co-radius BT will, in like manner, be given also; for, by the method of tangents, the normal of the curve B*m* is given, and therefore BT will be given by means of the similar triangles B*m*I, BQT.

But if we would have an expression for the co-radius independently of the radius, we may make BT $= z$. The triangle BTQ is similar to the triangle BCG, or BAF; for, the two angles TBG, QBC, being right ones, take away the common angle QBG, and there will remain the equal angles TBQ, CBG, and the angles at T and G are right ones. Therefore it will be $\dot{x} \,.\, \dot{s} :: z \,.\, \text{BQ} = \frac{z\dot{s}}{\dot{x}} = \frac{z\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}}{\dot{x}}$. But, by Theor. IV. § 15, BQ is equal to EQ, because they differ from each other only by an infinitesimal of the third degree; therefore the difference of QB shall be nothing; and, by differencing, without assuming a constant fluxion, $\frac{\dot{x}\dot{z} \times \overline{\dot{x}\dot{x} + \dot{y}\dot{y}} + z\dot{x}\dot{x}\ddot{x} + z\dot{x}\dot{y}\ddot{y} - z\ddot{x} \times \overline{\dot{x}\dot{x} + \dot{y}\dot{y}}}{\dot{x}\dot{x}\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}} = 0$. But $\dot{z} = \dot{y}$, because TB and IB have the same difference. Therefore $z = \frac{\dot{x} \times \overline{\dot{x}\dot{x} + \dot{y}\dot{y}}}{\dot{y}\ddot{x} - \dot{x}\ddot{y}} = \text{BT}$, a formula for the co-radius, in which no fluxion is yet assumed as constant. If $\dot{x}$ be constant, the term $\dot{y}\ddot{x}$ shall be nothing, and therefore the formula, on this supposition, will be $\frac{\dot{x}\dot{x} + \dot{y}\dot{y}}{-\ddot{y}} = \text{BT}$. If $\dot{y}$ be constant, the term $-\dot{x}\ddot{y}$ will be nothing, and therefore the formula, on this supposition, will be $\frac{\dot{x} \times \overline{\dot{x}\dot{x} + \dot{y}\dot{y}}}{\dot{y}\ddot{x}} = \text{BT}$. If the element of the curve be constant, it will be $-\ddot{y} = \frac{\dot{x}\ddot{x}}{\dot{y}}$, and therefore the formula, on this supposition, will be $\frac{\dot{x}\dot{y}}{\ddot{x}} = \text{BT}$, the value of $\ddot{y}$ being substituted: or else $-\frac{\dot{x}\dot{x}}{\ddot{y}} = \text{BT}$, the value of $\ddot{x}$ being substituted.

The co-radius being given, by the similitude of the triangles B*m*I, BQT, the radius QB will be given in a like manner.

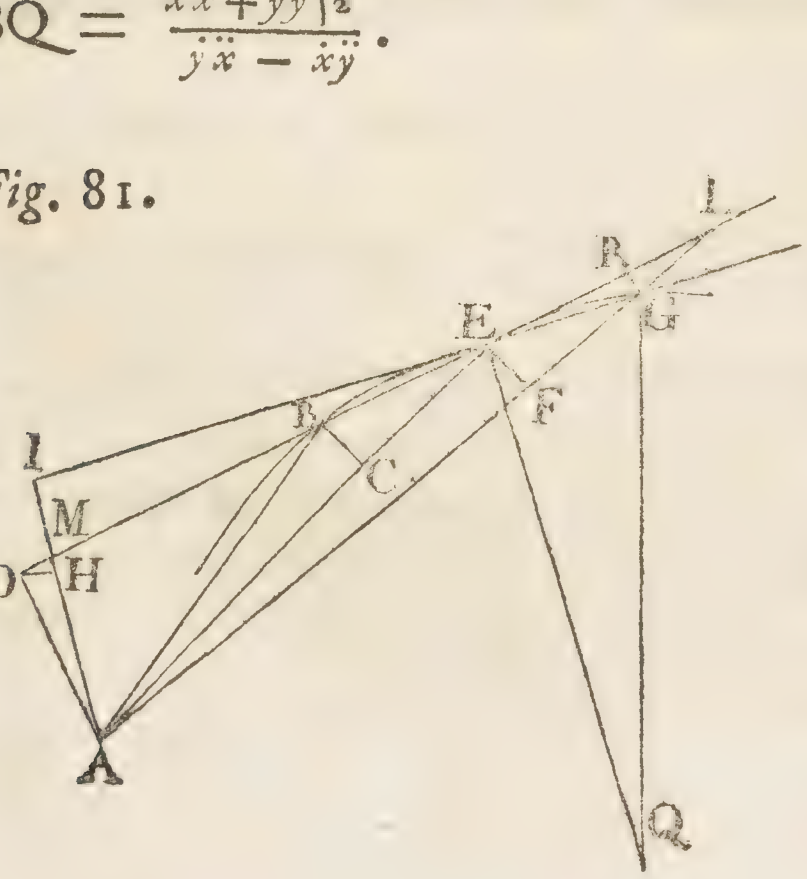
112. If the co-ordinates shall be at an oblique angle to each other, in the analogy $\dot{x} \,.\, \dot{s} :: z \,.\, \text{BQ}$, instead of $\dot{x}$ and $\dot{s}$, it will be enough to put the respective values, which in this case agree to AF, AB, and to do the rest as above; and then you will have the formula of the co-radius BT, in that case when the co-ordinates are at any oblique angle.

113. After

113. After ſeveral other manners the ſame formula of the radius of curvature may be had. As, with centre Q, diſtance Qm, deſcribe the little arch mn. Aſſuming the infiniteſimal arch mn by the tangent at n, the two triangles BCG, mnq, will be ſimilar, and therefore BC . BG :: mq . mn; that is, $\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}$. $\dot{x}$ :: mq . $mn = \frac{mq \times \dot{x}}{\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}}$. But mq is the fluxion of Dm, that is, of the ſubnormal Im, with the abſciſs DI or DH; that is, of $x + \frac{y\dot{y}}{\dot{x}}$. Therefore, by differencing in the hypotheſis, that no fluxion be conſtant, it will be $mq = \frac{\dot{x}^3 + y\dot{x}\ddot{y} + \dot{x}\dot{y}\dot{y} - y\dot{y}\ddot{x}}{\dot{x}\dot{x}}$. Therefore $mn = \frac{\dot{x}^3 + y\dot{x}\ddot{y} + \dot{x}\dot{y}\dot{y} - y\dot{y}\ddot{x}}{\dot{x}\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}}$. But, becauſe of ſimilar ſectors Qmn, QBE, it will be BE — mn . BE :: Bm $\left(\frac{y\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}}{\dot{x}}\right)$. QB; that is, ſubſtituting their analytical values, QB $= \frac{\overline{\dot{x}\dot{x} + \dot{y}\dot{y}}|^{\frac{3}{2}}}{\dot{y}\ddot{x} - \dot{x}\ddot{y}}$. Which formula, being modified according to the ſuppoſition of ſome conſtant fluxion, will give an expreſſion for the radius QB, correſponding to that ſuppoſition.

114. In another manner, thus. Let EM be produced to t, and BG to L. Becauſe the triangle EGL is ſimilar to the triangle BIm, the angles GEL, IBm, being different from each other only by the infiniteſimal angle BQE, it will be GL $= \frac{\dot{y}\dot{y}}{\dot{x}}$. Therefore BL $= \frac{\dot{x}\dot{x} + \dot{y}\dot{y}}{\dot{x}}$. But it has been ſeen, that $mq = \frac{\dot{x}^3 + y\dot{x}\ddot{y} + \dot{x}\dot{y}\dot{y} - y\dot{y}\ddot{x}}{\dot{x}\dot{x}}$. And the ſimilar triangles QBL, Qmq, give BL — mq . BL :: Bm . BQ. Therefore, ſubſtituting the analytical values, we ſhall have BQ $= \frac{\overline{\dot{x}\dot{x} + \dot{y}\dot{y}}|^{\frac{3}{2}}}{\dot{y}\ddot{x} - \dot{x}\ddot{y}}$.

Fig. 81.



115. Now let us reſume the curves which are referred to a *focus*. Therefore let the curve be BEG, the *focus* A. And taking the two infinitely little arches BE, EG, and drawing the ordinates AB, AE, AG, with centre A let the little arches BC, EF, be deſcribed, and to the chords GE, EB produced, let AI, AD, be perpendicular. Laſtly, let the chord DE, produced, meet the ordinate AG in L, and

and with centre E let the little arch GR be defcribed. Make AB $= y$, CE $= \dot{y}$, BC $= \dot{x}$, AD $= p$. The little arch DH being defcribed with centre A, it will be HI $= \dot{p}$. But HM is an infinitefimal quantity of the fecond degree; Theor. III. § 8. Therefore we may take as equal HI, IM, and thence it will be MI $= \dot{p}$. The triangles EBC, EAD, are fimilar, which gives ED $= \frac{y\dot{y}}{\dot{s}} =$ EI, as being different only by an infinitefimal. And, affuming the little arch GR by it's tangent, the triangles EIM, EGR, will be fimilar. Hence GR $= \frac{\dot{p}\dot{s}\dot{s}}{y\dot{y}}$. Now, drawing EQ, QG, perpendicular to the curve in the points E, G, the fectors QEG, EGR, are fimilar; fo that QE $= \frac{y\dot{y}}{\dot{p}}$. The fimilar triangles EBC, EAD, will give us $p = \frac{y\dot{x}}{\dot{s}} = \frac{y\dot{x}}{\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}}$; and, by differencing, without affuming any conftant fluxion, $\dot{p} =$ $\frac{\overline{y\ddot{x} + \dot{x}\dot{y}} \times \overline{\dot{x}\dot{x} + \dot{y}\dot{y}} - \overline{\dot{x}\ddot{x} + \dot{y}\ddot{y}} \times y\dot{x}}{\overline{\dot{x}\dot{x} + \dot{y}\dot{y}}^{\frac{3}{2}}}$; or $\dot{p} = \frac{\dot{x}^3\dot{y} + y\dot{y}\dot{y}\ddot{x} + \dot{x}\dot{y}^3 - y\dot{x}\dot{y}\ddot{y}}{\overline{\dot{x}\dot{x} + \dot{y}\dot{y}}^{\frac{3}{2}}}$. Whence, fubftituting this value inftead of $\dot{p}$ in the expreffion of QE, it will be QE $=$ $\frac{y \times \overline{\dot{x}\dot{x} + \dot{y}\dot{y}}^{\frac{3}{2}}}{\dot{x}^3 + y\dot{y}\ddot{x} + \dot{x}\dot{y}\dot{y} - y\dot{x}\ddot{y}}$, a general formula for the radius of curvature of curves referred to a *focus*, without taking any fluxion as conftant.

If we would have $\dot{x}$ conftant, taking the value of $\dot{p}$ in this hypothefis, and fubftituting; or, without any thing elfe but expunging the term $y\dot{y}\ddot{x}$ in the general formula, it will be QE $= \frac{y \times \overline{\dot{x}\dot{x} + \dot{y}\dot{y}}^{\frac{3}{2}}}{\dot{x}^3 + \dot{x}\dot{y}\dot{y} - y\dot{x}\ddot{y}}$.

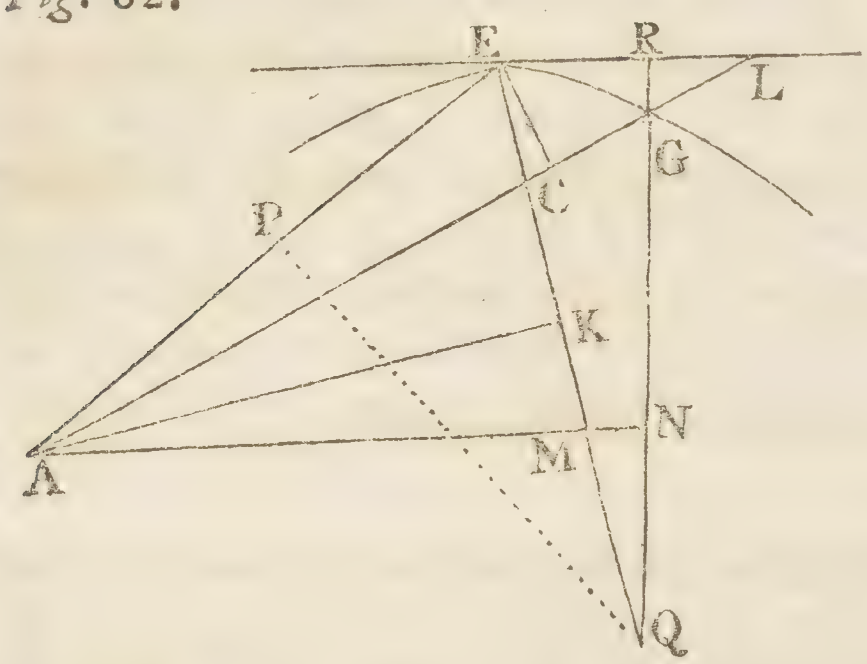
If we would have $\dot{y}$ conftant, expunging the term $- y\dot{x}\ddot{y}$ in the general formula, it will be QE $= \frac{y \times \overline{\dot{x}\dot{x} + \dot{y}\dot{y}}^{\frac{3}{2}}}{\dot{x}^3 + \dot{x}\dot{y}\dot{y} + y\dot{y}\ddot{x}}$.

And laftly, taking $\dot{s}$ for conftant, that is, $\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}$, we fhould have $\ddot{x} =$ $- \frac{\dot{y}\ddot{y}}{\dot{x}}$; and, inftead of $\ddot{x}$, fubftituting this value in the general formula, it will be QE $= \frac{y\dot{x}\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}}{\dot{x}\dot{x} - y\ddot{y}}$; or elfe, fubftituting the value of $\ddot{y}$, it is QE $=$ $\frac{y\dot{y}\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}}{\dot{x}\dot{y} + y\ddot{x}}$.

116. If

116. If, in any of these formulæ, we should suppose y infinite, all those terms would vanish in which it is not found, and the formulæ will be the same as those found for curves referred to an axis; which ought to obtain, because, if y be infinite, the point A will be at an infinite distance, and therefore the ordinates will be parallel.

Fig. 82.



117. After another manner. In the point E let ER be a tangent to the infinitely little arch EG, and let QE, QG, be the two radii of curvature, and produce QG to R. From the *focus* A draw AN perpendicular to QG, and AK perpendicular to QE, and make EK $= t$; then is KM $= \dot{t}$. Because the triangle AKM is similar to the triangle QNM, and this is similar to the triangle QER, it will be QE . ER :: AK . KM $= \dot{t}$. But, because of the similar triangles ELC, or EGC, EAK, it is AK $= \frac{\dot{y}y}{\dot{s}}$, and ER may be assumed for EG. Then it will be QE . $\dot{s}$ :: $\frac{y\dot{y}}{\dot{s}}$. $\dot{t}$, and therefore QE $= \frac{y\dot{y}}{\dot{t}}$. But EK $= t = \frac{y\dot{x}}{\dot{s}}$. Then doing the rest as before, that is, differencing the value of $\dot{t}$, and substituting in the expression of QE, we shall obtain the same formulæ as above.

118. Making QP perpendicular to EA produced to P, the triangles EAK, EQP, will be similar, and therefore EA . EK :: EQ . EP. But it has been shown, that EQ $= \frac{y\dot{y}}{\dot{t}}$. Then $y \,.\, t$ :: $\frac{y\dot{y}}{\dot{t}}$. EP $= \frac{t\dot{y}}{\dot{t}}$. And, instead of t, substituting it's value $\frac{y\dot{x}}{\dot{s}}$, and, instead of $\dot{t}$, the differential $\frac{\dot{x}^3\ddot{y} + y\dot{y}\dot{y}\ddot{x} + \dot{x}\dot{y}^3 - y\dot{x}\dot{y}\ddot{y}}{\overline{\dot{x}^2 + \dot{y}^2}|^{\frac{3}{2}}}$, without assuming a constant fluxion, it will be EP $= \frac{y\dot{x}\dot{s}\dot{s}}{\dot{x}\dot{s}\dot{s} + y\dot{y}\ddot{x} - y\dot{x}\ddot{y}} = \frac{y\dot{x}^3 + y\dot{x}\dot{y}\dot{y}}{\dot{x}^3 + \dot{x}\dot{y}\dot{y} + y\dot{y}\ddot{x} - y\dot{x}\ddot{y}}$, a general formula for the co-radius, in which no fluxion is made constant; from which, being modified, we obtain the other formulæ, which correspond to the supposition of a constant differential. And if in these we should suppose y to be infinite, that is, if we should cancel the terms in which it is not found, we should have the same formulæ which have been found for curves referred to an axis or diameter.

119. Now,

119. Now, whatever the curve may be, as we find but one expreffion only for the radius of curvature, and for the co-radius; and that as well in curves referred to an axis, as in thofe referred to a *focus*; it follows from hence, that, whatever the curve be, it can have but one evolute.

120. Therefore, any curve being given, expreffed by any equation whatever, of which curve the radius of curvature, or the co-radius is required; it will be neceffary to difference the equation, in order to have the values of $\dot{y}$, $\dot{y}\dot{y}$, and $\ddot{y}$ given by $\dot{x}$; or thofe of $\dot{x}$, &c. given by $\dot{y}$; and to fubftitute them in the formulas now found, by which we fhall have the expreffion in finite terms, and quite free from differentials, of the radius of curvature, or the co-radius of the propofed curve.

Fig. 83.

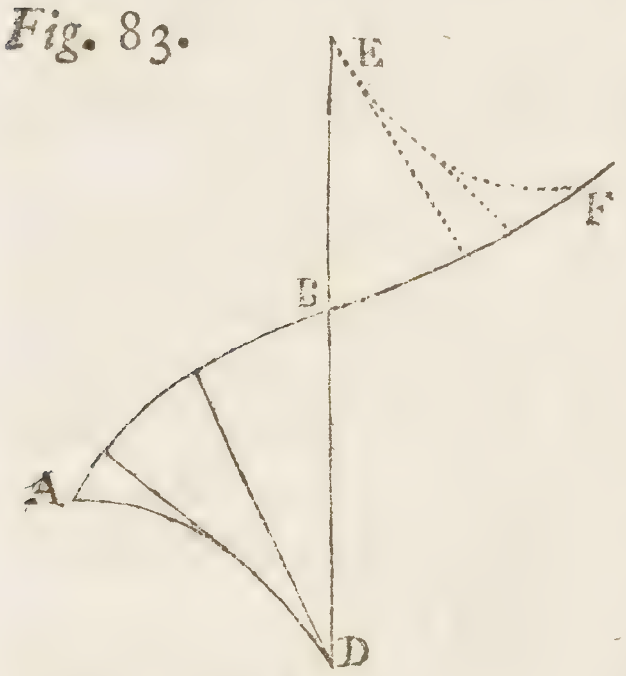


Fig. 84.

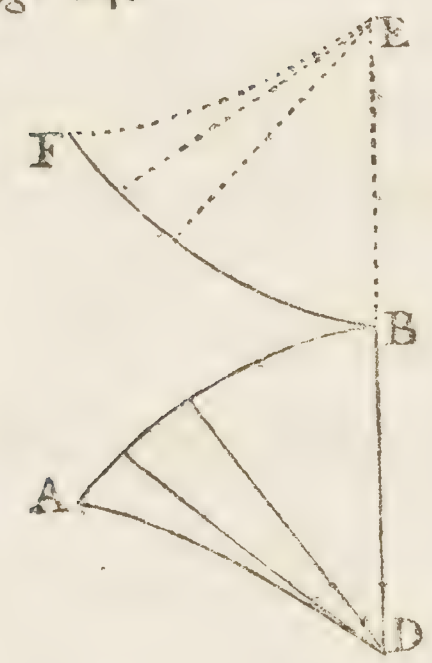
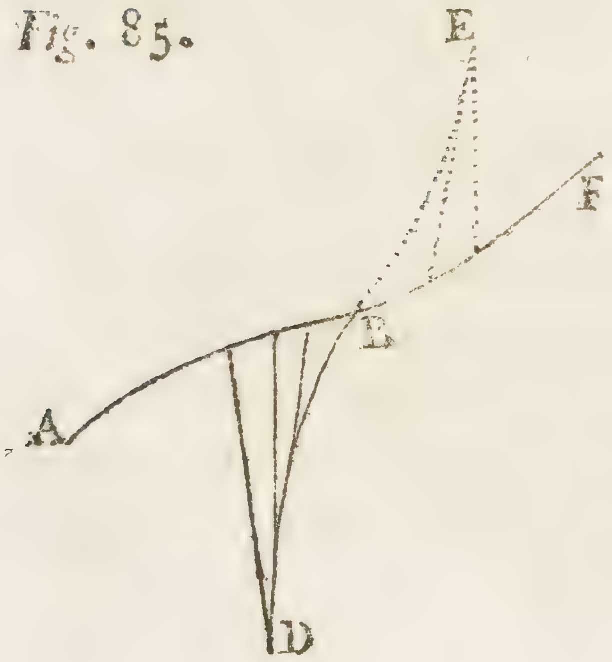
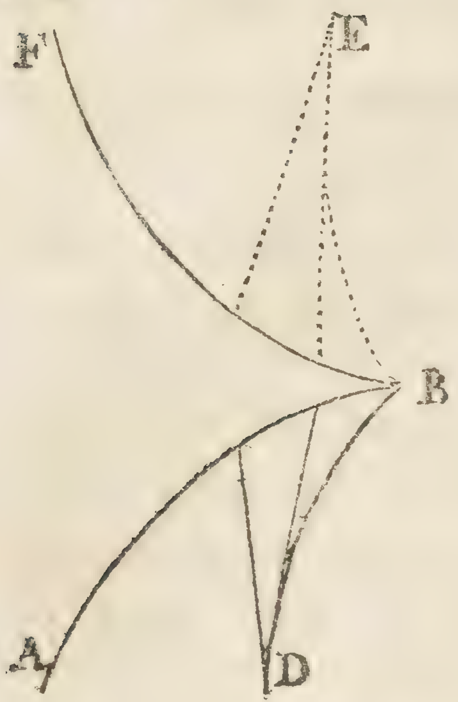


Fig. 85.



121. If the value of the radius of curvature, or of the co-radius, be pofitive, they ought to be taken on that fide of the axis DM, (Fig. 80.) or of the *focus*, (Fig. 81.) as has been hitherto fuppofed, and the curve will be concave to this axis or *focus*. But if it fhall be negative, they ought to be taken on the contrary fide, and, in this cafe, the curve will be convex. Hence it follows, that, in the point of contrary flexure or regreffion, if the curve have any, the co-radius, from pofitive, will become negative; and two radii of curvature that are infinitely near, from being convergent will become divergent. But this cannot be, without they firft become parallel, that is, the radius of the evolute muft be infinite in this point; or elfe they muft coincide one with the other, and thus make the radius of the evolute nothing. It is evident, that when the evolute is fuch, as that the radii go on always increafing, as they approach to the point B (Fig. 83, 84.) of contrary flexure or regreffion, to pafs from being converging to become diverging, they muft firft become parallel, being then AD, FE, the evolute of the curve ABF. But if the evolute of the curve ABF, (Fig. 85, 86.) fhall be DBE, the thread, unwinding itfelf from the point B, and proceeding towards A in refpect of the portion BA of the curve, and going on towards F, in refpect of the portion

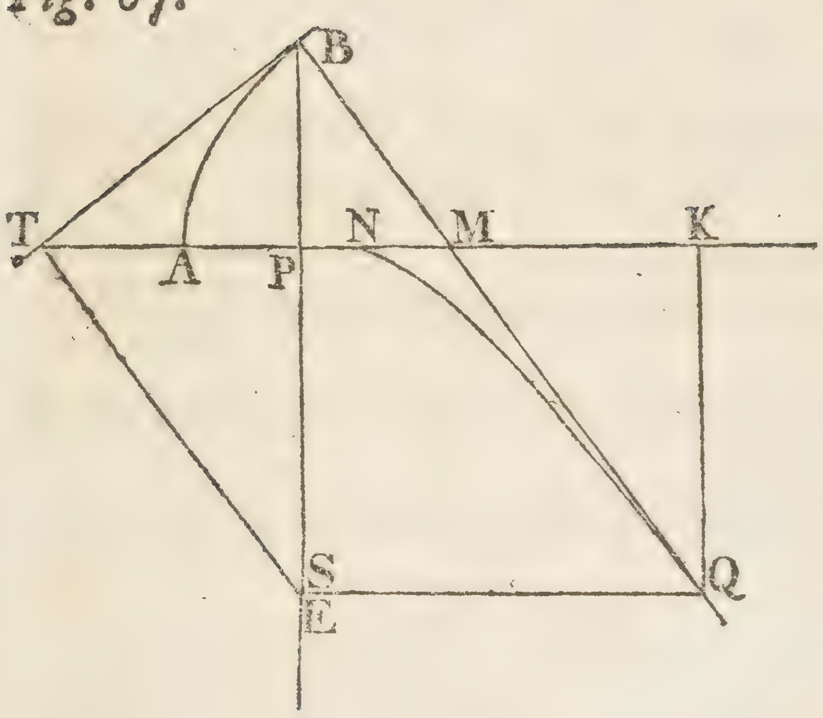
Fig. 86.



portion BF; becaufe, as the radius is always lefs, the nearer it is to the point B, it muft of neceffity become nothing before it paffes from being pofitive to become negative.

EXAMPLE I.

Fig. 87.



122. Let the curve AB be the *Apollonian* parabola of the equation $ax = yy$, of which we would find the radius of curvature at any point B. By taking the fluxions, it will be $a\dot{x} = 2y\dot{y}$; and taking the fluxions again, making, if you pleafe, $\dot{x}$ conftant, it will be $2\dot{y}\dot{y} + 2y\ddot{y} = 0$. But $\dot{y} = \frac{a\dot{x}}{2y}$, therefore $\ddot{y} = -\frac{aa\dot{x}\dot{x}}{4y^3}$. Wherefore, thefe values being fubftituted in the formula for the co-radius $\frac{\dot{x}\dot{x} + \dot{y}\dot{y}}{-\ddot{y}}$, it will be $\frac{4y^3 + aay}{aa} =$ BE; or elfe, by putting, inftead of y, it's value given by the equation of the curve, it will be BE $= \frac{4x\sqrt{ax}}{a} + \sqrt{ax}$.

From the point B let the tangent BT be drawn, which meets the axis in T, and from the point T is drawn TE parallel to the perpendicular BM: this will meet BP produced in the point required, E. For, becaufe of the right angle BTE, it will be BP . PT :: PT . PE; that is, by the property of the parabola $\sqrt{ax} \,.\, 2x :: 2x \,.\,$ PE $= \frac{4xx}{\sqrt{ax}} = \frac{4x\sqrt{ax}}{a}$. Therefore BP + PE = BE

=

$= \frac{4x\sqrt{ax}}{a} + \sqrt{ax}$. Now, BE being determined, draw EQ parallel to the axis AP; the normal BM, produced, will meet EQ in the point Q, which will be a point in the evolute.

Or elſe, becauſe of the ſimilar triangles BPM, BEQ, it will be BP . PM :: BE . EQ. But, by the property of the parabola, it is PM $= \frac{1}{2}a$. Then $\sqrt{ax} \,.\, \frac{1}{2}a :: \frac{4x\sqrt{ax}}{a} + \sqrt{ax} \,.\,$ EQ. Whence EQ $= 2x + \frac{1}{2}a =$ PK, and MK $= 2x$. Wherefore, taking MK double to AP, or PK = TM, and drawing KQ parallel to PB, it will meet the perpendicular BM produced in the point Q, which will be in the evolute. And, becauſe it is BP . BM :: BE . BQ, and BM $= \frac{\sqrt{4ax + aa}}{2}$, it will be $\sqrt{ax} \,.\, \frac{\sqrt{4ax + aa}}{2} :: \frac{4x\sqrt{ax}}{a} + \sqrt{ax} \,.\,$ BQ $= \frac{\overline{4ax + aa}|^{\frac{3}{2}}}{2aa}$, the radius of curvature.

Taking the formula $\frac{\overline{\dot{x}\dot{x} + \dot{y}\dot{y}}|^{\frac{3}{2}}}{-\dot{x}\ddot{y}}$ of the radius of curvature, and making the ſubſtitutions, it will be QB $= \frac{\overline{4yy + aa}|^{\frac{3}{2}}}{2aa} = \frac{\overline{4ax + aa}|^{\frac{3}{2}}}{2aa}$, as at firſt.

Proceeding to the ſecond fluxions of the equation $ax = yy$, without making any conſtant fluxion; becauſe $a\dot{x} = 2y\dot{y}$, it will be $a\ddot{x} = 2y\ddot{y} + 2\dot{y}\dot{y}$, or $\ddot{y} = \frac{a\ddot{x} - 2\dot{y}\dot{y}}{2y}$. Wherefore, taking the formula for the radius of curvature $\frac{\overline{\dot{x}\dot{x} + \dot{y}\dot{y}}|^{\frac{3}{2}}}{\dot{y}\ddot{x} - \dot{x}\ddot{y}}$, which belongs to this caſe, and making the ſubſtitution of the value of $\ddot{y}$, it will be QB $= \frac{2y \times \overline{\dot{x}\dot{x} + \dot{y}\dot{y}}|^{\frac{3}{2}}}{2y\dot{y}\ddot{x} - a\dot{x}\ddot{x} + 2\dot{x}\dot{y}\dot{y}}$; and laſtly, putting the values of y and $\dot{y}$, it is QB $= \frac{\overline{4ax + aa}|^{\frac{3}{2}}}{2aa}$, as above.

The ſame thing will be found in the other ſuppoſitions of $\dot{y}$ or $\dot{s}$ conſtant; which, conſulting brevity, I ſhall here omit.

If we would have the radius of curvature at any determinate point of the curve, it will be ſufficient to ſubſtitute, in the finite expreſſion already found for the radius of curvature for any point, the value of x agreeing to that determinate point. Thus, if we would have the radius of curvature in the vertex A, or in the point N in which the axis AN of the parabola touches the evolute NQ; ſince, at the vertex A, it is $x = 0$, by expunging the term $4ax$ in the expreſſion $\frac{\overline{4ax + aa}|^{\frac{3}{2}}}{2aa}$ of the radius of curvature, we ſhall have AN $= \frac{1}{2}a$;

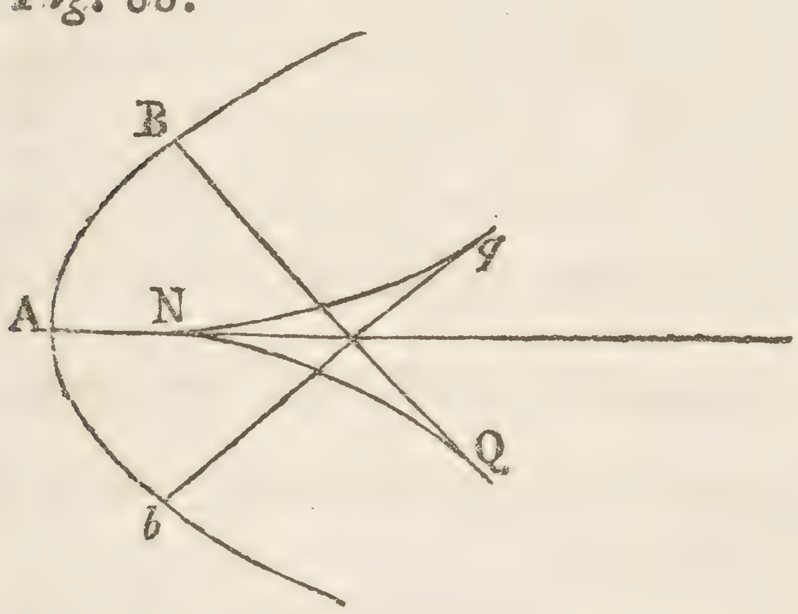
which

which cannot be otherwiſe, the radius AN in this caſe being the ſame as the ſubnormal, which, in the parabola, is known to be equal to half the parameter.

123. Now it will be eaſy to find the equation to the evolute NQ, after the manner of *Des Cartes*, or the relation of the ordinates NK, KQ, in the following manner.

Make $NK = u$, $KQ = t$. Since $KQ = PE = \frac{4x\sqrt{ax}}{a}$, we ſhall have the equation $t = \frac{4x\sqrt{ax}}{a}$. But $AK = AP + PK = 3x + \frac{1}{2}a$, and $AN = \frac{1}{2}a$. Then $NK = 3x = u$, and $x = \frac{1}{3}u$; therefore, putting, inſtead of x, this value in the equation $t = \frac{4x\sqrt{ax}}{a}$, we ſhall have $t = \frac{4u\sqrt{\frac{1}{3}au}}{3a}$, and, by ſquaring, $27att = 16u^3$, which is an equation to the ſecond cubic parabola, with a parameter $= \frac{27a}{16}$; which expreſſes the relation of the co-ordinates NK, KQ, and is the evolute of the propoſed *Apollonian* parabola.

Fig. 88.

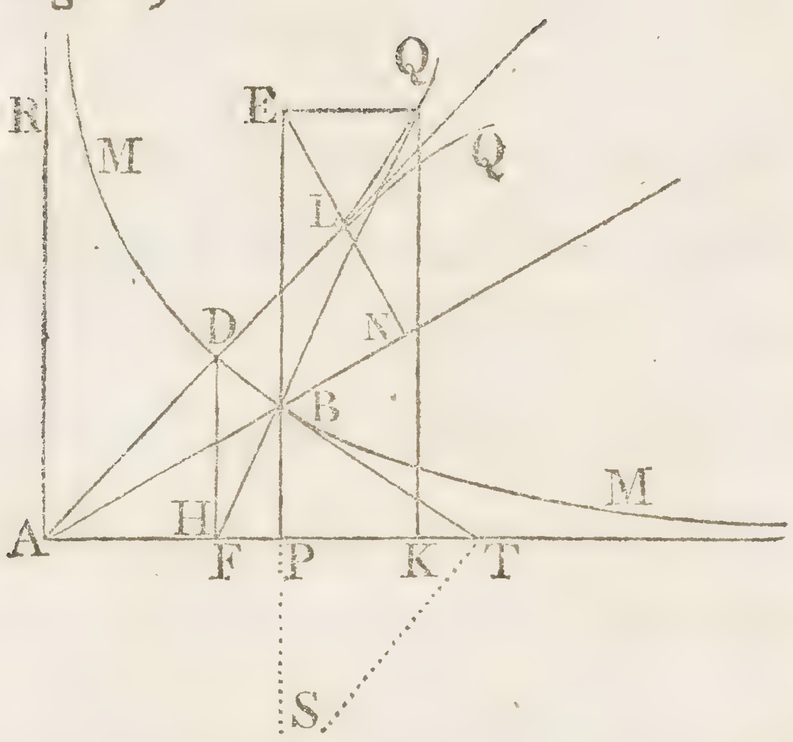


It is evident that the whole ſecond cubical parabola will be the evolute of the whole *Apollonian* parabola; that is, that the branch NQ will be the evolute of the upper part AB, and the branch Nq of the lower part Ab: and that the two branches Nq, NQ change their convexity, and have a regreſſion at N.

124. It is alſo evident, that if the propoſed curves be algebräical, their evolutes alſo will be algebräical curves, and that we may always have an equation in finite terms, expreſſing the relation of the co-ordinates; and that, beſides, thoſe evolutes will be rectifiable, or we may find right lines equal to any portion of the ſame; for example, to QN. For, if the propoſed curve AB be algebräical, we may always have the radii of curvature BQ, AN, in finite terms; and, from BQ ſubtracting AN, the remainder will be the arch NQ.

EXAMPLE II.

Fig. 89.



125. Let the curve MBM be the hyperbola between the asymptotes, whose equation is $aa = xy$. By differencing, it is $x\dot{y} + y\dot{x} = 0$, and by differencing again, and taking $\dot{x}$ as constant, it is $\ddot{y} = -\frac{2\dot{x}\dot{y}}{x}$. Substituting these values of $\dot{y}$ and $\ddot{y}$ in the formula $\frac{\dot{x}\dot{x} + \dot{y}\dot{y}}{-\ddot{y}}$ for the co-radius, we shall have BE $= \frac{xx + yy}{-2y}$, a negative value. If, therefore, it is AP $= x$, PB $= y$, in AB, produced, taking BN $= \frac{1}{2}$BA $= \frac{1}{2}\sqrt{xx + yy}$, and raising the perpendicular NE, which may meet the ordinate BP, produced in E, the co-radius will be BE, as was required. For, because of similar triangles BPA, BNE, it will be BP . BA :: BN . BE, that is, $y \,.\, \sqrt{xx + yy} :: \frac{1}{2}\sqrt{xx + yy} \,.\,$ BE $= \frac{xx + yy}{2y}$; and therefore, on the negative side, it must be BE $= \frac{xx + yy}{-2y}$. Wherefore, drawing EQ parallel to AP, and producing to Q the perpendicular to the curve FB in the point B, the radius of curvature will be BQ, and the point Q will be in the evolute.

To determine the radius of curvature at the vertex of the hyperbola D, make $x =$ AH $= a$, and therefore $y =$ HD $= a$. Then the co-radius $\frac{xx + yy}{-2y}$ at the vertex D will be equal to $-a$, and the radius equal to $-\sqrt{2aa}$.

If we do but consider a little the figure of the curve MBM, we shall find that the evolute will have two branches, with a point of regression at L, in which the radius DL will revert, and will be the least of all the radii BQ. Wherefore, by differencing the formula of the radius of curvature $\frac{(\dot{x}\dot{x} + \dot{y}\dot{y})^{\frac{3}{2}}}{-\dot{x}\ddot{y}}$, the difference or fluxion will be nothing, or infinite; that is, supposing $\dot{x}$ to be constant, it will be $\frac{-3\dot{x}\dot{y}\ddot{y}^2\sqrt{\dot{x}^2 + \dot{y}^2} + \dot{x}\dddot{y} \times (\dot{x}\dot{x} + \dot{y}\dot{y})^{\frac{3}{2}}}{\dot{x}\dot{x}\ddot{y}\ddot{y}} = 0$, or ∞. And, dividing by $\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}$, and multiplying by $\dot{x}\ddot{y}\ddot{y}$, it will be $\dot{x}\dot{x}\dddot{y} + \dot{y}\dot{y}\dddot{y} - 3\dot{y}\ddot{y}\ddot{y} = 0$,

$= 0$, or ∞. But, by the equation of the curve, it is $\dot{y} = -\frac{aa\dot{x}}{xx}$, $\ddot{y} = \frac{2aa\dot{x}\dot{x}}{x^3}$, $\dddot{y} = -\frac{6aa\dot{x}^3}{x^4}$. Therefore, making the fubftitutions, and fuppofing the faid quantity to be equal to nothing, we fhall have $x = a =$ AH. That is to fay, the regreffion will be in the radius of curvature at the vertex D of the curve. But it has been feen, that that radius is equal to $-\sqrt{2aa}$; therefore it will be DL $= -\sqrt{2aa} =$ DA.

In the formula of the radius of curvature, fubftituting the values of $\dot{y}$ and $\ddot{y}$, we fhall have BQ $= \frac{\overline{xx+yy}|^{\frac{3}{2}}}{-2xy} = \frac{\overline{xx+yy}|^{\frac{3}{2}}}{-2aa}$, and therefore, differencing, that we may have the leaft radius, that is, the point of regreffion L, it will be $\overline{3x\dot{x}+3y\dot{y}} \times \sqrt{xx+yy} = 0$; and, inftead of $\dot{y}$, putting it's value, it will be $\overline{3xx\dot{x}-3yy\dot{x}} \times \sqrt{xx+yy} = 0$, that is, $x = y = a$. And fubftituting this value in the expreffion for the radius of curvature, it will be $= -\sqrt{2aa} =$ DL, as found above.

The radius BQ may alfo be conftructed in another manner. For, becaufe $\ddot{y} = -\frac{2\dot{x}\dot{y}}{x}$, inftead of $\dot{x}$ and x, fubftituting their values by y, it will be $\ddot{y} = \frac{2\dot{y}\dot{y}}{y}$, and therefore the co-radius BE $= \frac{y\dot{x}\dot{x}+y\dot{y}\dot{y}}{-2\dot{y}\dot{y}}$. And, becaufe of fimilar triangles BPF, BEQ, we fhall have EQ $= -\frac{y\dot{y}}{2\dot{x}} - \frac{y\dot{x}}{2\dot{y}}$. Now draw the tangent BT to the point B, and from the point T the line TS perpendicular to BT, or parallel to BQ, and make BE $= \frac{1}{2}$BS, or PK $= \frac{1}{2}$FT. Now, if EQ be drawn parallel to AT, or KQ perpendicular to it, they will meet the line BQ in the point of the evolute Q. For it will be BS $= \frac{y\dot{x}\dot{x}+y\dot{y}\dot{y}}{\dot{y}\dot{y}}$, then BE $= \frac{y\dot{x}\dot{x}+y\dot{y}\dot{y}}{-2\dot{y}\dot{y}}$; it will be alfo FP + PT = FT $= -\frac{y\dot{y}}{\dot{x}} - \frac{y\dot{x}}{\dot{y}}$, and therefore EQ $= -\frac{y\dot{y}}{2\dot{x}} - \frac{y\dot{x}}{2\dot{y}}$.

If the equation be $y^m = x$, which expreffes all parabolas *ad infinitum*, when m denotes an affirmative number, and confequently the parabola of the firft example: (and it expreffes all hyperbolas between the afymptotes, when m ftands for a negative number, and therefore that of the prefent example.) By taking the fluxions, we fhall have $m\dot{y}y^{m-1} = \dot{x}$; and taking the fluxions again, fuppofing $\dot{x}$ conftant, it will be $\overline{mm-m} \times \dot{y}\dot{y}y^{m-2} + m\ddot{y}y^{m-1} = 0$.

O 2 Now,

Now, dividing by my^{m-1}, it will be $-\ddot{y} = \overline{m-1} \times \frac{\dot{y}\dot{y}}{y}$. Wherefore, taking the formula for the co-radius $\frac{\dot{x}\dot{x}+\dot{y}\dot{y}}{-\ddot{y}}$, and making the substitution of the value of $\ddot{y}$, we shall have BE $= \frac{y\dot{x}\dot{x}+y\dot{y}\dot{y}}{\overline{m-1}\,\dot{y}\dot{y}}$, and therefore EQ, or PK $=$ $\frac{y\dot{x}}{\overline{m-1}\,\dot{y}} + \frac{y\dot{y}}{\overline{m-1}\,\dot{x}}$.

Fig. 87.

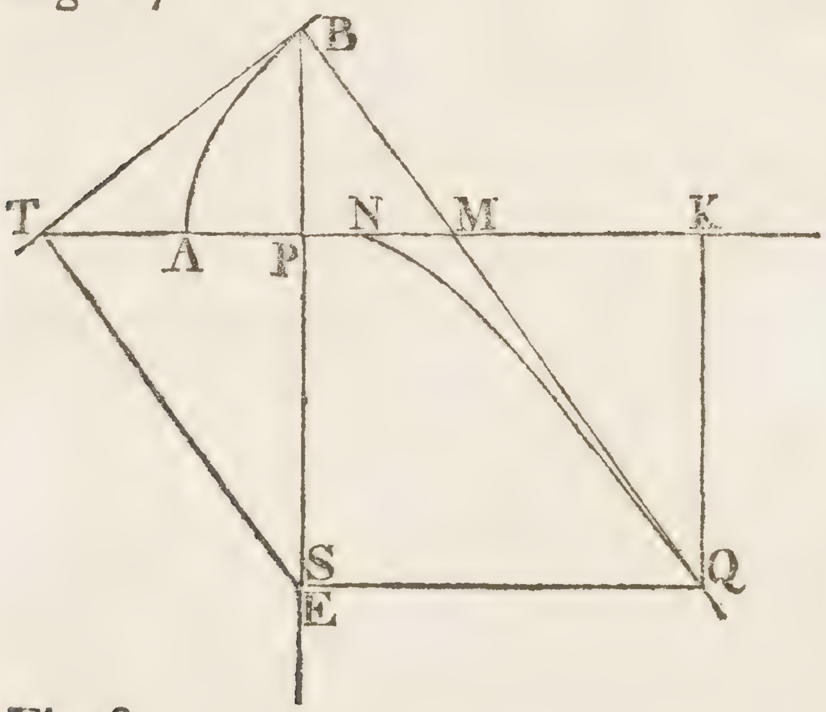
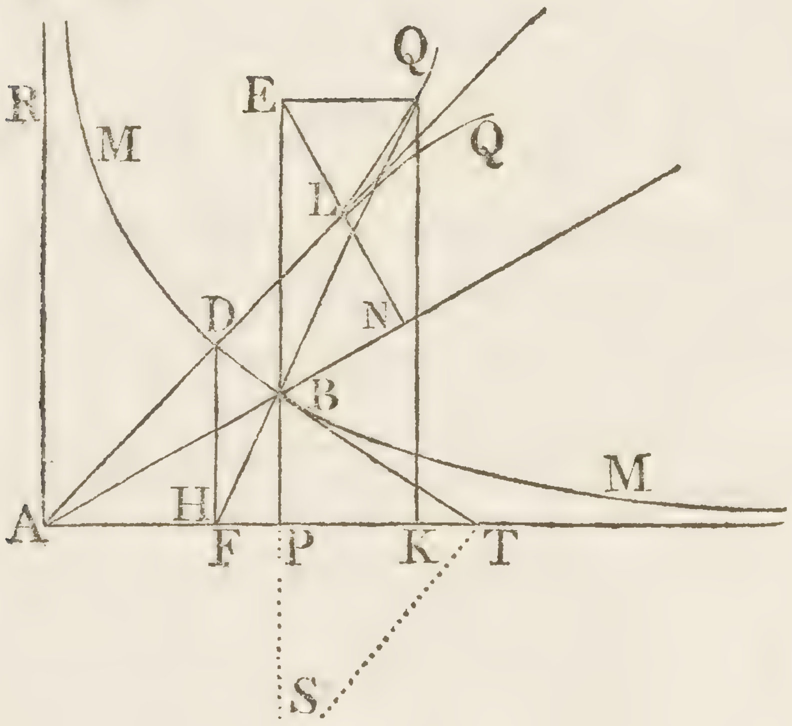


Fig. 89.

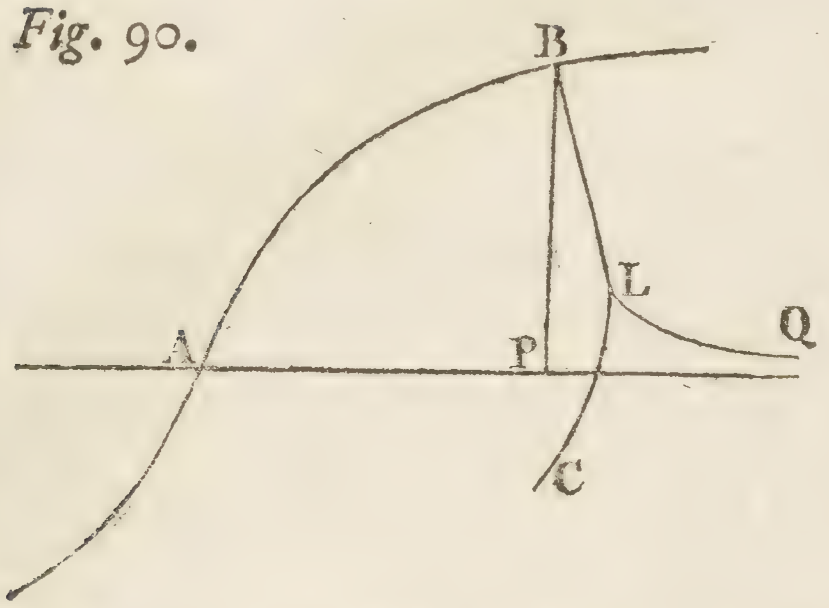


From the point T (Fig. 87, 89.) in which the tangent BT meets the axis AP, is drawn, in like manner, TS parallel to BQ, a perpendicular to the curve, which meets in S the ordinate BP produced. Then take BE $= \frac{BS}{m-1}$, on the negative side, if m be a negative number, as in the hyperbolas which are convex towards the the axis AP, (Fig. 89.) that is, to the asymptote. But BE must be taken on the positive side, if m be a positive number, and greater than unity, as in the parabolas (Fig. 87.) that are concave to the axis AP; and on the negative part, if m, being positive, be less than unity, in which case the parabolas are convex to the axis AP.

To determine the point in which the axis of the parabola touches the evolute, I take the formula of the radius of curvature, which is $\frac{\overline{\dot{x}\dot{x}+\dot{y}\dot{y}}|^{\frac{3}{2}}}{-\dot{x}\ddot{y}}$, from whence, by substituting the values of $\dot{x} = m\dot{y}y^{m-1}$, and of $-\ddot{y} = \frac{\overline{m-1} \times \dot{y}\dot{y}}{y}$, we shall have BQ $= \frac{\overline{mmy^{2m-2}+1}|^{\frac{3}{2}}}{m \times \overline{m-1} \times y^{m-2}}$. It is here understood, that unity may supply any powers required by the law of homogeneity. Whence, supposing m to be greater than unity, for that reason the parabolas will be concave to the axis AP; if m be less than 2, the y in the denominator will become a multiplier in the numerator, and therefore, making $y = 0$, as the present case requires, it will be BQ $= 0$, that is, the axis will be a tangent to the evolute in A, the vertex of

of the parabola, as it would be (for inſtance) in the ſecond cubic parabola $axx = y^3$, Fig. 70.

Fig. 90.



Now, if m be greater than 2, the y of the denominator would be raiſed to a poſitive power, and therefore, making $y = 0$, BQ would be infinite, that is, the axis of the parabola will be an aſymptote to the evolute; as in the firſt cubical parabola AB, (Fig. 90.) whoſe axis AP is an aſymptote to the evolute LQ.

The evolute CLQ of the cubical ſemiparabola ABM of the equation $aax = y^3$, has a point of regreſſion L, and therefore two branches LQ, LC; by evolving the branch LQ, the portion BA will be generated, and by evolving the branch LC, the infinite portion BM will be produced.

To determine the contrary flexure L, take the value of the radius of curvature, which in this curve is $\frac{\overline{9y^4 + a^4}|^{\frac{3}{2}}}{6a^4y}$, which ought to be a *minimum*; and therefore, by taking the fluxion, it will be $\frac{3 \times 18a^4y^4\dot{y} \times \overline{9y^4 + a^4}|^{\frac{1}{2}} - a^4\dot{y} \times \overline{9y^4 + a^4}|^{\frac{3}{2}}}{6a^8yy}$ $= 0$, that is, $45y^4 - a^4 = 0$; whence $y = \sqrt[4]{\frac{a^4}{45}}$. And this value, being ſubſtituted inſtead of y in the equation $aax = y^3$, we ſhall have $x = \sqrt[4]{\frac{a^4}{91125}}$. Taking, therefore, $AP = \sqrt[4]{\frac{a^4}{91125}}$, and drawing the ordinate PB, the point of regreſſion L will be in the perpendicular to the curve at the point B. And, in the expreſſion of the radius of curvature, putting $\sqrt[4]{\frac{a^4}{45}}$ inſtead of y, we ſhall have the value of BL.

After another manner. By differencing the equation $aax = y^3$, or $y = a^{\frac{2}{3}}x^{\frac{1}{3}}$, it will be $\dot{y} = \frac{1}{3}a^{\frac{2}{3}}\dot{x}x^{-\frac{2}{3}}$, $\ddot{y} = -\frac{2}{9}a^{\frac{2}{3}}\dot{x}\dot{x}x^{-\frac{5}{3}}$; $\dddot{y} = \frac{10}{27}a^{\frac{2}{3}}\dot{x}^3x^{-\frac{8}{3}}$, ſuppoſing $\dot{x}$ to be conſtant. Whence, taking the formula $\dot{x}\dot{x}\dddot{y} + \dot{y}\dot{y}\dddot{y} - 3\dot{y}\ddot{y}\ddot{y} = 0$, and ſubſtituting theſe values, we ſhall have $AP = \sqrt[4]{\frac{a^4}{91125}}$, as before.

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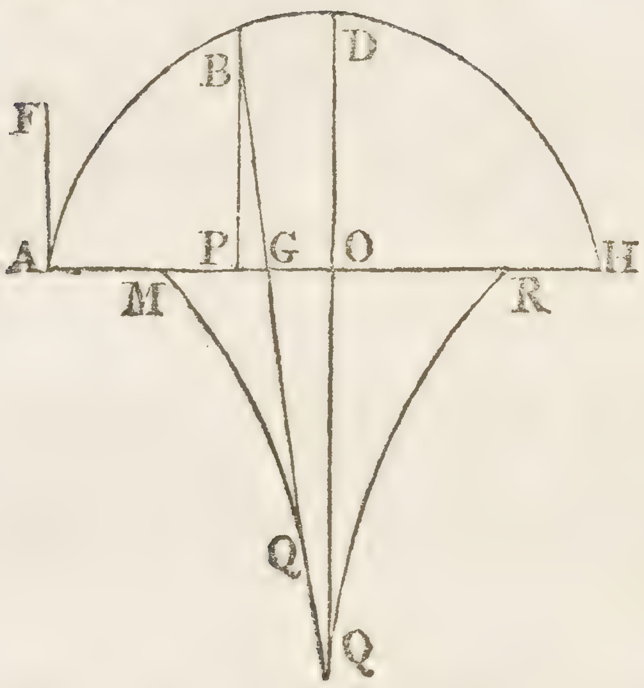
EXAMPLE III.

Fig. 91.



126. Let the curve ABD be an ellipsis or hyperbola, the axis of which is AH $= a$, the parameter AF $= b$, AP $= x$, PB $= y$, and the equation $y = \sqrt{\frac{abx \mp bxx}{a}}$. By differencing, it will be $\dot{y} = \frac{ab\dot{x} \mp 2bx\dot{x}}{2\sqrt{aabx \mp baxx}}$, and $\ddot{y} = \frac{-a^3bb\dot{x}\dot{x}}{4 \times \overline{aabx \mp abxx}|^{\frac{3}{2}}}$, taking $\dot{x}$ for constant. Making the substitutions in the formula $\frac{\overline{\dot{x}\dot{x} + \dot{y}\dot{y}}|^{\frac{3}{2}}}{-\dot{x}\ddot{y}}$ of the radius of curvature, it will be BGQ $= \frac{\overline{4aabx \mp 4abxx + aabb \mp 4abbx + 4bbxx}|^{\frac{3}{2}}}{2a^3bb}$. But the normal will be found to be BG $= \frac{\overline{4aabx \mp 4abxx + aabb \mp 4abbx + 4bbxx}|^{\frac{1}{2}}}{2a}$. Therefore the radius will be BQG $= \frac{4\text{BG cub.}}{bb}$; so that, taking the parameter b for the first term, the normal BG for the second, and continuing the geometrical proportion, the quadruple of the fourth term will be the radius of curvature BQ.

Making $x = 0$ in the expression for the radius of curvature, it will be BGQ $=$ AM $= \frac{1}{2}b$. And making $x =$ AO $= \frac{1}{2}a$, we shall have in the ellipsis BGQ $=$ DOQ $= \frac{a\sqrt{ab}}{2b}$, that is, equal to half the parameter of the conjugate axis; and in Q will be a regression; and the evolute of the portion AD $=$ DH will be MQ,—of the portion DH, will be RQ. But, in the hyperbola, the radius is extended *in infinitum*.

In the ellipsis, if we make $a = b$, the radius of curvature BGQ will be $= \frac{1}{2}a$, wherever the point B be situate. Therefore the radii will all be equal to one another, and the evolute will become a point; that is to say, that the ellipsis, in this case, degenerates into a circle, having the centre for it's evolute.

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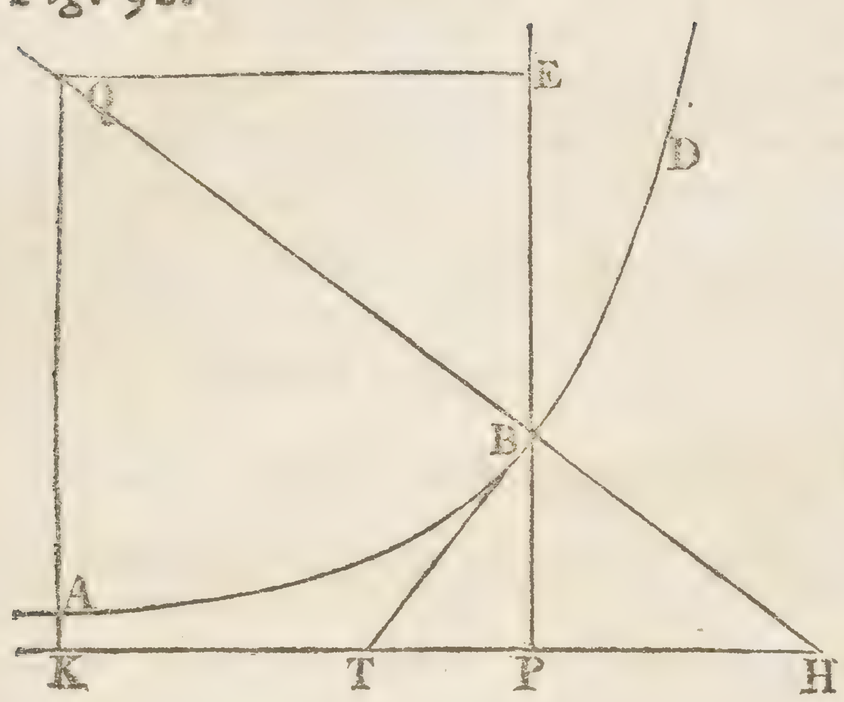
EXAMPLE IV.

Fig. 92.



127. Let the curve ABD be the common logarithmic curve, the equation of which is $\frac{a\dot{y}}{y} = \dot{x}$.

By taking the fluxions, making $\dot{x}$ conſtant, it will be $\ddot{y} = \frac{\dot{x}\dot{y}}{a} = \frac{y\dot{x}\dot{x}}{aa}$, by ſubſtituting the value of $\dot{y}$. Making the uſual ſubſtitutions in the formula $\frac{\dot{x}\dot{x} + \dot{y}\dot{y}}{-\ddot{y}}$ of the co-radius, we ſhall have BE $= \frac{-aa-yy}{y}$; and becauſe, in the logarithmic, it is found that the ſubnormal PH $= \frac{yy}{a}$, it will be EQ $= -a - \frac{yy}{a}$. Therefore, taking PK $=$ TH, and raiſing KQ at right angles, it will meet the normal HBQ in Q, the point of the evolute required.

If we would determine the point of greateſt curvature in the logarithmic, that is, the point where there is the leaſt radius of curvature; making the ſubſtitutions in the formula $\frac{\overline{\dot{x}\dot{x}+\dot{y}\dot{y}}^{\frac{3}{2}}}{-\dot{x}\ddot{y}}$ of the radius of curvature, it will be $\frac{\overline{aa+yy}^{\frac{3}{2}}}{-ay}$; and taking the fluxions, it will be $\frac{-3ayy\dot{y} \times \overline{aa+yy}^{\frac{1}{2}} + a\dot{y} \times \overline{aa+yy}^{\frac{3}{2}}}{aayy} = 0$, and therefore PB $= y = \sqrt{\tfrac{1}{2}aa}$.

Or, taking the formula of § 125, $\dot{x}\dot{x}\dddot{y} + \dot{y}\dot{y}\dddot{y} - 3\dot{y}\ddot{y}\ddot{y} = 0$, and making the ſubſtitutions of $\dot{y} = \frac{y\dot{x}}{a}$, $\ddot{y} = \frac{y\dot{x}\dot{x}}{aa}$, and $\dddot{y} = \frac{y\dot{x}^3}{a^3}$, we ſhall come to the ſame concluſion of PB $= y = \sqrt{\tfrac{1}{2}aa}$.

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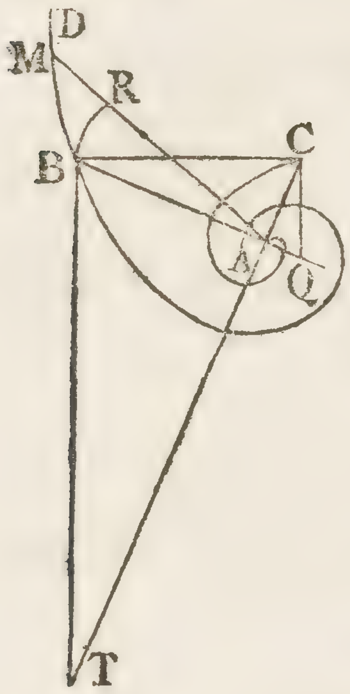
EXAMPLE V.

Fig. 93.



128. Let ABD be the logarithmic ſpiral, the property of which is, that, at any point B, drawing the tangent BT, and from the pole A the ordinate AB, the angle ABT may always be the ſame : therefore, making AM to be infinitely near AB, the ratio of MR to RB will be conſtant. Wherefore, putting $AB = y$, the little arch $BR = \dot{x}$, the equation will be $a\dot{x} = b\dot{y}$; and, by taking the fluxions, and making $\dot{x}$ conſtant, it will be $\ddot{y} = 0$. Therefore, taking the formula of the co-radius, § 118, $\frac{y\dot{x}^3 + y\dot{x}\dot{y}\dot{y}}{\dot{x}^3 + \dot{x}\dot{y}\dot{y} + y\dot{y}\ddot{x} - y\dot{x}\ddot{y}}$, for curves that are referred to a *focus*, which, being managed on the ſuppoſition of $\dot{x}$ being conſtant, will be $\frac{y\dot{x}\dot{x} + y\dot{y}\dot{y}}{\dot{x}\dot{x} + \dot{y}\dot{y} - y\ddot{y}}$. And in this, expunging the term $y\ddot{y}$, becauſe the curve gives us here $\ddot{y} = 0$, and making the ſubſtitution of the value of $\dot{x}$ or $\dot{y}$, or, dividing the numerator and denominator by $\dot{x}\dot{x} + \dot{y}\dot{y}$, the co-radius will be $BA = y$.

Therefore, drawing AC perpendicular to AB, it will meet the perpendicular BC in C, the point of the evolute required ; and, becauſe the ſubnormal $AC = \frac{ay}{b}$, it will be $BC = \frac{y\sqrt{aa + bb}}{b}$.

Drawing BT, a tangent to the curve in the point B, the triangles TCB, CBA, will be ſimilar, and therefore the angles TBA, ACB, will be equal. But the angle TBA is a conſtant angle, ſo that the angle ACB will be ſo too. Therefore the evolute AC will be the ſame logarithmic ſpiral ABD, but in an inverted ſituation.

EXAMPLE VI.

129. Let ABD (Fig. 93.) be the hyperbolical ſpiral, the property of which is, that the ſubtangent is a conſtant line.

Do the ſame things as in the foregoing example, and the equation of the curve will be $\frac{y\dot{x}}{\dot{y}} = a$, or $y\dot{x} = a\dot{y}$. Then, by differencing, making $\dot{x}$ conſtant,

ſtant, $\ddot{y} = \frac{\dot{x}\dot{y}}{a}$. Wherefore, taking the formula of the co-radius, correſponding to the hypotheſis of $\dot{x}$ conſtant, that is, $\frac{y\dot{x}\dot{x} + y\dot{y}\dot{y}}{\dot{x}\dot{x} + \dot{y}\dot{y} - y\ddot{y}}$, and, inſtead of $\ddot{y}$, ſubſtituting it's value $\frac{\dot{x}\dot{y}}{a}$, and, inſtead of $\dot{y}$, it's value $\frac{y\dot{x}}{a}$ given by the equation, the co-radius will be $= \frac{y \times \overline{aa + yy}}{aa}$.

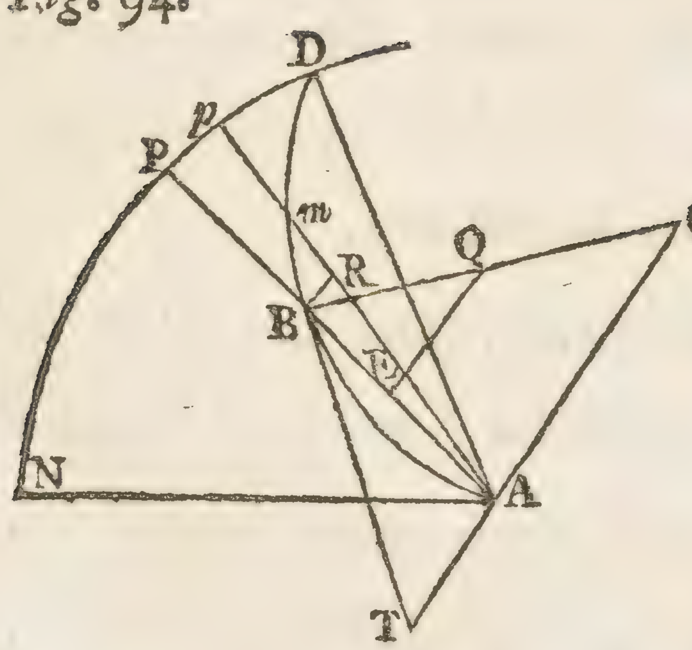
But, becauſe the ſubtangent AT $= a$, and the ſubnormal AC $= \frac{yy}{a}$, it will be TC $= \frac{aa + yy}{a}$. Therefore the fourth proportional to the ſubtangent TA, and TC, and the ordinate AB, here determines the co-radius. Whence, from the point C drawing CQ parallel to the tangent BT, which cuts in Q the ordinate BA produced, BQ will be the co-radius required.

For the triangles BAT, CAQ, are ſimilar; ſo that we ſhall have CA . AQ :: TA . AB; and, by permutation, CA . TA :: AQ . AB. And, by compounding, TC . AT :: QB . AB; and, by inverſion, TA . TC :: BA . BQ.

Q. E. I.

EXAMPLE VII.

Fig. 94.

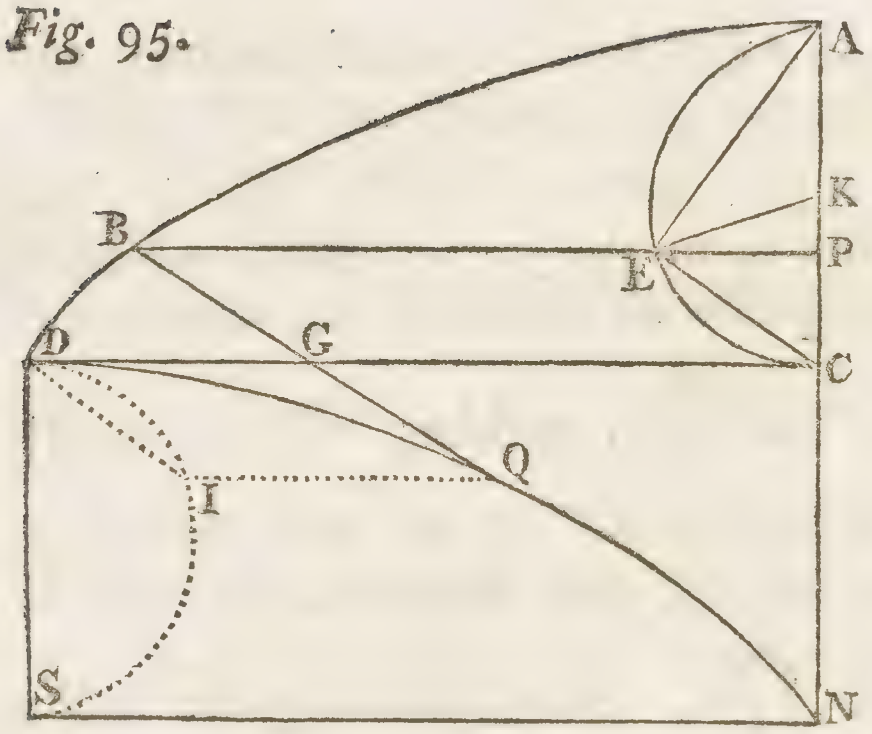


130. Let ADN be a ſector of a circle, and from the centre A drawing any radius ABP, let it be ND . NP :: $\overline{AP}|^m . \overline{AB}|^m$. The point B ſhall be in the curve ABD, which is one of the ſpirals *ad infinitum*, the equation of which is $y^m = \frac{a^m z}{b}$, making NPD $= b$, NP $= z$, the radius AP $= a$, and AB $= y$. Then, by taking the fluxions, it will be $m\dot{y}y^{m-1} = \frac{a^m \dot{z}}{b}$. Now, drawing the radius Ap infinitely near to AP, and making BR $= \dot{x}$; becauſe of ſimilar ſectors APp, ABR, it will be $\dot{z} = \frac{a\dot{x}}{y}$. Wherefore, putting the value, inſtead of $\dot{z}$, in

in the fluxional equation, it will be $m\dot{y}y^m = \frac{a^{m+1}\dot{x}}{b}$; and therefore, taking the fluxions again, making $\dot{x}$ conftant, we fhall have $mm\dot{y}\dot{y}y^{m-1} + my^m\ddot{y} = 0$, that is, $y\ddot{y} = -m\dot{y}\dot{y}$. Wherefore, making a fubftitution of this value, and of the value of $\dot{x}$, in the formula of the co-radius, it will be BE $= \frac{y \times \overline{mmbby^{2m} + a^{2m+2}}}{mmbby^{2m} + \overline{m+1} \times a^{2m+2}}$. Make TAC perpendicular to AB, and draw BT a tangent to the curve in B, and BC perpendicular to it; it will be AT $= \frac{mby^{m+1}}{a^{m+1}}$, AC $= \frac{a^{m+1}}{mby^{m-1}}$, and therefore TC $= \frac{mmbby^{2m} + a^{2m+2}}{mba^{m+1}y^{m-1}}$. Whence the fourth proportional to TA $+ \overline{m+1} \times$ AC, to TC, and to AB, will be $\frac{y \times \overline{mmbby^{2m} + a^{2m+2}}}{mmbby^{2m} + \overline{m+1} \times a^{2m+2}} =$ BE. And therefore, drawing EQ parallel to TC, it will meet the perpendicular BC in the point Q, which will be a point in the evolute.

EXAMPLE VIII.

Fig. 95.



131. Let the curve ABD be half of the common cycloid, the equation of which is $\dot{y} = \dot{x}\sqrt{\frac{2a - x}{x}}$; making AC $= 2a$, AP $= x$, PB $= y$.

By differencing, and taking $\dot{x}$ for conftant, it will be $\ddot{y} = \frac{-a\dot{x}\dot{x}}{x\sqrt{2ax - xx}}$; and fubftituting thefe values in the formula for the radius of curvature $\frac{\overline{\dot{x}\dot{x} + \dot{y}\dot{y}}|^{\frac{3}{2}}}{-\dot{x}\ddot{y}}$, it will be BQ $= 2\sqrt{4aa - 2ax}$. But the normal BG $= \sqrt{4aa - 2aa}$, which is equal to the chord EC. Therefore the radius of curvature BQ $=$ 2BG $=$ 2EC.

Making

3

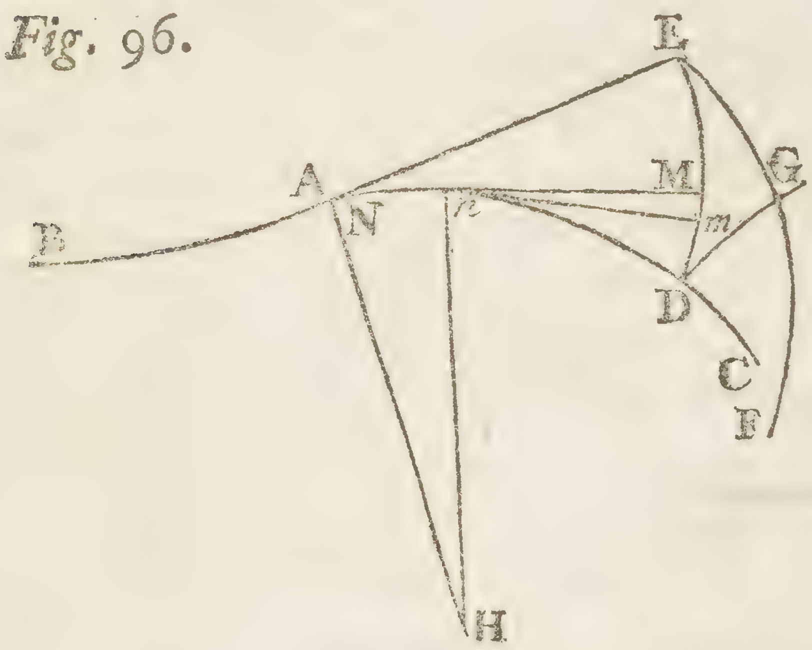
Making $x = 0$, to have the radius of curvature in the point A, it will be $BQ = AN = 4a$, and therefore $CN = CA = 2a$.

Making $x = 2a$, the radius of curvature in the point D will be $= 0$, and therefore the evolute begins in D, and terminates in N.

Becaufe the tangent of the cycloid in B is parallel to the chord EA, (§ 47.) the normal BQ will be parallel to the chord EC. This fuppofed, complete the rectangle DCNS, and with the diameter $DS = CN = AC$ defcribe the femicircle DIS, and draw the chord DI parallel to BQ, or to EC. The angles CDI, DCE, will be equal, and confequently the arches DI, CE, and their chords. Therefore DI, GQ, are equal and parallel; and drawing IQ, it will be equal and parallel to DG. But, by the property of the cycloid, DG is equal to the arch EC, and therefore to the arch DI. Then the arch $DI = IQ$, and the femicircle $DIS = SN$. Whence the evolute DQN is the fame cycloid, DBA, in an inverted fituation.

132. The radius of curvature and it's formula being now fufficiently explained, it will not be difficult to find the formula for the regreffions of the fecond fpecies, mentioned before at § 98.

Fig. 96.



Let the curve be BAC, with a contrary flexure at A, and let it be evolved by the thread beginning at any point D, different from the point of contrary flexure A. The evolution of the portion DC generates the curve DG, and that of the portion AB generates the curve EF; in fuch manner, that the evolution of the whole curve BAC will form the entire curve FEDG, which has two regreffions; one at D of the ufual form, becaufe the two branches DE, DG, turn their convexity; the other at E of the fecond fort, becaufe the two branches ED, EF, are concave towards the fame parts. Let NM, Nnm, be any two rays infinitely near, of the evolute DA, and NH, nH, two perpendiculars to the fame; the two infinitefimal fectors NmM, HNn, will be fimilar, and therefore $HN . NM :: Nn . Mm$. But, in the point of contrary flexure A, the radius HN (§ 121.) ought to be either infinite, or equal to nothing, and the radius NM, which becomes AE, continues finite. Therefore, in the cafe of contrary flexure A, that is, in the point of regreffion E, of the fecond fort, the ratio of Nn, Mm, that is, the ratio of the differential of the radius MN to the element of the curve, ought to be either infinitely great or infinitely little. But the formula of the radius

MN is $\frac{\overline{\dot{x}\dot{x} + \dot{y}\dot{y}}|^{\frac{3}{2}}}{-\dot{x}\ddot{y}}$, taking $\dot{x}$ for conſtant; the differential of which is $\frac{-3\dot{x}\dot{y}\ddot{y}\ddot{y}\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}} + \dot{x}\dddot{y} \times \overline{\dot{x}\dot{x} + \dot{y}\dot{y}}|^{\frac{3}{2}}}{\dot{x}\dot{x}\ddot{y}\ddot{y}}$, and $Mm = \sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}$. Therefore $\frac{Nn}{Mm} = \frac{\dot{x}\dot{x}\dddot{y} + \dot{y}\dot{y}\dddot{y} - 3\dot{y}\ddot{y}\ddot{y}}{\dot{x}\ddot{y}\ddot{y}} = 0$, or ∞, the formula for the points of regreſſion of the ſecond ſort.

This formula is the ſame as that already found, § 125; but in that place it ſerved for the regreſſions of the firſt ſort of evolutes, and here for the regreſſion of the ſecond ſort of curves, derived from evolutes; x and y, in both caſes, being the co-ordinates of the curves ſo produced.

END OF THE SECOND BOOK.

ANA-

ANALYTICAL INSTITUTIONS.

BOOK III.

OF THE INTEGRAL CALCULUS.

THE *Integral Calculus*, which is alſo uſed to be called the *Summatory* Introduction. *Calculus*, is the method of reducing a differential or fluxional quantity, to that quantity of which it is the difference or fluxion. Whence the operations of the Integral Calculus are juſt the contrary to thoſe of the Differential; and therefore it is alſo called *The Inverſe Method of Fluxions*, or of *Differences*. Thus, for example, the fluxion or differential of y is $\dot{y}$, and conſequently the *fluent* or *integral* of $\dot{y}$ is y. Hence it will be a ſure proof that any integral is juſt and true, if, being differenced again, it ſhall reſtore the given fluxion, or the quantity whoſe integral was to be found. Differential formulæ have two different manners, by which their integrals are inveſtigated. One is, by the help of finite Algebräical expreſſions, or by being reduced to quadratures which are granted or ſuppoſed. In the other, we are allowed the uſe of infinite ſeries. In this firſt Section, I ſhall deliver the rules required in the firſt manner. In the ſecond Section, I ſhall treat of the ſecond manner; to which I ſhall add a third Section, to ſhow the uſe of theſe Rules in the Rectification of Curve-lines, the Quadrature of Curve-ſpaces, &c. And laſtly, I ſhall add a fourth, which ſhall teach the Rules of the *Exponential Calculus*.

SECT.

SECT. I.

The Rules of Integrations expressed by finite Algebräical Formulæ, or which are reduced to supposed Quadratures.

1. As in simple quantities raised to any power, their differential or fluxion is the product of the exponent of the variable into the variable itself, raised to the same power lessened by unity, and multiplied by it's fluxion or difference; so the fluent or integral of the product of a variable raised to any power, into the difference of the same variable, is the variable raised to a power the exponent of which is increased by unity, divided by the same exponent so increased. And this obtains, whatever the exponent shall be of the power of the variable, whether positive or negative, integer or fraction. Thus, for example, the fluent of $mx^{m-1}\dot{x}$ will be $\frac{m x^{m-1+1}}{m-1+1}$, or x^m. The integral of $x^{\pm\frac{m}{n}}\dot{x}$ will be $\frac{x^{\pm\frac{m}{n}+1}}{\pm\frac{m}{n}+1}$, that is, $\frac{nx^{\frac{\pm m+n}{n}}}{\overline{\pm m+n}}$; and so of others.

2. Any constant quantities, simple or complicate, by which the fluxions may be multiplied or divided, will make no alteration in the rule; for they remain in the fluents just as they were in the fluxions. Therefore the fluent of $\frac{aax^n\dot{x}}{mb-cc}$ will be $\frac{aax^{n+1}}{\overline{n+1}\times\overline{mb-cc}}$.

3. Thus, if the differential formula were a fraction, the denominator of which were also any power of the variable, multiplied (if you please) by any constant quantity; as the formula $\frac{x^m\dot{x}}{aax^n - bbx^n}$, or $\frac{x^m\dot{x}}{\overline{aa-bb}\times x^n}$, which will be the same as $\frac{\dot{x}x^{m-n}}{aa-bb}$, and therefore subject to the general rule.

4. But

4. But here we are to obferve, that, in order to have the integrals complete, we ought always to add to them, or to fubtract from them, fome conftant quantity at pleafure, which, in particular cafes, is afterwards to be determined as occafion may require. Of this we fhall take further notice in it's due place.

Thus, the complete integral of $\dot{x}$, for example, will be $x \pm a$, where a fignifies fome conftant quantity. That of $x^2\dot{x}$ will be $\frac{1}{3}x^3 \pm a^3$; and fo of others. The reafon of which is, that, as conftant quantities have no differentials, but $\dot{x}$ may as well be the differential of $x + a$, or of $x - b$, &c. as of x; fo x, or $x + a$, or $x - b$, &c. may be the integral of $\dot{x}$. The fame obtains in any other formula.

5. The fame rule of integration ferves for complicate differential formulæ, or thofe compounded of many terms; whether they have a denominator, whether that be wholly conftant, or contains the variable in it, whether it be fimple and of one term, or whether it be reducible to fuch.

For, in thefe cafes, the complicate differential formula may be refolved into as many fimple ones, as are the terms of the complicate, and then each of thefe comes under the given rule. Let the formula be $\frac{bx^m\dot{x} + aax^{m-1}\dot{x}}{aa - bb}$; this will be equivalent to thefe two, $\frac{bx^m\dot{x}}{aa - bb}$ and $\frac{aax^{m-1}\dot{x}}{aa - bb}$, and therefore the integral of thefe two formulæ will be the integral of the firft; that is, $\frac{bx^{m+1}}{\overline{m+1} \times \overline{aa - bb}}$ $+ \frac{aax^m}{m \times \overline{aa - bb}} \pm f$.

Let it be $\frac{bx^3\dot{x} - a^4\dot{x}}{axx - cxx}$; this is the fame as thefe two, $\frac{bx^3\dot{x}}{\overline{a - c} \times x^2} - \frac{a^4\dot{x}}{\overline{a - c} \times x^2}$, or as thefe, $\frac{bx\dot{x}}{a - c} - \frac{a^4x^{-2}\dot{x}}{a - c}$, and therefore the integral will be $\frac{bx^2}{2 \times \overline{a - c}} -$ $\frac{a^4x^{-1}}{-1 \times \overline{a - c}} \pm f$, that is, $\frac{bxx}{2a - 2c} + \frac{a^4}{\overline{a - c} \times x} \pm f$.

Let it be $\frac{bx^m\dot{x} - aax^{m-1}\dot{x}}{xx}$; this is equivalent to thefe two, $bx^{m-2}\dot{x} - aax^{m-3}\dot{x}$, and therefore the integral will be $\frac{bx^{m-1}}{m - 1} - \frac{aax^{m-2}}{m - 2} \pm f$.

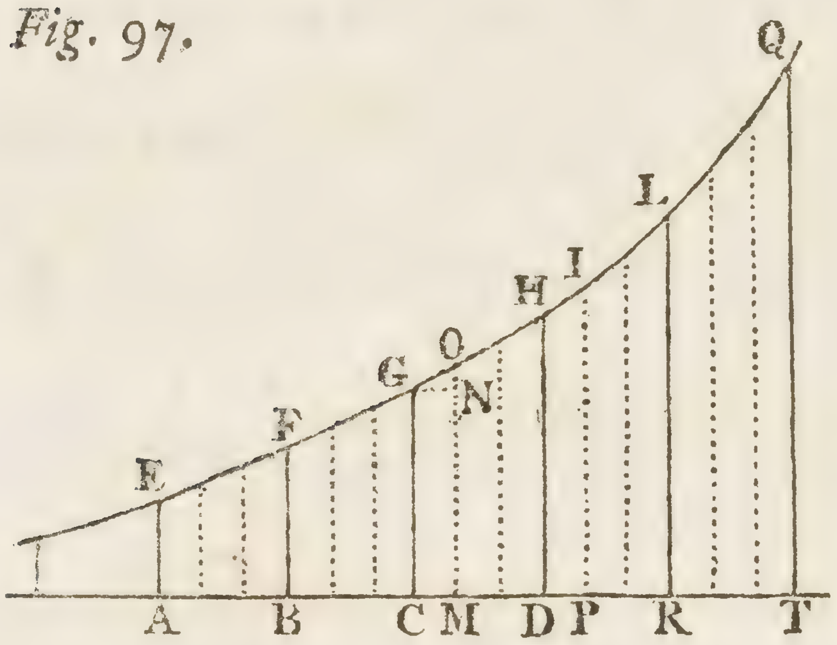
6. Befides,

6. Befides, if the complicate differential formula be raifed to any power, the exponent of which is a pofitive integer, it being actually reduced to the given power, every term may be integrated by the fame rule.

7. All that I have hitherto faid will obtain, when in the differential formula there is no term in which the exponent of the variable is negative unity, fuch as $\frac{a\dot{x}}{x}$, or $ax^{-1}\dot{x}$; for, according to the rule, the integral would be $\frac{ax^{-1+1}}{-1+1}$, or $\frac{ax^{0}}{0}$, that is, infinite; and which therefore teaches us nothing.

8. In thefe cafes, therefore, we are obliged to have recourfe to other methods. There are two of thefe which will affift us. One is, by means of a curve which is called the *Logarithmic Curve*, or the *Logiftic*. The other is, by means of infinite feries. As to infinite feries, of which we fhall make very great ufe in many other cafes alfo, I fhall treat of them hereafter, as may be feen in the next Section.

Fig. 97.



9. Now, as to the logarithmic curve, it is a curve of fuch a property, that, in the axis, taking the abfciffes in arithmetical progreffion, the correfponding ordinates will be in geometrical progreffion. Therefore let the axis AD be divided into equal parts, AB, BC, CD, DE, &c. At the points A, B, C, D, &c. erect the perpendiculars AE, BF, CG, DH, &c. fuch, that they may be to each other in geometrical proportion. The points E, F, G, H, &c. will be in the curve. And again dividing the lines AB, BC, &c. into equal parts, and at the divifions raifing perpendiculars in the fame geometrical proportion, we fhall have other intermediate points. And laftly, multiplying the divifions *in infinitum*, we fhall have infinite points, or the very curve itfelf.

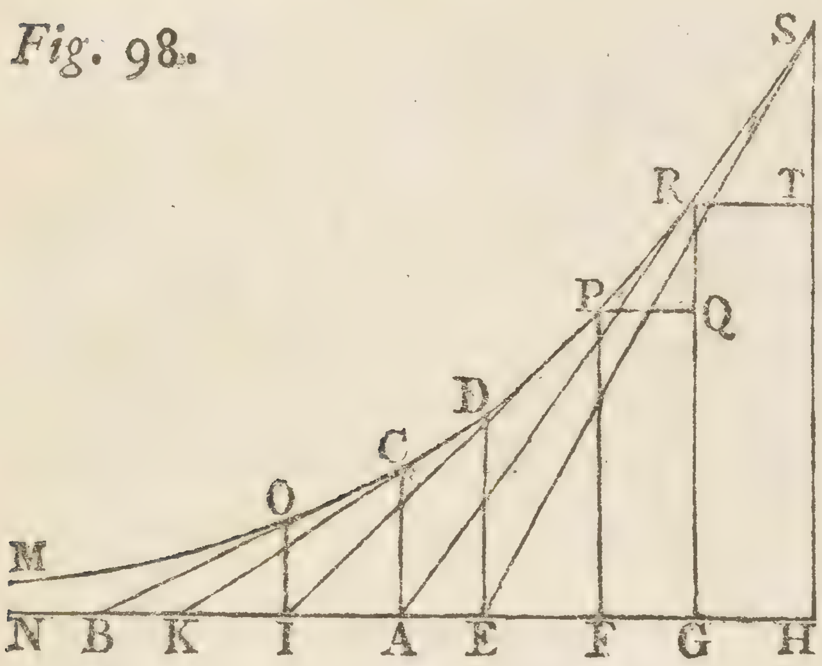
Therefore, the axis being divided into infinitefimal equal parts, let one of thefe be CM $= \dot{x}$, the ordinate CG $= y$, and MO infinitely near it; therefore it will be NO $= \dot{y}$. Let there be another ordinate DH $= z$, and others as many as you pleafe, correfponding to the abfciffes that are arithmetically proportionals. Therefore thefe ordinates will have the fame proportion to each other, and, by confequence, their differentials alfo will be in the fame proportion. So that it will be $\dot{y} \,.\, \dot{z} :: y \,.\, z$; or $\dot{y} \,.\, y :: \dot{z} \,.\, z$; whence the ratio of $\dot{y}$ to y will be a conftant ratio. And therefore, affuming $\dot{x}$ conftant, it will be $\dot{y} \,.\, y :: \dot{x} \,.\, a$, or $\frac{a\dot{y}}{y} = \dot{x}$; which is the equation to the curve.

9 Here

Here it will be eaſy to perceive that the ſubtangent of this curve will always be conſtant; for, in the general formula of the ſubtangent $\frac{y\dot{x}}{\dot{y}}$, inſtead of y, ſubſtituting it's value given from the equation of the curve, we ſhall have $\frac{y\dot{x}}{\dot{y}} = \frac{ay\dot{x}}{x\dot{y}} = a$. Now, as the increaſing geometrical progreſſion of the ordinates may be continued *in infinitum*, the abſciſſes alſo increaſing arithmetically *in infinitum*; therefore the curve will go on infinitely, always receding further from the axis. And as the ſame progreſſion, decreaſing, may be alſo continued *in infinitum*, the axis ſtill increaſing the contrary way, the other part of the curve will go on infinitely, but always approaching towards the axis without ever touching it, and therefore that axis will be an aſymptote to the curve.

9. Among many other ways, the logarithmic curve may be conceived to be deſcribed in this manner alſo.

Fig. 98.



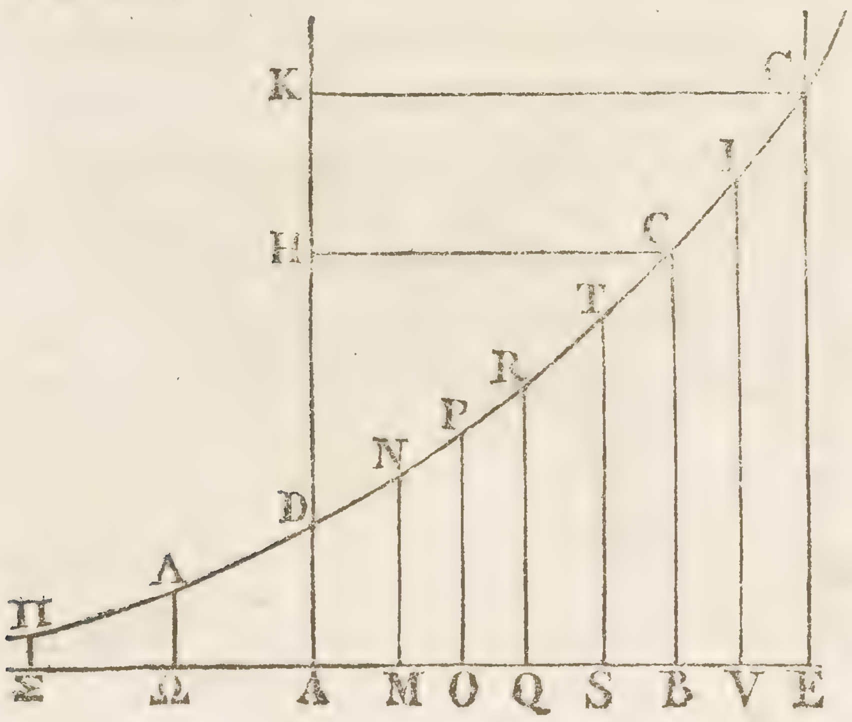
Let the indefinite right line MH be divided into equal parts MN, NB, BK, &c.; and taking NI at pleaſure, at the point I let the perpendicular IO be erected of any magnitude; then draw NO, and at the point A let the perpendicular AC be erected till it meets NO produced to C. From the point B draw BC, and at the point E let the perpendicular ED be erected, which meets BC produced in D. From the point K draw KD, and at the point F let the perpendicular FP be raiſed, which meets KD produced in the point P. After the ſame manner, let the operation be continued *in infinitum*, and the points O, C, D, P, &c. will be in the logarithmic curve. To have the intermediate points between O, C, D, P, &c. let the portions MN, NB, &c. be biſected, and the ſame operation being repeated, we ſhall have other points. And finally, by multiplying the equal diviſions infinitely in the right line MH, that is, by ſuppoſing the equal portions MN, NB, &c. to become infiniteſimals, we ſhall have an infinite number of points which will mark out the logarithmic curve, the ſubtangent of which ſhall be always a conſtant line, as appears from the conſtruction. Making, therefore, NI $= a$, and ſuppoſing the infiniteſimal conſtant portion of the axis to be $\dot{x}$; make the ordinate GR $= y$, GH $= \dot{x}$, TS $= \dot{y}$; by the ſimilar triangles STR, RGA, it will be $\dot{y} \,.\, \dot{x} :: y \,.\, a$; that is, $\frac{a\dot{y}}{y} = \dot{x}$, the equation of the curve.

From this construction we deduce also this, which the first supposes; that is, the primary property of the logarithmic curve, that the ordinates are in geometrical proportion, which correspond to the abscisses in arithmetical proportion. For, supposing the equal portions of the axis to be infinitesimals, the little arch OC, produced, will be the tangent NO, the little arch CD, produced, will be the tangent BC, the little arch BD, produced, the tangent KD; and so of all the others. Therefore the triangles OIN, CAN, will be similar, and therefore it will be OI . CA :: NI . NA. Thus, also, by the similitude of the triangles CAB, DEB, it will be CA . DE :: BA . BE. But NI = BA, NA = BE; therefore it will be OI . CA :: CA . DE; and so successively. Therefore the ordinates will be in continual geometrical proportion. Hence, also, if we conceive the axis to be divided, not into infinitely little parts, but into finite and equal parts; because the intermediate proportional ordinates, for example, between IO and CA, are neither more nor fewer in number than the intermediate between CA and DE, and thus of others; therefore IO, CA, DE, will be in geometrical proportion, corresponding to the abscisses in arithmetical proportion. Therefore, taking any two ordinates at pleasure, and other two also where you please, provided the distance between the first and second be the same as the distance between the third and fourth, as would be IO, CA, RG, SH; then the first will be to the second, as the third to the fourth.

The logarithmic curve cannot be described geometrically, but only organically, and therefore it is called a mechanical curve; and the impossibility of being geometrically described is the same as the impossibility of the quadrature of the hyperbolical space, as will be seen in it's place. Wherefore the integrals of such differential formulæ as belong to the logarithmic curve, are also said to depend on the quadrature of the hyperbola.

Hence, in the logarithmic curve, the portions of the axis, or the abscisses taken from some fixed point, correspond to the ordinates just in the same manner as, in the trigonometrical tables, the logarithms correspond to the natural series or progression of numbers.

Fig. 99.



10. This supposed, let DC be the logarithmic curve, the subtangent of which is equal to unity, or, if you please, is equal to the constant line a; and let the ordinate AD be equal to the subtangent, that is, equal to unity, or to the constant line a, which is in the place of unity. Taking any abscis AB $= x$, make BC $= y$. But the equation of the curve is $\frac{a\dot{y}}{y} = \dot{x}$, and therefore the integral

integral or fluent of $\frac{a\dot{y}}{y}$ will be x. But $x = AB$, and AB is the logarithm of BC, or of y. Now, to make uſe of the mark $\int$ to ſignify the integral, ſum, or fluent, all which mean the ſame thing; and of the mark l, which means the logarithm, it will be $\int \frac{a\dot{y}}{y} = ly$, in the logarithmic curve, the ſubtangent of which is a. After the ſame manner, it will be $\int \frac{\dot{y}}{y} = ly$, in the logarithmic whoſe ſubtangent $= 1$; $\int \frac{b\dot{y}}{y} = ly$, in the logarithmic whoſe ſubtangent is b; $\int \frac{a\dot{y}}{b+y} = l\,\overline{b+y}$, in the logarithmic whoſe ſubtangent is equal to a. That is, taking, in the logarithmic, the ordinate $BC = AH = y$, if to it we ſhall add $HK = b$, and if we draw KG parallel to the aſymptote, and draw GE parallel to AD, it will be $GE = y + b$, and then $AE = l\,\overline{b+y}$.

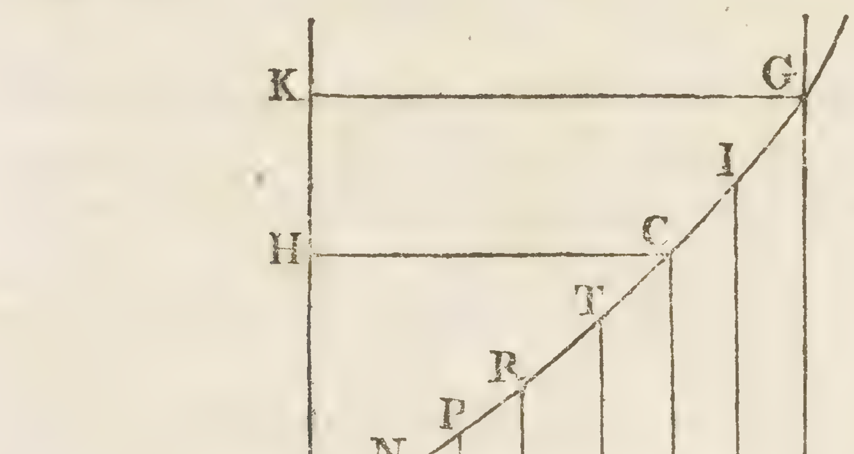
11. From the nature of the logarithmic it is plainly ſeen, that whenever the quantity is infinite, of which we would have the logarithm; which quantity will be repreſented by an infinite ordinate in the logarithmic; then the line intercepted in the axis, between that ordinate and the point A, will alſo be infinite, that is, the logarithm will be infinite. And if it ſhall be equal to the firſt ordinate AD, that is, to the ſubtangent, the logarithm will then be equal to nothing. And if it ſhall be leſs than AD, as if it were ΩA, the logarithm will be ΩA, and therefore negative. And laſtly, if the ordinate were $= 0$, the logarithm would be negative and infinite. If the differential formula were $-\frac{\dot{y}}{y}$, the integral would be $-ly$. And if it were $-\frac{\dot{y}}{a+y}$, the integral would be $-l\,\overline{a+y}$. If it were $-\frac{\dot{y}}{a-y}$, the integral would be $l\,\overline{a-y}$; and if it were $\frac{\dot{y}}{a-y}$, the integral would be $-l\,\overline{a-y}$. Theſe logarithms are to be underſtood in the logarithmic of which the ſubtangent is unity. The reaſon of this is, that as the integral of $\frac{\dot{y}}{y}$ is ly, ſo the differential of ly is $\frac{\dot{y}}{y}$. And, to ſpeak in general, the differential of a logarithmic quantity is that fraction, the numerator of which is the product of the ſubtangent into the differential of the quantity, and the denominator is the ſame quantity. Thus, the differential of $-l\,\overline{a+y}$ will be $-\frac{\dot{y}}{a+y}$. The differential of $l\,\overline{a-y}$ will be $-\frac{\dot{y}}{a-y}$. The differential of $-l\,\overline{a-y}$ will be $\frac{\dot{y}}{a-y}$, ſuppoſing the ſubtangent

ſubtangent of the logarithmic $= 1$: and whenever it is not ſo, the numerators of the differentials muſt be multiplied by the given ſubtangent.

12. But, becauſe the logarithmic has no negative ordinates, it would ſeem that we cannot find the quantity which correſponds to the expreſſion $l\,\overline{a-y}$, that is, what is the logarithm of $a-y$, when $a-y$ is a negative quantity, or when y is greater than a. But, in this caſe, it may be obſerved, that $l\,\overline{a-y}$ and $l\,\overline{y-a}$ are the ſame thing; and that in ſuch a ſuppoſition, when $y-a$ is poſitive, it may be the ordinate in the logarithmic; and, indeed, if we difference the firſt logarithm, we ſhall have $-\frac{\dot{y}}{a-y}$, and if we difference the ſecond, we ſhall have $\frac{\dot{y}}{y-a}$; and changing the ſigns of the numerator and denominator, it will be $-\frac{\dot{y}}{a-y}$, the ſame as the firſt.

13. Other properties concerning logarithmic quantities may be deduced from theſe of the logarithmic curve; and firſt, that the multiple or ſubmultiple of a logarithm ſhall be the logarithm of the quantity raiſed to the power of the given number. Thus, $2lx = lx^2$; $3lx = lx^3$; $\frac{1}{2}lx = lx^{\frac{1}{2}}$; $\frac{1}{3}lx = lx^{\frac{1}{3}}$; $nlx = lx^n$; $\frac{1}{n}lx = lx^{\frac{1}{n}}$; and the reaſon of this is, becauſe, in the logarithmic curve, if we take any ordinate whatever, $OP = y$, (Fig. 3.) whoſe logarithm is AO; if AO, OS, SV, &c. be equal to each other, then AO, AS, AV, &c. will be arithmetical proportionals, and the ordinates AD, OP, ST, VI, &c. will be geometrical proportionals. Wherefore, putting AD equal to unity, $OP = y$, it will be $ST = y^2$, $VI = y^3$, &c. But AS, the double of AO, is the logarithm of y^2, or ly^2; and AV, the triple of AO, is ly^3. So that $2ly = ly^2$, $3ly = ly^3$, &c. Thus, alſo, making $AO = ly$, and biſecting it at M, it will be $MN = y^{\frac{1}{2}}$, and therefore $AM = \frac{1}{2}AO$, that is, $\frac{1}{2}ly = ly^{\frac{1}{2}}$. In the ſame manner, making $QR = y$, and dividing AQ into three equal parts in M and O, it will be $MN = \sqrt[3]{y} = y^{\frac{1}{3}}$. But $AM = \frac{1}{3}ly$, and therefore $\frac{1}{3}ly = ly^{\frac{1}{3}}$; and, in like manner, of all others.

Fig. 99.



We

We muſt here obſerve, that the integral of $-\frac{\dot{y}}{y}$ is not only $-ly$, as was ſeen before, but may be thus expreſſed alſo, $l\frac{1}{y}$, or ly^{-1}; for, taking in the logarithmic any ordinate OP $= y$, and making AΩ = AO, it will be, by the nature of the curve, OP . AD :: AD . ΩA; that is, $y . 1 :: 1 . \Omega A = \frac{1}{y}$. But ΩA is the negative logarithm of OP, that is, of y, and is alſo the logarithm of ΩA. Therefore it will be $-ly = l\frac{1}{y} = ly^{-1}$; that is to ſay, the negative logarithm of any quantity whatever will be the ſame with the poſitive logarithm of the fraction, of which the ſame quantity is the denominator, or of the ſame quantity with a negative exponent. Thus it will be $-mly = l\frac{1}{y^m} = ly^{-m}$.

14. Moreover, the ſum of two, three, &c. logarithms will be equal to the logarithm of the product of the quantities, of which they are the poſitive logarithms; and the difference of two, three, &c. logarithms ſhall be equal to the logarithm of the fraction, the numerator of which is the product of the quantities, of which they are the poſitive logarithms, and the denominator is the product of the quantities, of which they are the negative logarithms. For, becauſe it is OP $= y$, QR $= z$, it will be AO $= ly$, AQ $= lz$. Take QB = AO, it will be AB $= ly + lz$. But AB is alſo the logarithm of BC, and, by the property of the logarithmic, BC is the fourth proportional to AD, OP, QR, that is, $= yz$; therefore it will be AB $= ly + lz = lzy$. Let there be another ordinate MN $= p$, and take BV = AM; it will be AV = AM + AB $= lp + lyz$; but AV is the logarithm of VI, and VI $= pyz$. Therefore $lp + ly + lz = lpyz$.

Now make QR $= z$, OP $= y$, and take QM = AO; it will be AM = AQ − AO $= lz - ly$. But AM is the logarithm of MN, and, by the ſame property of the logarithmic, it is MN $= \frac{z}{y}$. Therefore AM $= lz - ly = l\frac{z}{y}$. Let there be another ordinate BC $= p$, and take ΣA = BM. It will be ΣA = − AB + AM $= -lp + l\frac{z}{y}$. But ΣA is the logarithm of ΣΠ, and ΣΠ $= \frac{z}{py}$, (becauſe it is the fourth proportional to BC, MN, AD,) therefore $lz - ly - lp = l\frac{z}{py}$.

15. As

15. As in other cafes, fo alfo in thefe integrations by means of the logarithms, fome conftant quantity fhould always be added, that is, the logarithm of an arbitrary conftant quantity, which is to be determined afterwards as particular cafes may require.

16. But when the differential formulæ propofed to be integrated are fractions with a complicated denominator, fome cafes may be given in which it is eafy to have their integrals by means of the logarithmic, and this will be as often as the numerator of the fraction fhall be the exact differential of the denominator, or as often as it is proportional to it. And, in this cafe, the integral of the formula will be the logarithm of the denominator, or it's multiple, or fubmultiple, or proportional to that logarithm.

Thus, the integral of $\frac{2x\dot{x}}{aa + xx}$ will be $l\,\overline{aa + xx}$; the integral of $-\frac{2x\dot{x}}{aa - xx}$ will be $l\,\overline{aa - xx}$; the integral of $\frac{3x^2\dot{x}}{a^3 + x^3}$ will be $l\,\overline{a^3 + x^3}$; the integral of $\frac{4x\dot{x}}{aa + xx}$ will be $2l\,\overline{aa + xx}$, that is, $l\,\overline{aa + xx}|^2$; the integral of $\frac{x\dot{x}}{aa + xx}$ will be $\frac{1}{2}l\,\overline{aa + xx}$, or $l\,\overline{aa + xx}|^{\frac{1}{2}}$; the integral of $\frac{x^2\dot{x}}{a^3 + x^3}$ will be $\frac{1}{3}l\,\overline{a^3 + x^3}$, or $l\sqrt[3]{a^3 + x^3}$; and, in general, the integral of $\frac{mx^{n-1}\dot{x}}{a^n \pm x^n}$ will be $\pm\frac{m}{n}\,l\,\overline{a^n \pm x^n}$; that is, $\pm m\,l\,\overline{a^n \pm x^n}|^{\frac{1}{n}}$, or $\pm\,l\,\overline{a^n \pm x^n}|^{\frac{m}{n}}$. Thus the integral of $\frac{a\dot{x} - 2x\dot{x}}{ax - xx}$ will be $l\,\overline{ax - xx}$; the integral of $\frac{\frac{1}{2}a\dot{x} - x\dot{x}}{ax - xx}$ will be $l\sqrt{ax - xx}$; and thus of all others whatever, taking thefe logarithms from the logarithmic, the fubtangent of which is $= 1$.

17. But if the numerator of the fraction be not of the form we have now confidered, though the denominator may be fuch; and that no one of its linear components is imaginary; that is, when all the roots of the product from whence it arifes are real ones; then we may proceed in the following manner.

18. And, firft, the roots of the denominator are all equal to each other, or they are not. If they be all equal, as in the formula $\frac{x^m\dot{x}}{\overline{x \pm a}|^n}$, make $x \pm a = z$, and therefore $\dot{x} = \dot{z}$, $x^m = \overline{z \mp a}|^m$, $\overline{x \pm a}|^n = z^n$; and fubftituting

thefe

these values in the formula, it will be $\frac{\overline{z \mp a}|^m \times \dot{z}}{z^n}$. Wherefore, actually raising $z \mp a$ to the power m, each term can be integrated, either algebräically, or, at least, transcendentally, by means of the logarithmic. Whence, instead of z, restoring it's value given by x, we shall have the integral of the formula proposed $\frac{x^m \dot{x}}{\overline{x \pm a}|^n}$.

Let it be, for example, $\frac{x^3\dot{x}}{\overline{x - a}|^3}$. Put $x - a = z$, and therefore $\dot{x} = \dot{z}$, $x^3 = z^3 + 3az^2 + 3aaz + a^3$, $\overline{x - a}|^3 = z^3$; and, making the substitutions, it will be $\frac{z^3\dot{z} + 3az^2\dot{z} + 3aaz\dot{z} + a^3\dot{z}}{z^3}$; and, by integration, $z + 3lz - \frac{3aa}{z} - \frac{a^3}{2zz}$; and, instead of z, restoring it's value given by x, we shall have at last $\int \frac{x^3\dot{x}}{\overline{x - a}|^3} = x - a + l\,\overline{x - a}|^3 - \frac{3aa}{x - a} - \frac{a^3}{2 \times \overline{x - a}|^2}$; which integral, being differenced again, will restore the formula proposed to be integrated.

19. Now, if the roots of the denominator shall not be all equal, but either all unequal, or mixed of equal and unequal; then it will be necessary, first, to prepare the formula, by making the term of the highest power of the variable in the denominator to be positive, if it should happen to be negative, and then to free it from co-efficients, if it have any. Then, if the variable in the numerator, when there is any, be raised to a greater or equal power to the highest in the denominator, the numerator must be divided by the denominator so long, as that the exponent of the variable in that may be less than in this. Lastly, the roots of the denominator are to be found algebräically. Take this formula $-\frac{aa\dot{x}}{aa - 4xx}$ for an example. Changing the signs, and dividing by 4, it will become $\frac{\frac{1}{4}aa\dot{x}}{xx - \frac{1}{4}aa}$, that is, $\frac{\frac{1}{4}aa\dot{x}}{\overline{x - \frac{1}{2}a} \times \overline{x + \frac{1}{4}a}}$. Again, let the formula proposed be $\frac{aa\dot{x}}{2x^2 + 4ax + 2bx + 4ab}$; dividing by 2, it will be $\frac{\frac{1}{2}aa\dot{x}}{xx + 2ax + bx + 2ab}$, that is, $\frac{\frac{1}{2}aa\dot{x}}{\overline{x + 2a} \times \overline{x + b}}$. If the variable should be in the numerator, and raised to a higher power than in the denominator, we must make an actual division, by which we shall have both integers and fractions. The integers must be treated in the manner before explained; the fractions in the manner following.

20. Let the fraction be $\frac{\frac{1}{2}aa\dot{x}}{\overline{x+2a}\times\overline{x+b}}$; I ſay, this will be equal to two fractions, the numerators of which will be the ſame as of the firſt, and the denominators will be theſe: Of the firſt, it will be the product of one of the roots into the difference of the conſtant quantity of the other root, and of the conſtant quantity of the ſame root: Of the ſecond, it will be the product of the other root into the difference of the conſtant quantity of the firſt root, and of the conſtant quantity of this ſecond root. Thus, $\frac{\frac{1}{2}aa\dot{x}}{\overline{x+2a}\times\overline{x+b}} = \frac{\frac{1}{2}aa\dot{x}}{\overline{x+2a}\times\overline{b-2a}} + \frac{\frac{1}{2}aa\dot{x}}{\overline{x+b}\times\overline{2a-b}}$. And if the roots ſhall be three, four, &c. proceed always in the ſame method. And if the fractions found after this manner ſhall be reduced to a common denominator, they will reſtore the firſt fraction from which they were derived.

Now the integrals of ſuch fractions ſo ſplit, which will always be in our power to find, ſuppoſing the logarithmic curve to be given, will be the integrals of the formula propoſed. Thus, it will be $\int\frac{\frac{1}{2}aa\dot{x}}{\overline{x+2a}\times\overline{x+b}} = \frac{\frac{1}{2}a}{2a-b}\times l\,\overline{x+b} - \frac{\frac{1}{2}a}{2a-b}\times l\,\overline{x+2a}$; that is, $\frac{\frac{1}{2}a}{2a-b}\times l\frac{x+b}{x+2a}$, or $\frac{a}{2a-b}l\sqrt{\frac{x+b}{x+2a}}$, in the logarithmic whoſe ſubtangent $= a$.

Let it be $\frac{\frac{1}{4}aa\dot{x}}{\overline{x+\frac{1}{2}a}\times\overline{x-\frac{1}{2}a}}$; this may be ſplit into theſe two, $\frac{\frac{1}{4}aa\dot{x}}{\overline{x+\frac{1}{2}a}\times\overline{-\frac{1}{2}a-\frac{1}{2}a}} + \frac{\frac{1}{4}aa\dot{x}}{\overline{x-\frac{1}{2}a}\times\overline{\frac{1}{2}a+\frac{1}{2}a}}$, or $\frac{\frac{1}{4}a\dot{x}}{x-\frac{1}{2}a} - \frac{\frac{1}{4}a\dot{x}}{x+\frac{1}{2}a}$, and therefore it will be $\int\frac{\frac{1}{4}aa\dot{x}}{\overline{x+\frac{1}{2}a}\times\overline{x-\frac{1}{2}a}} = \frac{1}{4}l\frac{x-\frac{1}{2}a}{x+\frac{1}{2}a}$, or $= l\sqrt[4]{\frac{x-\frac{1}{2}a}{x+\frac{1}{2}a}}$, in the logarithmic of which the ſubtangent $= a$.

Let it be $\frac{a^3\dot{x}}{\overline{x+a}\times\overline{x-b}\times\overline{x+c}}$; this may be ſplit into three,

$$\frac{a^3\dot{x}}{\overline{x+a}\times\overline{-b-a}\times\overline{c-a}} + \frac{a^3\dot{x}}{\overline{x-b}\times\overline{a+b}\times\overline{c+b}} + \frac{a^3\dot{x}}{\overline{x+c}\times\overline{a-c}\times\overline{-b-c}},$$

and therefore $\int\frac{a^3\dot{x}}{\overline{x+a}\times\overline{x-b}\times\overline{x+c}} = \frac{aa}{\overline{a+b}\times\overline{a-c}}\times l\,\overline{x+a} + \frac{aa}{\overline{a+b}\times\overline{c+b}}\times l\,\overline{x-b} - \frac{aa}{\overline{a-c}\times\overline{b+c}}\times l\,\overline{x+c}$, in the logarithmic whoſe ſubtangent $= a$.

Let

Let it be $\frac{-a^3\dot{x}}{x^3 - aax}$, that is, $\frac{-a^3\dot{x}}{\overline{x+a} \times \overline{x-a} \times \overline{x+0}}$; this may be ſplit into theſe three, $\frac{-a^3\dot{x}}{\overline{x+a} \times \overline{-2a} \times \overline{0-a}} + \frac{-a^3\dot{x}}{\overline{x-a} \times 2a \times \overline{0+a}} + \frac{-a^3\dot{x}}{\overline{x+0} \times \overline{a-0} \times \overline{-a-0}}$, that is, $\frac{-a\dot{x}}{2 \times \overline{x+a}} - \frac{a\dot{x}}{2 \times \overline{x-a}} + \frac{a\dot{x}}{x}$; and therefore it will be $\int \frac{-a^3\dot{x}}{x^3 - aax} = lx - \frac{1}{2}l\,\overline{xx - aa}$, that is, $l\frac{x}{\sqrt{xx - aa}}$, in the logarithmic of ſubtangent $= a$.

22. If the denominator of the formula ſhall be mixed of equal and unequal roots, as, for example, $\frac{a^3\dot{x}}{\overline{x-b}|^2 \times \overline{x+c}}$, then the formula muſt be conſidered as if it were $\frac{a^3\dot{x}}{\overline{x-b} \times \overline{x+c}}$, and being ſplit as uſual, it will be $\frac{a^3\dot{x}}{\overline{x-b} \times \overline{x+c}} = \frac{a^3\dot{x}}{\overline{x-b} \times \overline{c+b}} + \frac{a^3\dot{x}}{\overline{x+c} \times \overline{-b-c}}$; and then, multiplying the denominators by $\overline{x-b}$, the other root of the propoſed formula, it will be $\frac{a^3\dot{x}}{\overline{x-b}|^2 \times \overline{x+c}} = \frac{a^3\dot{x}}{\overline{x-b}|^2 \times \overline{c+b}} + \frac{a^3\dot{x}}{\overline{x+c} \times \overline{x-b} \times \overline{-b-c}}$; but the firſt term of the *homogeneum comparationis* has all the roots of it's denominator equal, and the ſecond term conſiſts of roots all unequal; ſo that, both of them being managed as before, we may have the integral of $\frac{a^3\dot{x}}{\overline{x-b}|^2 \times \overline{x+c}}$, which will be partly algebräical, and partly logarithmical, that is, $\frac{aa}{\overline{b+c}|^2} \times l\frac{x+c}{x-b} - \frac{a^3}{\overline{x-b} \times \overline{b+c}}$; taking the logarithm from the logarithmic, whoſe ſubtangent $= a$.

If there ſhall be a greater number of equal roots, the operation muſt be repeated in the ſame manner, as often as ſhall be neceſſary.

23. That caſe remains to be conſidered, in which the fractions have alſo in the numerator the variable raiſed to any power; always meaning, as has been already obſerved, that the power of this variable in the numerator be leſs than the greateſt which is in the denominator; and not being ſo, it muſt be made ſuch by actually dividing.

In theſe caſes the formula muſt be treated in the ſame manner, as if in the numerator there were no power of the variable, ſplitting it, in the manner before explained, into ſo many parts, as are the roots of the denominator. Then, if the exponent of the variable in the numerator of the given formula be an odd

 number,

number, let the ſigns be changed in the numerators of the fractions found; and if it be an even number, their own ſigns muſt remain to the numerators. After which, every numerator muſt be multiplied by ſuch a power of the conſtant quantity of that root, which is in the denominator, as is the power of the variable in the numerator of the propoſed formula, prefixing ſuch a ſign to that conſtant, raiſed to that power, as it's natural ſign requires, which it has in the denominator.

Let the example be $\frac{bbx\dot{x}}{\overline{x+a} \times \overline{x-a}}$. This being conſidered as if there were no variable in the numerator, it will be ſplit into theſe two, $\frac{bb\dot{x}}{\overline{x+a} \times -2a} + \frac{bb\dot{x}}{\overline{x-a} \times 2a}$; but, becauſe in the numerator there is the variable raiſed to the power denominated by unity, or the firſt power, the ſigns are changed in the numerators, and are multiplied relatively by the conſtant of that root which is in it's denominator, that is, the firſt by a, and the ſecond by $-a$, and we ſhall have $\frac{bbx\dot{x}}{\overline{x+a} \times \overline{x-a}} = \frac{-bb\dot{x} \times a}{\overline{x+a} \times -2a} - \frac{bb\dot{x} \times -a}{\overline{x-a} \times 2a}$, that is, $\frac{bb\dot{x}}{2 \times \overline{x+a}} + \frac{bb\dot{x}}{2 \times \overline{x-a}}$; and therefore it will be $\int \frac{bbx\dot{x}}{\overline{x+a} \times \overline{x-a}} = bl\sqrt{x+a} + bl\sqrt{x-a}$, or $bl\sqrt{xx-aa}$, in the logarithmic of the ſubtangent $= b$. Or otherwiſe, it will be $bbl\sqrt{xx-aa}$, in the logarithmic of the ſubtangent $= 1$.

But it was needleſs to reduce this formula to two fractions; for, as it was $\frac{bbx\dot{x}}{xx-aa}$, the numerator is exactly half the differential of the denominator, and therefore, without any other operation, the integral will be $bbl\sqrt{xx-aa}$, (as is ſaid at § 17.) in the logarithmic whoſe ſubtangent is unity.

Let it be $\frac{x^4\dot{x}}{\overline{xx-aa} \times \overline{x+b}}$, that is, $\frac{x^4\dot{x}}{x^3+bx^2-aax-aab}$; and dividing the numerator by the denominator, we ſhall have $x\dot{x} + \frac{-bx^3\dot{x} + aax^2\dot{x} + aabx\dot{x}}{x^3+bx^2-aax-aab}$; and dividing again the term $-bx^3\dot{x}$ by the denominator, we ſhall have $\frac{x^4\dot{x}}{\overline{xx-aa} \times \overline{x+b}} = x\dot{x} - b\dot{x} + \frac{aax^2\dot{x} + bbx^2\dot{x} - aabb\dot{x}}{\overline{xx-aa} \times \overline{x+b}}$. Now the two firſt terms are integers, and the laſt has not the variable in the laſt term of the numerator, and therefore may be managed; ſo that there only remains the term $\frac{\overline{aa+bb} \times x^2\dot{x}}{\overline{xx-aa} \times \overline{x+b}}$ ſtill to be reduced. This being conſidered as not having the variable

variable in the numerator, will be $\frac{\overline{aa+bb}\times\dot{x}}{\overline{xx-aa}\times\overline{x+b}} = \frac{\overline{aa+bb}\times\dot{x}}{\overline{x+b}\times\overline{-aa+bb}} + \frac{\overline{aa+bb}\times\dot{x}}{\overline{x+a}\times\overline{-2ab+2aa}} + \frac{\overline{aa+bb}\times\dot{x}}{\overline{x-a}\times\overline{2ab+2aa}}$; and therefore it will be $\frac{\overline{aa+bb}\times x^2\dot{x}}{\overline{x+b}\times\overline{xx-aa}} = \frac{\overline{aa+bb}\times bb\dot{x}}{\overline{x+b}\times\overline{-aa+bb}} + \frac{\overline{aa+bb}\times aa\dot{x}}{\overline{x+a}\times\overline{-2ab+2aa}} + \frac{\overline{aa+bb}\times a^2\dot{x}}{\overline{x-a}\times\overline{2ab+2a^2}}$. Whence, laſtly, $\frac{x^4\dot{x}}{\overline{xx-aa}\times\overline{x+b}} = x\dot{x} - b\dot{x} - \frac{aabb\dot{x}}{\overline{x+b}\times\overline{xx-aa}} + \frac{\overline{aa+bb}\times bb\dot{x}}{\overline{x+b}\times\overline{-aa+bb}} + \frac{\overline{aa+bb}\times aa\dot{x}}{\overline{x+a}\times\overline{-2ab+2aa}} + \frac{\overline{aa+bb}\times aa\dot{x}}{\overline{x-a}\times\overline{2ab+2aa}}$; and if we would ſtill ſplit the term $-\frac{aabb\dot{x}}{\overline{x+b}\times\overline{xx-aa}}$, in order to have, finally, the integral of the propoſed formula, it will be $\frac{x^4\dot{x}}{\overline{xx-aa}\times\overline{x+b}} = x\dot{x} - b\dot{x} + \frac{b^4\dot{x}}{\overline{x+b}\times\overline{-aa+bb}} + \frac{a^4\dot{x}}{\overline{x+a}\times\overline{2aa-2ab}} + \frac{a^4\dot{x}}{\overline{x-a}\times\overline{2ab+2aa}}$. Then, by integration, we ſhall have $\int\frac{x^4\dot{x}}{\overline{xx-aa}\times\overline{x+b}} = \frac{1}{2}xx - bx - \frac{b^4}{aa-bb}\times l\overline{x+b} + \frac{a^4}{2aa-2ab}\times l\overline{x+a} + \frac{a^4}{2aa+2ab}\times l\overline{x-a}$; taking ſuch logarithms in the logarithmic of the ſubtangent $= 1$.

Now in this, as well as in all other integrations that can be made, we are to conceive a conſtant quantity is to be added, though, for the ſake of brevity, I here omit it; but it will be enough to mention it here.

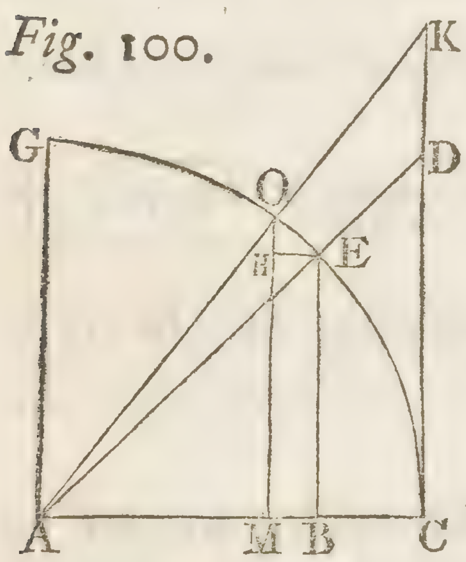
24. But differential formulæ may have, and often have, ſuch denominators, of which we cannot find the roots algebräically; yet, notwithſtanding this, we may make good uſe of the Rule of Fractions in theſe caſes alſo. For we may treat the denominator as if it were an equation, and, by means of the interſections of curves, may be found geometrically, in lines, the values of the variable, juſt after the ſame manner as ſolid problems are conſtructed. And ſuch values or lines may be called A, B, C, &c. with poſitive or negative ſigns, according as they come out poſitive or negative. Every one of theſe, being ſubtracted from the variable, will form a root of the denominator in ſuch manner, that the propoſed differential formula will be converted into one of this form, $\frac{x^n\dot{x}}{\overline{x-A}\times\overline{x+B}\times\overline{x-C},\ \&c}$, and with this we may proceed in the ſame manner, as the operation has been performed in the caſe of algebräical roots.

25. It may be eafily obferved, that the rule here produced ferves only in fuch cafes, when the roots of the denominator are real ; for when it is otherwife, the formula being fplit into other fractions, fo many of thefe will be imaginary, (and confequently the integrals will be imaginary,) as are the imaginary roots in the denominator of the differential formula propofed.

26. Therefore, when the denominator of the propofed differential formula is compofed of imaginary roots, either wholly or in part, there is a neceffity of having recourfe to other means. And, in the firft place, let the given formulæ have their denominators of two dimenfions only, that is, of two imaginary roots; and let it be, for example, $\frac{bb\dot{x}}{xx + aa}$.

The integral of this formula, and of all others like it, depends on the rectification or quadrature of the circle; I fay rectification or quadrature, becaufe, one of them being given, the other is reciprocally given alfo.

Fig. 100.



Wherefore let ACG be a quadrant of a circle, the radius AC $= a$, the tangent CD $= x$; it will be AB $= \frac{aa}{\sqrt{aa + xx}}$, CB $= a - \frac{aa}{\sqrt{aa + xx}}$, EB $= \frac{ax}{\sqrt{aa + xx}}$.

Drawing AK infinitely near to AD, then EO will be the fluxion or difference of the arch CE. And from the point O drawing the right line OM parallel to EB, and EH parallel to AC, then will HE be the differential of CB, and HO the differential of EB, and therefore EH $= \frac{aax\dot{x}}{\overline{aa + xx}|^{\frac{3}{2}}}$, and HO $= \frac{a^3\dot{x}}{\overline{aa + xx}|^{\frac{3}{2}}}$. Thence the little arch EO $= \sqrt{\overline{HEq + OHq}}$, will be $= \sqrt{\frac{a^6\dot{x}\dot{x} + a^4xx\dot{x}\dot{x}}{\overline{aa + xx}|^3}} = \frac{aa\dot{x}}{aa + xx}$. Whence the integral of the formula $\frac{aa\dot{x}}{aa + xx}$ will be the arch CE of the tangent CD $= x$, and of radius CA $= a$.

Now I refume the formula $\frac{bb\dot{x}}{aa + xx}$; multiplying the numerator and denominator by aa, it will be $\frac{bb}{aa} \times \frac{aa\dot{x}}{aa + xx}$; but the integral of $\frac{aa\dot{x}}{aa + xx}$ is the circular arch, which has for it's tangent x, and it's radius $= a$; therefore $\int \frac{bb\dot{x}}{aa + xx}$ $=$ to the fourth proportional of aa, of bb, and of the arch of the circle with radius $= a$, and tangent $= x$.

Let the formula be $\frac{aam\dot{x}}{nxx + nab}$; as, by multiplying the numerator and denominator by b, it will be equivalent to this other, $\frac{am}{nb} \times \frac{ab\dot{x}}{xx + ab}$; it will be $\int \frac{aam\dot{x}}{nxx + nab} =$ to a fourth proportional to nb, to am, and to the arch of a circle, with radius $= \sqrt{ab}$, and tangent $= x$. And ſo of all others of a like kind.

27. On the contrary, therefore, the differential of any arch of a circle is the product of the ſquare of the radius into the fluxion of the tangent, divided by the ſum of the ſquares of the ſaid radius, and the ſquare of the tangent.

And, as a conſtant quantity is always to be joined to other integrals or fluents, ſo alſo to this of the rectification of the circle; to have the integral complete, we muſt add a conſtant arch of the ſame circle; for the difference by which the arch, thus compoſed of a variable and a conſtant, can increaſe or diminiſh, can never be any other than what belongs to the differential of the variable arch; ſo that to the ſame differential may belong, by way of integral, the ſum of the variable arch, together with any conſtant arch of the ſame circle. Let us ſuppoſe that x is the tangent of an arch of a circle whoſe radius is a, and that b is the tangent of another conſtant arch of the ſame circle; we know that the tangent of the ſum of theſe two arches (Vol. I. § 108.) will be $= \frac{aab + aax}{aa - bx}$. But the differential of this, multiplied by the ſquare of the radius, and the product divided by the ſquare of the radius, adding the ſquare of the ſame tangent, is found to be $\frac{aa\dot{x}}{aa + xx}$, which is the differential of the variable arch.

Let the formula be $\frac{aa\dot{x}}{aa + xx - 2bx + bb}$, in which $xx - 2bx + bb$ is a ſquare. Make $x - b = z$, and, by ſubſtitution, we ſhall have $\frac{aa\dot{z}}{aa + zz}$. Therefore $\int \frac{aa\dot{z}}{aa + zz} =$ arch of a circle with radius $= a$, and tangent $= z$. But $z = x - b$; therefore $\int \frac{aa\dot{x}}{aa + xx - 2bx + bb} =$ arch of a circle with radius $= a$, with tangent $= x - b$, when x is greater than b. But, taking x leſs than b, the integral will be *minus* the arch of the circle, with the ſame radius and tangent. And, indeed, by differencing, we ſhould have $\frac{aa\dot{x}}{aa + bb - 2bx + xx}$, the ſame formula as at firſt.

Let

Let this formula be propofed, $\frac{4ab\dot{x} + 3bx\dot{x}}{xx - 4ax + 6aa}$. Make the fecond term of the denominator to vanifh, by putting $x = y + 2a$. Making the fubftitutions, it will be $\frac{4ab\dot{y} + 3by\dot{y} + 6ab\dot{y}}{yy + 2aa}$, that is, $\frac{10ab\dot{y}}{yy + 2aa} + \frac{3by\dot{y}}{yy + 2aa}$.

Therefore the integral of the firft term will be a third proportional to a, to $5b$, and to the arch of a circle with radius $= \sqrt{2aa}$, and with tangent $= y$: Of the fecond, it will be $l\,\overline{yy + 2aa}|^{\frac{3}{2}}$, in the logarithmic of fubtangent $= b$. Then, inftead of y, fubftituting it's value $x - 2a$, the integral of the formula $\frac{4ab\dot{x} + 3bx\dot{x}}{xx - 4ax + 6aa}$ will be the third proportional of a, $5b$, and the arch of the circle with radius $= \sqrt{2aa}$, with tangent $= x - 2a$; with $l\,\overline{xx - 4ax + 6aa}|^{\frac{3}{2}}$ alfo, in the logarithmic of fubtangent $= b$.

28. We will proceed now to fuch differential formulæ, as contain radical figns, that is, quantities raifed to a power with a fraction for it's exponent. If the formula either is, or may be reduced to fuch, that the variable quantity under the radical does not exceed the firft dimenfion; and out of the radical is a pofitive power; then fuch formulæ will always be integrable algebräically, and will obtain their integrations by making ufe of a very fimple fubftitution; and that is, by putting the quantity under the vinculum equal to a new variable.

Wherefore let the formula be $a\dot{x}\sqrt{ax - aa}$. Put $\sqrt{ax - aa} = z$, and therefore $x = \frac{zz + aa}{a}$, $\dot{x} = \frac{2z\dot{z}}{a}$; and, making the fubftitutions, we fhall have $2zz\dot{z}$, and, by integration, $\frac{2}{3}z^3$; and, inftead of z, reftoring it's value given by x, it will be $\frac{2}{3} \times \overline{ax - aa}|^{\frac{3}{2}}$, the integral of the propofed formula.

If the given formula were $\frac{a\dot{x}}{\sqrt{ax - aa}}$, by proceeding after the fame manner we fhould have $2 \times \overline{ax - aa}|^{\frac{1}{2}}$ for the integral.

Let it be $x\dot{x}\sqrt[4]{a - x}$; putting $\sqrt[4]{a - x} = z$, and therefore $x = a - z^4$, and $\dot{x} = -4z^3\dot{z}$; and making the fubftitutions, we fhould have $4z^8\dot{z} - 4az^4\dot{z}$; and by integrating, $\frac{4}{9}z^9 - \frac{4}{5}az^5$; and, inftead of z, reftoring it's value given by x, it will be $\frac{4}{9} \times \overline{a - x}|^{\frac{9}{4}} - \frac{4a}{5} \times \overline{a - x}|^{\frac{5}{4}}$.

If

If the formula were $\frac{x\dot{x}}{\sqrt[4]{a-x}}$, proceeding after the ſame manner, we ſhould have the integral $\frac{4}{7} \times \overline{a-x}|^{\frac{7}{4}} - \frac{4a}{3} \times \overline{a-x}|^{\frac{3}{4}}$.

Let it be $x^2\dot{x}\sqrt{a+x}$; make $\sqrt{a+x} = z$, and therefore $x = z^2 - a$, and $\dot{x} = 2z\dot{z}$, and $xx = \overline{zz-a}|^2$; and making the ſubſtitutions, we ſhall have $\overline{zz-a}|^2 \times 2zz\dot{z}$, that is, $2z^6\dot{z} - 4az^4\dot{z} + 2aaz^2\dot{z}$; and, by integration, $\frac{2}{7}z^7 - \frac{4}{5}az^5 + \frac{2}{3}aaz^3$; and, inſtead of z, reſtoring it's value given by x, it will be, laſtly, $\frac{2}{7} \times \overline{a+x}|^{\frac{7}{2}} - \frac{4}{5}a \times \overline{a+x}|^{\frac{5}{2}} + \frac{2}{3}aa \times \overline{a+x}|^{\frac{3}{2}}$, the integral required.

If the formula were $\frac{xx\dot{x}}{\sqrt{a+x}}$, the integral would be $\frac{2}{5} \times \overline{a+x}|^{\frac{5}{2}} - \frac{4a}{3} \times \overline{a+x}|^{\frac{3}{2}} + 2a^2 \times \overline{a+x}|^{\frac{1}{2}}$.

Let it be $x\dot{x}\sqrt{\overline{a+x}|^3}$, that is, $x\dot{x} \times \overline{a+x}|^{\frac{3}{2}}$. Make, as uſual, $\overline{a+x}|^{\frac{3}{2}} = z$, and therefore $x = z^{\frac{2}{3}} - a$, $\dot{x} = \frac{2}{3}z^{\frac{2}{3}-1}\dot{z}$; and making the ſubſtitutions, it will be $\overline{z^{\frac{2}{3}} - a} \times \frac{2}{3}z^{\frac{2}{3}}\dot{z}$, that is, $\frac{2}{3}z^{\frac{4}{3}}\dot{z} - \frac{2}{3}az^{\frac{2}{3}}\dot{z}$; and integrating, $\frac{2}{7}z^{\frac{7}{3}} - \frac{2}{5}az^{\frac{5}{3}}$; and, inſtead of z, ſubſtituting it's value, it will be $\frac{2}{7} \times \overline{a+x}|^{\frac{7}{2}} - \frac{2}{5}a \times \overline{a+x}|^{\frac{5}{2}}$.

If the formula were $\frac{x\dot{x}}{\overline{a+x}|^{\frac{3}{2}}}$, we ſhould have for it's integral $2\sqrt{a+x} + \frac{2a}{\sqrt{a+x}}$.

29. In general, let it be $ax^t\dot{x} \times \overline{a+x}|^{\frac{m}{n}}$, and let the exponents t, m, n, be poſitive integers; make, as uſual, $\overline{a+x}|^{\frac{m}{n}} = z$, and therefore $a + x = z^{\frac{n}{m}}$, $\dot{x} = \frac{n}{m}z^{\frac{n}{m}-1}\dot{z}$, $x^t = \overline{z^{\frac{n}{m}} - a}|^t$; and making the ſubſtitutions, the formula will be $\overline{z^{\frac{n}{m}} - a}|^t \times \frac{n}{m}az^{\frac{n}{m}}\dot{z}$; and actually raiſing $z^{\frac{n}{m}} - a$ to the power t, it is plain that every term will be algebräically integrable; in which terms, being integrated, inſtead of z, reſtore it's value given by x, and we ſhall have the algebräical integral of the propoſed formula.

30. If

30. If the exponent m were negative, ſo that the quantity under the vinculum would paſs into the denominator, in which caſe the exponent m would then become poſitive; that is, if the formula were $\dfrac{ax^t\dot{x}}{\overline{a+x}|^{\frac{m}{n}}}$; making the ſame ſubſtitutions, we ſhould have $\overline{z^{\frac{n}{m}} - a}|^t \times \frac{n}{m} az^{\frac{n}{m}-2}\dot{z}$; and actually raiſing $z^{\frac{n}{m}} - a$ to the power t, every term would then be algebräically integrable, excepting ſuch caſes in which the power $z^{-1}\dot{z}$ ſhould inſinuate itſelf, and then we ſhould be obliged to have recourſe to the logarithms.

But if the exponent t were negative, the two foregoing formulæ would not then be algebräically integrable, but might be freed from their radicals, and reduced to the quadrature of the circle and hyperbola, as will be ſeen in it's place.

31. But when the variable under the vinculum is raiſed to any power greater than unity, provided the quantity out of the vinculum is the exact differential, or any proportional to the differential, of the quantity under the vinculum; then, by means of the ſaid very ſimple ſubſtitution, we might have the integral of the differential formula, which ſaid integrals will always be algebräical.

Wherefore let the formula be $2x\dot{x}\sqrt{xx + aa}$; make $\sqrt{xx + aa} = z$, whence $xx + aa = zz$, $2x\dot{x} = 2z\dot{z}$; and making the ſubſtitutions, we ſhall have $2zz\dot{z}$, and integrating, $\frac{2}{3}z^3$; and reſtoring the value of z, it will be $\frac{2}{3} \times \overline{xx + aa}|^{\frac{3}{2}}$.

If the formula were $\dfrac{2x\dot{x}}{\sqrt{xx + aa}}$, we ſhould have for the integral $2\sqrt{xx + aa}$.

Let it be $\overline{2a\dot{x} - 4x\dot{x}} \times \sqrt{ax - xx + bb}$, that is, $2 \times \overline{a\dot{x} - 2x\dot{x}} \times \sqrt{ax - xx + bb}$; put $\sqrt{ax - xx + bb} = z$, and therefore $ax - xx + bb = zz$, and $a\dot{x} - 2x\dot{x} = 2z\dot{z}$; and making the ſubſtitutions, we ſhall have $4zz\dot{z}$, and integrating, it will be $\frac{4}{3}z^3$; and, inſtead of z, reſtoring it's value, it is $\frac{4}{3} \times \overline{ax - xx + bb}|^{\frac{3}{2}}$.

Let the formula be $\dfrac{2a\dot{x} - 4x\dot{x}}{\sqrt{ax - xx + bb}}$; it's integral will be $4 \times \overline{ax - xx + bb}|^{\frac{1}{2}}$.

Let it be $\overline{xx\dot{x} - \frac{2}{3}ax\dot{x}} \times \sqrt[4]{x^3 - ax^2}$, that is, $\dfrac{3xx\dot{x} - 2ax\dot{x}}{3} \times \sqrt[4]{x^3 - ax^2}$; make $\sqrt[4]{x^3 - ax^2} = z$, and therefore $z^4 = x^3 - ax^2$, and $3xx\dot{x} - 2ax\dot{x}$ =

$= 4z^3\dot{z}$; and making the fubftitutions, we fhall have $\frac{4}{3}z^4\dot{z}$, and by integrating, $\frac{4}{15}z^5$; and, inftead of z, reftoring it's value, it will be $\frac{4}{15} \times \overline{x^3 - axx}|^{\frac{5}{4}}$.

If the formula were $\frac{3xx\dot{x} - 2ax\dot{x}}{3\sqrt[4]{x^3 - axx}}$, the integral would be $\frac{4}{9} \times \overline{x^3 - axx}|^{\frac{3}{4}}$.

Let it be $2x\dot{x}\sqrt[3]{\overline{xx + aa}|^2}$, that is, $2x\dot{x} \times \overline{xx + aa}|^{\frac{2}{3}}$; put $\overline{xx + aa}|^{\frac{2}{3}} = z$, and therefore $xx + aa = z^{\frac{3}{2}}$, and $2x\dot{x} = \frac{3}{2}z^{\frac{3}{2}-1}\dot{z}$; and making the fubftitutions, we fhall have $\frac{3}{2}z^{\frac{3}{2}}\dot{z}$, and by integration, $\frac{3}{5}z^{\frac{5}{2}}$; and, inftead of z, reftoring it's value, $\frac{3}{5} \times \overline{xx + aa} \times \sqrt[3]{\overline{xx + aa}|^2}$.

If the formula were $\frac{2x\dot{x}}{\sqrt[3]{\overline{xx + aa}|^2}}$, the integral would be $3\sqrt[3]{xx + aa}$.

And, in general, let the formula be $px^{m-1}\dot{x} \times \overline{x^m + a^m}|^{\frac{n}{u}}$, in which p and m may alfo be fractions; put $\overline{x^m + a^m}|^{\frac{n}{u}} = z$, and therefore $z^{\frac{u}{n}} = x^m + a^m$, and $mx^{m-1}\dot{x} = \frac{u}{n}z^{\frac{u}{n}-1}\dot{z}$; and making the fubftitutions, we fhall have $\frac{pu}{mn}z^{\frac{u}{n}}\dot{z}$, and by integration, $\frac{pu}{mu + mn} \times z^{\frac{u+n}{n}}$; and, inftead of z, reftoring it's value, the integral will be $\frac{pu}{mu + mn} \times \overline{x^m + a^m} \times \overline{x^m + a^m}|^{\frac{n}{u}}$.

If n were negative, or if the formula were $\frac{px^{m-1}\dot{x}}{\overline{x^m + a^m}|^{\frac{n}{u}}}$, in which n is now pofitive, we fhould have the integral $\frac{pu}{mu - mn} \times \overline{x^m + a^m}|^{\frac{u-n}{u}}$.

Hence we may form this general rule, that the integral of fuch a formula will be the quantity under the vinculum, the exponent being increafed by unity, and dividing it by the exponent fo increafed; or the integral will be a proportional to this, according to the proportion which the differential quantity out of the vinculum will have to the precife differential.

32. But ſtill in a more general manner: Let the formula be $px^{rm-1}\dot{x} \times \overline{x^m + a^m}|^{\frac{n}{u}}$, ſuppoſing r to be a poſitive integer. It will be equivalent to this other, $px^{rm-m} \times x^{m-1}\dot{x} \times \overline{x^m + a^m}|^{\frac{n}{u}}$; make, as uſual, $z = \overline{x^m + a^m}|^{\frac{n}{u}}$, and therefore $x^m + a^m = z^{\frac{u}{n}}$, and $mx^{m-1}\dot{x} = \frac{u}{n} z^{\frac{u}{n}-1}\dot{z}$; and, becauſe $x^m = z^{\frac{u}{n}} - a^m$, it will be $x^{rm-m} = \overline{z^{\frac{u}{n}} - a^m}|^{r-1}$. Therefore, making the ſubſtitutions, we ſhall have $p \times \overline{z^{\frac{u}{n}} - a^m}|^{r-1} \times \frac{u}{mn} z^{\frac{u}{n}}\dot{z}$. Now, ſuppoſing r to be a poſitive integer number, then alſo $r - 1$ will be a poſitive integer number; and actually raiſing $z^{\frac{u}{n}} - a^m$ to the power $r - 1$, each term will be algebräically integrable, in which integral reſtoring, inſtead of z, it's value given by x, we ſhall have the integral required.

If n were negative, that is, if the formula were $\frac{px^{rm-1}\dot{x}}{\overline{x^m + a^m}|^{\frac{n}{u}}}$, in which n is now poſitive, making the ſubſtitutions, it will be $p \times \overline{z^{\frac{u}{n}} - a^m}|^{r-1} \times \frac{u}{mn} z^{\frac{u}{n}-2}\dot{z}$, which is likewiſe integrable.

In all theſe caſes, if the quantity under the vinculum, inſtead of being $x^m + a^m$, had been $x^m - a^m$, or $a^m - x^m$, we might proceed after the ſame manner, without hindering the operation.

By this method we may find likewiſe, that it will be

$$\int ax^{m-1}\dot{x} \times \sqrt{e + fx^m} = \frac{2a}{3mf} \times \overline{e + fx^m}|^{\frac{3}{2}}.$$

$$\int \frac{ax^{m-1}\dot{x}}{\sqrt{e + fx^m}} = \frac{2a}{mf} \times \overline{e + fx^m}|^{\frac{1}{2}}.$$

$\int ax$

$$\int ax^{2m-1}\dot{x}\sqrt{e+fx^m} = -\frac{4e-6fx^m}{15mff} \times a \times \overline{e+fx^m}|^{\frac{3}{2}}.$$

$$\int \frac{ax^{2m-1}\dot{x}}{\sqrt{e+fx^m}} = -\frac{4e-2fx^m}{3mff} \times a \times \overline{e+fx^m}|^{\frac{1}{2}}.$$

$$\int ax^{3m-1}\dot{x}\sqrt{e+fx^m} = a \times \frac{16ee-24efx^m+30ffx^{2m}}{105f^3m} \times \overline{e+fx^m}|^{\frac{3}{2}}.$$

$$\int \frac{ax^{3m-1}\dot{x}}{\sqrt{e+fx^m}} = \frac{16ee-8efx^m+6ffx^{2m}}{15mf^3} \times a \times \overline{e+fx^m}|^{\frac{1}{2}}.$$

And ſo we might go on as far as we pleaſe.

33. Likewiſe in the caſe, in which the variable out of the vinculum ſhall be in the denominator, the formula will be algebräically integrable by the help of two ſubſtitutions, provided the exponent of that variable out of the vinculum ſhall have a certain condition; thus, let the formula be $\frac{\dot{x} \times \overline{x^m+a^m}|^{\frac{n}{u}}}{x^{rm+\frac{mn}{u}+1}}$.

Then make $x = \frac{aa}{y}$, $\dot{x} = -\frac{aa\dot{y}}{yy}$, $x^m = \frac{a^{2m}}{y^m}$, $\overline{x^m+a^m}|^{\frac{n}{u}} = \frac{\overline{a^{2m}+a^my^m}|^{\frac{n}{u}}}{y^{\frac{mn}{u}}}$.

Then making the ſubſtitutions, the formula will be

$$\frac{-\frac{aa\dot{y}}{yy} \times \frac{\overline{a^{2m}+a^my^m}|^{\frac{n}{u}}}{y^{\frac{mn}{u}}} \times y^{rm+\frac{mn}{u}+1}}{a^{2rm+\frac{2mn}{u}+2}}, \text{ that is, } -y^{rm-1}\dot{y} \times \frac{\overline{a^{2m}+a^my^m}|^{\frac{n}{u}}}{a^{2rm+\frac{2mn}{u}}};$$

a formula which has the conditions here required, and which may be integrated algebräically, by means of the ſubſtitution mentioned at § 32.

If the formula propoſed were $\frac{a^5\dot{x}}{x^4\sqrt{ax+xx}}$, that is, $\frac{a^5\dot{x}}{x^{\frac{9}{2}}\sqrt{a+x}}$; this having the conditions required, will be algebräically integrable; which is alſo to be obſerved of others.

34. But

34. But here it may be observed, that, in the general formula, it may also be $n = 1$, in which case the power of $x^m + a^m$ will be rational, that is, integrable. [qu. integral.]

Also, in this case, supposing n to be a negative number, (for when it is affirmative there will be no difficulty,) we may make use of the same substitution, and of the same method, by which the integrals may be found of such formulæ, the integrals of which will not always be algebräical. For very often they will depend in part upon the quadrature of the hyperbola, that is, on the logarithmic curve.

Therefore, by a known method, we shall find that

$$\int \frac{x^{m-1}\dot{x}}{\overline{x^m + a^m}|^2} = -\frac{1}{m} \times \overline{x^m + a^m}|^{-1}.$$

$$\int \frac{x^{2m-1}\dot{x}}{\overline{x^m + a^m}|^2} = \frac{1}{m} l\,\overline{a^m + x^m} + \frac{a^m}{m \times \overline{a^m + x^m}}.$$

$$\int \frac{x^{3m-1}\dot{x}}{\overline{x^m + a^m}|^2} = \frac{a^m + x^m}{m} - \frac{2a^m\, l\,\overline{a^m + x^m}}{m} - \frac{a^{2m}}{m \times \overline{a^m + x^m}}.$$

$$\int \frac{x^{m-1}\dot{x}}{\overline{a^m + x^m}|^3} = -\frac{1}{2m \times \overline{a^m + x^m}|^2}.$$

$$\int \frac{x^{2m-1}\dot{x}}{\overline{a^m + x^m}|^3} = -\frac{1}{m \times \overline{a^m + x^m}} + \frac{a^m}{2m \times \overline{a^m + x^m}|^2}.$$

$$\int \frac{x^{3m-1}\dot{x}}{\overline{a^m + x^m}|^3} = \frac{1}{m} l\,\overline{a^m + x^m} + \frac{2a^m}{m \times \overline{a^m + x^m}} - \frac{a^{2m}}{2m \times \overline{a^m + x^m}|^2}. \quad \&c.$$

35. But the manner of proceeding will be very different when the proposed differential formulæ containing the radical, are not such as that the quantity out of the vinculum shall have those conditions before mentioned. These formulæ may always be delivered from their radical, provided they contain but one, which is that of the square-root, and that the variable under the same does not exceed two dimensions. Now, for these there will be occasion for some caution in the choice of such substitutions as are to be made, that they may be freed from radical signs. When this is done, we may go on to integrations, either algebräical, or such as depend on the quadrature of the circle or hyperbola, after the manner already explained, if they come under the given rules. If not, we must have recourse to other methods, which are to be given hereafter.

If the radical of the propofed formula were $\sqrt{ax \pm xx}$, or $\sqrt{xx \pm ax}$; this radical may be made equal to $\frac{xz}{b}$, meaning by z a new variable, and by b any conftant quantity whatever.

If the radical were $\sqrt{aa \pm xx}$, make it $= x + z$, or $x - z$.

If the radical were $\sqrt{aa - xx}$, or $\sqrt{fp - xx}$, put the radical $= \sqrt{fp} + \frac{xz}{b}$, or $= \sqrt{fp} - \frac{xz}{b}$. From fuch equations the values of x and $\dot{x}$ may be derived, expreffed by z and conftant quantities; which values are to be fubftituted in the given formulæ, and we fhall have other formulæ free from radicals, and given by z. In the integrations of which, if they can be had, the value of z by x being reftored, we fhall have the integrations of the propofed formulæ.

36. If the quantity fhould have three terms, that is, the fquare of the variable with the rectangle of the fame into a conftant, and befides, a term which is wholly conftant; then either the fecond term muft be taken away, after the ufual manner, as in the common Algebra; or, if the conftant term be pofitive, as in $\sqrt{xx + ax + aa}$ for inftance, however the others may be pofitive or negative, provided the quantity be not imaginary; make $\sqrt{xx + ax + aa} = a + \frac{xz}{b}$; and if the conftant term be negative, as, fuppofe $\sqrt{xx + ax - aa}$, it may be made $\sqrt{xx + ax - aa} = x + z$.

From hence it may be feen, that the whole artifice confifts in comparing the radical quantity to fuch other quantity compofed of the given variable, and of a new one with conftant quantities, as that an equation may refult from thence, from whence we may have the value of x and of $\dot{x}$, free from radical figns.

Let there be propofed to be integrated the differential formula $x^3\dot{x}\sqrt{ax - xx}$. Put $\sqrt{ax - xx} = \frac{xz}{b}$, and therefore $a - x = \frac{xzz}{bb}$, that is, $x = \frac{abb}{zz + bb}$, and $\dot{x} = -\frac{2abbz\dot{z}}{\overline{zz + bb}|^2}$, $x^3 = \frac{a^3b^6}{\overline{zz + bb}|^3}$, and $\sqrt{ax - xx} = \frac{xz}{b} = \frac{abz}{zz + bb}$. Make the fubftitutions in the propofed formula, and it will be $-\frac{2a^5b^9zz\dot{z}}{\overline{zz + bb}|^6}$, a formula which, though free from radical figns, yet, as to it's integration, will not fubmit to the ufual methods.

Let

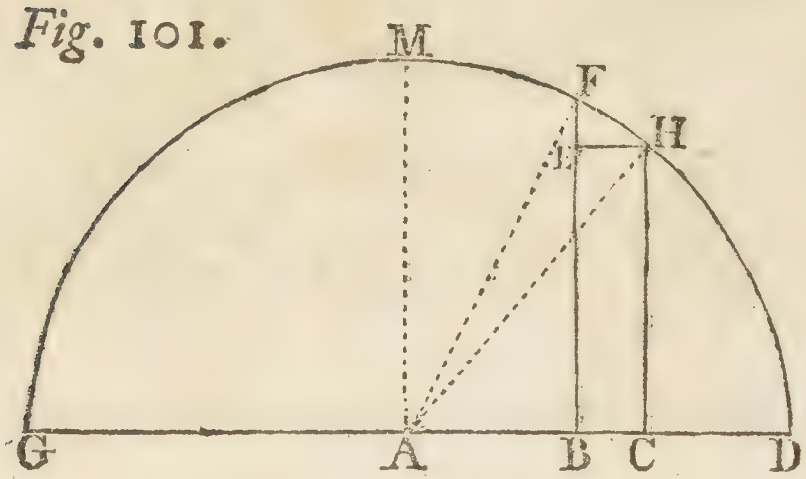
Let it be $\frac{aa\dot{x}}{x\sqrt{ax+xx}}$. Make $\sqrt{ax+xx} = \frac{xz}{b}$, and therefore it will be $x = \frac{abb}{zz-bb}$, $\dot{x} = -\frac{2abbz\dot{z}}{\overline{zz-bb}|^2}$, $\sqrt{ax+xx} = \frac{xz}{b} = \frac{abz}{zz-bb}$. Making the ſubſtitutions in the propoſed formula, it will be $-\frac{2a\dot{z}}{b}$, and by integration, $-\frac{2az}{b}$; and, inſtead of z, reſtoring it's value by x, it is $\int\frac{aa\dot{x}}{x\sqrt{ax+xx}} = -\frac{2a\sqrt{ax+xx}}{x}$.

Let it be $\frac{x\dot{x}}{\sqrt{ax+xx}}$; put $\sqrt{ax+xx} = \frac{xz}{b}$, and making the neceſſary ſubſtitutions as before, the formula will be $-\frac{2ab^3\dot{z}}{\overline{zz-bb}|^2}$, that is, $-\frac{2ab^3\dot{z}}{\overline{z+b}|^2\times\overline{z-b}|^2}$. But we have already ſeen how to manage this by the Rule of Fractions, and it will have for it's fluent $\frac{abz}{zz-bb} + \frac{1}{2}a\,l\,\frac{z-b}{z+b}$, in the logarithmic the ſubtangent of which is unity. And, inſtead of z, reſtoring it's value by x, it will be $\int\frac{x\dot{x}}{\sqrt{ax+xx}} = \sqrt{ax+xx} + \frac{1}{2}al\,\frac{\sqrt{ax+xx}-x}{\sqrt{ax+xx}+x}$, in the logarithmic of the ſame ſubtangent $= 1$.

Let it be $\frac{x\dot{x}}{\sqrt{xx+ax-aa}}$. Make $\sqrt{xx+ax-aa} = x+z$, and therefore it will be $x = \frac{zz+aa}{a-2z}$, $\dot{x} = \frac{2az\dot{z}-2zz\dot{z}+2aa\dot{z}}{\overline{a-2z}|^2}$, and $\sqrt{xx+ax-aa} = x+z = \frac{aa+az-zz}{a-2z}$. Make the ſubſtitutions, and the propoſed formula will be $\frac{\overline{zz+aa}\times 2\dot{z}}{\overline{a-2z}|^2}$, that is, $\frac{2zz\dot{z}+2aa\dot{z}}{\overline{a-2z}|^2}$; and by integration, (which may be performed by the foregoing rules, it is $\frac{5aa}{4\times\overline{a-2z}} - \frac{1}{4}a + \frac{1}{2}z + \frac{1}{2}al\,\overline{a-2z}$, in the logarithmic with ſubtangent $= 1$. And, inſtead of z, reſtoring it's value by x, it will be, laſtly, $\int\frac{x\dot{x}}{\sqrt{xx+ax-aa}} = \frac{5aa}{4a+8x-8\sqrt{xx+ax-aa}} - \frac{1}{4}a - \frac{1}{2}x + \frac{1}{2}\sqrt{xx+ax-aa} + \frac{1}{2}a\,l\,\overline{a+2x-2\sqrt{xx+ax-aa}}$, in the logarithmic whoſe ſubtangent is unity.

37. As to ſome radical differential formulæ, the trouble, indeed, would be ſuperfluous to tranſmute them, by means of theſe ſubſtitutions, into others that are free from radical ſigns, in order to prepare them for integration ; and ſuch are all thoſe which of their own nature require the quadrature or rectification of the circle. Wherefore let there be a ſemicircle GMD, (Fig. 101.) it's radius AD $= a$, AB $= x$, whence BF $= \sqrt{aa - xx}$; and drawing CH infinitely near to BF, it will be BC $= \dot{x}$, EF $= \frac{x\dot{x}}{\sqrt{aa - xx}}$. Therefore the expreſſion of the infiniteſimal rectangle BCHE will be $\dot{x}\sqrt{aa - xx}$, and therefore $\int \dot{x}\sqrt{aa - xx}$ is equal to the ſpace ABFM.

Fig. 101.



Alſo, $\frac{a\dot{x}}{\sqrt{aa - xx}}$ will be the expreſſion of the infinitely little arch FH, and therefore $\int \frac{a\dot{x}}{\sqrt{aa - xx}} =$ arch MF. And if the little arch FH be drawn into half the radius, then $\frac{aa\dot{x}}{2\sqrt{aa - xx}}$ will be the expreſſion of the infinitely little ſector AFH, and therefore $\int \frac{aa\dot{x}}{2\sqrt{aa - xx}} =$ to the ſector AFM.

In the ſame circle let it be now DC $= x$, and CB $= \dot{x}$. It will be CH $= \sqrt{2ax - xx}$, EF $= \frac{a\dot{x} - x\dot{x}}{\sqrt{2ax - xx}}$. Wherefore $\int \dot{x}\sqrt{2ax - xx}$ will be equal to the ſpace HCD. And thus $\int \frac{a\dot{x}}{\sqrt{2ax - xx}} =$ arch HD, and $\int \frac{aa\dot{x}}{2\sqrt{2ax - xx}} =$ ſector AHD. In ſuch as theſe, therefore, the trouble [of transformation] would be needleſs; for, in the firſt caſe, we ſhould make $\sqrt{aa - xx} = a - \frac{xz}{b}$, and therefore $x = \frac{2abz}{zz + bb}$, $\dot{x} = \frac{2ab^3\dot{z} - 2abzz\dot{z}}{(zz + bb)^2}$, $\sqrt{aa - xx} = a - \frac{xz}{b} = \frac{abb - azz}{zz + bb}$. Now, making theſe ſubſtitutions, it will be $\frac{a\dot{x}}{\sqrt{aa - xx}} = \frac{2ab\dot{z}}{zz + bb}$; a formula for the rectification of the circle, the tangent of which is equal to z, as has been ſeen already at § 26.

Alſo,

Alſo, let it be $\frac{aa\dot{x}}{2\sqrt{ax-xx}} = \frac{aab\dot{z}}{zz+bb}$, a formula which requires the ſame rectification. In like manner, it will be $\dot{x}\sqrt{aa-xx} = \frac{2aab\dot{z} \times \overline{bb-zz}|^2}{\overline{zz+bb}|^3}$, a formula which, though at preſent we cannot manage, yet afterwards we ſhall find to depend on the ſame circle.

In the ſecond caſe, I put $\sqrt{2ax-xx} = \frac{xz}{b}$, and therefore $x = \frac{2abb}{zz+bb}$, $\dot{x} = -\frac{4abbz\dot{z}}{\overline{zz+bb}|^2}$, and $\sqrt{2ax-xx} = \frac{xz}{b} = \frac{2abz}{zz+bb}$. Making the ſubſtitutions, it will be $\frac{a\dot{x}}{\sqrt{2ax-xx}} = -\frac{2ab\dot{z}}{zz+bb}$, the rectification of the circle.

Let it be alſo $\frac{aa\dot{x}}{2\sqrt{2ax-xx}} = -\frac{aab\dot{z}}{zz+bb}$, the rectification of the circle, as before.

In like manner, it will be $\dot{x}\sqrt{2ax-xx} = -\frac{8a^2b^3z^2\dot{z}}{\overline{zz+bb}|^3}$, which includes the ſame circle.

38. If our differential formulæ ſhall be compoſed of two radical quantities, in this caſe the operation will be double, but ſtill it will ſucceed as well. For, in the radical quantities, the ſecond term may be wanting, or it may be taken away, and the formula may be multiplied by an odd power of the variable; and that by putting one of the radical quantities equal to a new variable. And thus the propoſed formula will be reduced to another, which will contain one radical only, and which conſequently may be managed in the uſual manner.

Let it be, for example, $\frac{x^3\dot{x}\sqrt{aa+xx}}{\sqrt{bb+xx}}$. I put $\sqrt{aa+xx} = y$, and therefore $xx = yy - aa$, $x\dot{x} = y\dot{y}$. Making the ſubſtitutions, it will be $\frac{yy\dot{y} \times \overline{yy-aa}}{\sqrt{yy-aa+bb}}$, that is, $\frac{y^4\dot{y}}{\sqrt{yy-aa+bb}} - \frac{aayy\dot{y}}{\sqrt{yy-aa+bb}}$, each of which we know how to manage.

39. If we conſider a little this manner of operation, we may eaſily perceive, that, in theſe radical formulæ, it will not ſucceed in general, that we ſhall be able to free them from their radical vinculum, except when it is a ſquare-root, and the invariable under the vinculum does not exceed the ſecond dimenſion. I ſay in general; becauſe, in ſeveral caſes, it may ſucceed, whatever the radical may

may be, and whatever the power of the variable may be, which is under the vinculum. And certainly it will, in all cafes, be comprehended in the two following formulæ, the firft of which is this, $\frac{\dot{y} \times \overline{y^m + b^m}|^{\pm \frac{1}{n}}}{y^{tm+1}}$, in which m, n, t, are pofitive integers, and may alfo be nothing; and this obtains, by making $\overline{y^m + b^m}|^{\frac{1}{n}} = z$, whence $y^m = z^n - b^m$, $\dot{y} = \frac{nz^{n-1}\dot{z}}{my^{m-1}}$; and making the fubftitutions, it will be $\frac{nz^{n-1}\dot{z} \times z^{\pm 1}}{my^{tm+m}}$, that is, $\frac{nz^{n-1}\dot{z} \times z^{\pm 1}}{m\overline{y^{t+1 \times m}}}$. But $\overline{y^{t+1 \times m}} = \overline{z^n - b^m}|^{t+1}$; and when t is an integer, the power $t + 1$ will be an integer, fo that the propofed formula will be free from radicals.

If t were negative, the formula would be the cafe confidered above at § 32, which has an algebräical integration.

In other cafes, the integral will depend on the quadrature of the circle, and of the hyperbola, as will be feen in it's place.

The fecond formula is $y^n\dot{y} \times \overline{y^m + b^m}|^{\pm \frac{t}{p}}$, which, when $\frac{n+1}{m}$ is a whole number, may always be freed from it's radical figns, either in the whole, or, at leaft, from radicals of the complicate quantity, which will be fufficient. Wherefore, make $\overline{y^m + b^m}|^{\frac{t}{p}} = z$, and then it will be $y^m = z^{\frac{p}{t}} - b^m$, $y = \overline{z^{\frac{p}{t}} - b^m}|^{\frac{1}{m}}$, $\dot{y} = \frac{\frac{p}{t} z^{\frac{p}{t}-1}\dot{z} \times \overline{z^{\frac{p}{t}} - b^m}|^{\frac{1}{m}-1}}{m}$, and $y^n = \overline{z^{\frac{p}{t}} - b^m}|^{\frac{n}{m}}$; and making the fubftitutions, we fhall have the formula $\frac{pz^{\frac{p}{t}-1}\dot{z}}{tm} \times z^{\pm 1} \times \overline{z^{\frac{p}{t}} - b^m}|^{\frac{1}{m}+\frac{n}{m}-1}$. But when $\frac{n+1}{m}$ is an integer, the power $\frac{1+n}{m} - 1$ will always be an integer, [or 0,] fo that the formula will have

 only

only radical ſigns of the complicate quantities. And therefore, when $\frac{1+n}{m} - 1$ is a poſitive integer number, the integration, at moſt, will depend on the quadrature of the hyperbola, or on the logarithmic, and may be had by the given rules. And when $\frac{1+n}{m} - 1$ is a negative integer, the integration will depend on the quadrature of the circle, and of the hyperbola, and may be had by the rules which will be given in due place.

40. Now let us go on to ſuch formulæ, which being fractions free from radicals, the variable is raiſed to any power in the denominator, which I will ſuppoſe to be compoſed of imaginary roots, becauſe in theſe only there is any difficulty. I ſay, that as often as the denominator is reducible to real components, in which the variable does not exceed the ſecond dimenſion, the formula may always be ſplit into ſo many fractions, as are the forementioned real components, each of which will be integrable, ſuppoſing the quadrature of the circle and hyperbola; and conſequently the propoſed formula will always be reducible to the ſaid quadratures. To do this, let there be propoſed this formula, $\frac{aa\dot{x}}{\overline{xx+ax+bb} \times \overline{xx+cx+bc}}$. Take a fictitious equation,

$$\frac{aa\dot{x}}{\overline{xx+ax+bb} \times \overline{xx+cx+bc}} = \frac{Ax\dot{x}+B\dot{x}}{xx+ax+bb} + \frac{Cx\dot{x}+D\dot{x}}{xx+cx+bc},$$

in which formula the capitals A, B, C, D, are conſtant arbitrary quantities, which are to be determined by the proceſs.

Thus, if the formula were $\frac{ab\dot{x}}{\overline{xx+ax+bb} \times \overline{xx \pm aa} \times \overline{x \pm c}}$, we ſhould make it equal to $\frac{Ax\dot{x}+B\dot{x}}{xx+ax+bb} + \frac{Cx\dot{x}+D\dot{x}}{xx \pm aa} + \frac{H\dot{x}}{x \pm c}$. And thus we may proceed in the ſame order, if the components in the denominator were more in number. When this is done, the terms of this equation are to be reduced to a common denominator, and laſtly, by tranſpoſition, the equation muſt be made equal to nothing. Then, by comparing the firſt terms to nothing, the value of the aſſumed quantity A may be found. And ſo, by comparing the ſecond, third, fourth, &c. terms in the ſame manner, the values of the other capitals B, C, D, &c. may be found, expreſſed by the given quantities of the propoſed formula; which values, being ſubſtituted in the places of the aſſumed capitals A, B, C, D, &c. in the equation, will ſupply us with ſo many fractions as are equivalent to the propoſed formula; and which, being reduced to a common denominator, will exactly reſtore the formula at firſt propoſed.

Of this we will take an example. Let it be propofed to find the integral of this formula $\frac{aa\dot{x}}{xx+2ax-aa \times xx+aa}$. Therefore I affume this fictitious equation $\frac{aa\dot{x}}{xx+2ax-aa \times xx+aa} = \frac{Ax\dot{x}+B\dot{x}}{xx+2ax-aa} + \frac{Cx\dot{x}+D\dot{x}}{xx+aa}$. Then I reduce the equation to a common denominator, and, by tranfpofing the term $aa\dot{x}$, I reduce it to o, and find it to be

$$\left.\begin{array}{l} Ax^3\dot{x} + Bx^2\dot{x} + Aaax\dot{x} + Baa\dot{x} \\ + Cx^3\dot{x} + Dx^2\dot{x} + 2Dax\dot{x} - Daa\dot{x} \\ \quad + 2Cax^2\dot{x} - Caax\dot{x} - aa\dot{x} \end{array}\right\} = 0.$$

Wherefore, from the comparifon of the firft terms with o, we fhall have $A + C = 0$, or $A = -C$. From the fecond, $B + D + 2Ca = 0$, that is, putting $-A$ inftead of C, $B = 2Aa - D$. From the third, $Aa^2 + 2Da - Ca^2 = 0$, that is, $C = a + \frac{2D}{a}$. From the laft, $Baa - Daa - aa = 0$, that is, putting, inftead of B, it's value given by D and A, it will be $D = Aa - \frac{1}{2}$, and therefore it will be $C = \frac{3Aa-1}{a}$; but $C = -A$, and therefore $A = \frac{1}{4a}$, $D = -\frac{1}{4}$, $B = \frac{3}{4}$, $C = -\frac{1}{4a}$; whence we fhall have at laft $\frac{aa\dot{x}}{xx+2ax-aa \times xx+aa} = \frac{x\dot{x}+3a\dot{x}}{4a \times xx+2ax-aa} - \frac{x\dot{x}+a\dot{x}}{4a \times xx+aa}$.

But, by making the fecond term of the denominator to vanifh, where there is occafion, the *homogeneum comparationis* is integrable by the quadrature of the circle and hyperbola; the integral of which, by the given rules, will be found to be $\frac{1}{4a} l\sqrt{xx+2ax-aa} + \frac{1}{2\sqrt{2aa}} l\sqrt{x+a-\sqrt{2aa}} - \frac{1}{2\sqrt{2aa}} l\sqrt{x+a+\sqrt{2aa}} - \frac{1}{4a} l\sqrt{xx+aa}$, fubtracting, befides, from thefe logarithms the fourth proportional of $4aa$, of unity, and of the arch of the circle, the radius of which is a, and the tangent $= x$. Therefore the integration of this formula depends on no higher quadratures than thofe of the circle and hyperbola.

41. If, befides, the fraction fhall be multiplied into any power of the variable, which power is pofitive; as if the formula were $\frac{aax^n\dot{x}}{xx+2ax-aa \times xx+aa}$; make it equal to $\frac{Ax^{n+1}\dot{x}+Bx^n\dot{x}}{xx+2ax-aa} + \frac{Cx^{n+1}\dot{x}+Dx^n\dot{x}}{xx+aa}$, and let the values of the capitals A, B, C, &c. be found in the fame manner as above, or you may work as if the faid power were not there; and the refulting fractions may be multiplied by the

 faid

faid power, and we fhall have, in like manner, fo many fractions, which will not require any higher quadratures than thofe of the circle and hyperbola, and which may be managed by the rules already given.

42. And if the power of the variable fhall be negative, that is, if it fhall be pofitive in the denominator, all the denominators of the refulting fractions may be multiplied by this power, and they will acquire the form following.

As, for example, $\dfrac{x^{-n}\dot{x}}{\overline{ax+ax+bb}\times\overline{xx\pm aa}\times\overline{x\pm c}}$. This being refolved as if x^{-n} were abfent, and then multiplying every term by x^{-n}, it will be $\dfrac{x^{-n}\dot{x}}{\overline{xx+ax+bb}\times\overline{xx\pm aa}\times\overline{x\pm c}} = \dfrac{Ax\dot{x}+B\dot{x}}{\overline{xx+ax+bb}\times x^{n}} + \dfrac{Cx\dot{x}+D\dot{x}}{\overline{xx\pm aa}\times x^{n}} + \dfrac{H\dot{x}}{\overline{x\pm c}\times x^{n}}$, underftanding now by the capitals fuch values, as, being found by the foregoing method, fhall make the fum of thefe fractions equal to the propofed formula.

The laft fraction will have no occafion for any particular artifice, becaufe it's integration is known by the common rules.

As to the firft, to clear up the example, let it be $A = aa$, and $B = abb$, whence it will be thus expreffed, $\dfrac{aax\dot{x}+abb\dot{x}}{\overline{xx+ax+bb}\times x^{n}}$, which is to be made equal to $\dfrac{Mx\dot{x}+N\dot{x}}{xx+ax+bb} + \dfrac{Px^{n-1}\dot{x}+Hx^{n-2}\dot{x}+Ex^{n-3}\dot{x},\ \&c.}{x^{n}}$. And thus we muft go on till the laft term becomes conftant, that is, the laft power of the variable x muft have it's index $= 0$. When thefe fractions are reduced to a common denominator, and all made $= 0$, we fhall have the values of the capitals, as was done before. The fame thing muft be done in regard to the other fraction $\dfrac{Cx\dot{x}+D\dot{x}}{\overline{xx\pm aa}\times x^{n}}$, and thus, finally, the integral will be found of the propofed formula.

Wherefore generally, fuppofing only the quadratures of the circle and hyperbola, we may always have the integral of the foregoing formula, if the components of the denominator be real, provided in them the unknown quantity do not exceed the fecond dimenfion.

43. But if the denominator of the propofed formula, or fraction, may not be refolvable into it's real components, in which the variable does not exceed two dimenfions, nor can be reduced to fuch by the common rules of Algebra; yet it may always be reduced to fuch by a little further artifice, as often as it is a convertible

convertible formula, or the product of feveral convertible terms. I fhall call that a convertible formula, in which the variable has the greateft exponent of it's dimenfions an even-affirmative number; as, fuppofe n were fuch, then the laft term would be a^n, and the terms equidiftant from that in the middle muft have the fame co-efficient, and be affected by the fame fign, fupplying the dimenfions by that conftant quantity, of which the laft term is formed. Such would be the formula $x^6 + a^6$, or this, $x^4 + bx^3 + ccxx + aabx + a^4$, or this other, $x^6 - bx^5 + b^3x^3 - a^4bx + a^6$. Now, if it were $x^5 + bx^4 + a^4x + a^4b$, it would be written in this equivalent form, $\overline{x^4 + a^4} \times \overline{x + b}$, in which $x^4 + a^4$ is a convertible formula, and $x + b$ is linear, which does not increafe the difficulty. The fame thing is to be underftood of infinite others.

A convertible formula, what.

44. Therefore now let us have $x^m - a^m$ to be refolved into it's real components, in which x may not exceed two dimenfions, and which fhall not have fractions for their exponents; and, in the firft place, let m be an even affirmative whole number. In this cafe, it will be divifible into $x^{\frac{1}{2}m} + a^{\frac{1}{2}m}$ and $x^{\frac{1}{2}m} - a^{\frac{1}{2}m}$, without any fractions in the exponents, becaufe of m being an even whole number. The firft divifor may be refolved by the rules which will be foon given for the binomial $x^m + a^m$. The fecond, $x^{\frac{1}{2}m} - a^{\frac{1}{2}m}$, if $\frac{1}{2}m$ fhall be an even number, may be again refolved into $x^{\frac{1}{4}m} + a^{\frac{1}{4}m}$ and $x^{\frac{1}{4}m} - a^{\frac{1}{4}m}$, without a fraction in the exponents. But, if $\frac{1}{2}m$ fhall be an odd number, it will be refolved by the rules that will be prefcribed for the binomial $x^m - a^m$, when m is an odd number.

In the fecond place, let it be $x^m + a^m$, and let m be an even affirmative whole number, in which cafe the formula is convertible. Let us fuppofe $x^m + a^m = 0$, and then let there be formed a convertible formula, in which the greateft exponent of x may be $m - 2$, and which may have all it's terms, and the laft term may be a^{m-2}, and the co-efficient of the fecond term may be b, for example, that of the third cc, that of the fourth d^3, and fo on; and let this be compared to o, whence refults an equation. Let this equation be multiplied by $xx + fx + aa$; the product will be another convertible equation, in which the greateft exponent of x will be $= m$. Let this equation be compared, term by term, with the fictitious equation $x^m + a^m = 0$, in which the co-efficients of the intermediate terms are $= 0$; and, by the comparifon of the fecond terms having the value of the affumed quantity b, from the comparifon of the third terms the value of cc, from that of the fourth terms the value of d^3, and fo on to the middle term, taking this in alfo; now, from that of the middle the other

other equations will become the ſame, becauſe of their being convertible equations which are compared. From this laſt term will be found the value of f expreſſed by an equation, which will have $\frac{1}{2}m$ for the number of it's dimenſions, of which all the roots will be real, and will give us the values of f; which being ſubſtituted in the trinomial $xx + fx + aa$, will give us ſo many trinomials, the products of which will reſtore the propoſed binomial $x^m + a^m$.

Let the example be $x^4 + a^4$. I take a convertible equation of the ſecond degree, $xx + bx + aa = 0$, which I multiply by $xx + fx + aa = 0$, from whence I have another convertible equation,

$$\left.\begin{array}{l} x^4 + bx^3 + 2aax^2 + aafx + a^4 \\ \quad + fx^3 + \ \ bfx^2 + aabx \end{array}\right\} = 0.$$

I compare this with the fictitious equation $x^4 + a^4 = 0$, and from the compariſon of the ſecond terms I find $b + f = 0$, or $b = -f$. From the compariſon of the middle terms I find $2aa + bf = 0$, and, inſtead of b, ſubſtituting it's value $-f$, it will be $ff - 2aa = 0$, or $f = \pm\sqrt{2aa}$.

Let it be $x^6 + a^6$. I take the convertible equation $x^4 + bx^3 + c^2x^2 + a^2bx + a^4 = 0$, which I multiply by $x^2 + fx + aa = 0$, and the reſulting equation is

$$\left.\begin{array}{l} x^6 + bx^5 + ccx^4 + 2aabx^3 + a^4x^2 \ \ + a^4fx + a^6 \\ \quad + fx^5 + bfx^4 + \ fccx^3 + a^2bfx^2 + a^4bx \\ \quad\quad\quad + a^2x^4 \quad\quad\quad\quad + a^2c^2x^2 \end{array}\right\} = 0.$$

I compare this with the fictitious equation $x^6 + a^6 = 0$, and from the compariſon of the ſecond terms I find $b + f = 0$; from the compariſon of the third terms I find $cc + bf + aa = 0$, that is, ſubſtituting the value of b, $cc - ff + aa = 0$; from the compariſon of the middle terms I find $2aab + fcc = 0$, that is, inſtead of b and cc, ſubſtituting their values, $f^3 - 3aaf = 0$.

Now, by actually performing theſe operations, we ſhall find that

If $m = 4$, it will be $ff - 2aa = 0$.

If $m = 6$, then $f^3 - 3aaf = 0$.

If $m = 8$, then $f^4 - 4aaf^2 + 2a^4 = 0$.

If $m = 10$, then $f^5 - 5aaf^3 + 5a^4f = 0$.

If $m = 12$, then $f^6 - 6aaf^4 + 9a^4f^2 - 2a^6 = 0$.

If $m = 14$, then $f^7 - 7aaf^5 + 14a^4f^3 - 7a^6f = 0$.

And ſo we might proceed to the other even values of m.

Inſtead

Inftead of $x^4 + a^4$, let it be $x^4 + 2bx^3 + 2aabx + a^4$, which is alfo a convertible formula. I multiply the convertible equation $xx + hx + aa = 0$ by $xx + fx + aa = 0$, and I fhall have, as above,

$$\left.\begin{array}{l} x^4 + hx^3 + 2aax^2 + aafx + a^4 \\ \quad + fx^3 + \;\; hfx^2 + aahx \end{array}\right\} = 0.$$

I compare this with the fictitious equation $x^4 + 2bx^3 + 2aabx + a^4 = 0$, and from the comparifon of the fecond terms I find $h + f = 2b$, that is, $h = 2b - f$; from the comparifon of the middle terms I find $2aa + hf = 0$, and, inftead of h, fubftituting it's value, we fhall have $2aa + 2bf - ff = 0$, that is, $ff - 2bf - 2aa = 0$.

Let it be $x^6 + a^3x^3 + a^6$. I take the convertible equation $x^4 + bx^3 + ccx^2 + aabx + a^4 = 0$, which I multiply by $xx + fx + aa$, and I fhall have this product,

$$\left.\begin{array}{l} x^6 + bx^5 + ccx^4 + 2aabx^3 + a^4x^2 \;\; + a^4fx + a^6 \\ \quad + fx^5 + bfx^4 + \;\; ccfx^3 + a^2bfx^2 + a^4bx \\ \qquad\quad + aax^4 \qquad\qquad\quad + a^2c^2x^2 \end{array}\right\} = 0.$$

This being compared with the equation $x^6 + a^3x^3 + a^6 = 0$, I find, from the comparifon of the fecond terms, $b + f = 0$; from the comparifon of the third terms, $cc + bf + aa = 0$; and, inftead of b, putting it's value, it will be $cc - ff + aa = 0$; from the comparifon of the middle terms, $2aab + ccf = a^3$; and, inftead of b and cc, putting their values, it will be $f^3 - 3aaf - a^3 = 0$. And fo for as many others as you pleafe.

Now let us have $x^4 + 2bx^3 + 2aabx + a^4$ to refolve into it's real components, in which x has no fraction for it's exponent, and does not exceed the fecond dimenfion. The equation which fhould give us the values of f is therefore $ff - 2bf = 2aa$, from which we obtain both the real values of f, that is, $f = b + \sqrt{2aa + bb}$, and $f = b - \sqrt{2aa + bb}$. Wherefore, fubftituting each of thefe values inftead of f, in the trinomial $xx + fx + aa$, we fhall find that $x^4 + 2bx^3 + 2aabx + a^4$ is the product of the two real components $xx + bx + x\sqrt{2aa + bb} + aa$, and $xx + bx - x\sqrt{2aa + bb} + aa$.

Thus, if it were $x^6 + aax^4 + a^4x^2 + a^6 = 0$. The equation which gives the values of f being $f^3 - 2aaf = 0$, from thence we fhall have the values of f all real, that is, $f = 0$, $f = \sqrt{2aa}$, and $f = -\sqrt{2aa}$; fo that $x^6 + aax^4 + a^4x^2 + a^6$ is the product of the three real components $xx + aa$, $xx + x\sqrt{2aa} + aa$, and $xx - x\sqrt{2aa} + aa$.

Let us have $x^{10} + a^{10}$. The equation which ought to give the values of f is $f^5 - 5aaf^3 + 5a^4f = 0$. From whence we derive the values of f all real, that

that is, $f = 0$, $f = a\sqrt{\frac{5+\sqrt{5}}{2}}$, $f = -a\sqrt{\frac{5+\sqrt{5}}{2}}$, $f = a\sqrt{\frac{5-\sqrt{5}}{2}}$, and $f = -a\sqrt{\frac{5-\sqrt{5}}{2}}$. Wherefore, ſubſtituting every one of theſe values inſtead of f in the trinomial $xx + fx + aa$, we ſhall find that $x^{10} + a^{10}$ is the product of theſe five real components, $xx + aa$, $xx + ax\sqrt{\frac{5+\sqrt{5}}{2}} + aa$, $xx - ax\sqrt{\frac{5+\sqrt{5}}{2}} + aa$, $xx + ax\sqrt{\frac{5-\sqrt{5}}{2}} + aa$, and $xx - ax\sqrt{\frac{5-\sqrt{5}}{2}} + aa$. Whence it is to be concluded, that the integral of any differential formula, whoſe numerator is $\dot{x}$ multiplied into any conſtant quantity, and the denominator is of a like nature with theſe here conſidered, will not depend on quadratures higher than thoſe of the circle and hyperbola, and may be had from the rules here given.

45. Now let $x^m \pm a^m$ be given to reſolve as above, and let m be any affirmative integer, but odd.

The formula may be divided by $x \pm a$, and the quotient (which in the firſt caſe will be $x^{m-1} - ax^{m-2} + a^2x^{m-3} - a^3x^{m-4}$, &c. to the laſt term, which will be $+ a^{m-1}$; and, in the ſecond caſe, it will be $x^{m-1} + ax^{m-2} + a^2x^{m-3} + a^3x^{m-4}$, &c. to the laſt term, which will be $+ a^{m-1}$,) may be ſuppoſed $= 0$; and let this fictitious equation, which is a convertible one, be compared, term by term, with the product of a convertible equation, in which the number of dimenſions of the variable x is $m - 3$, into the trinomial $xx + fx + aa$; and, from the compariſon of the ſecond terms, we ſhall have the value of the aſſumed quantity, for example b; from the third the value of cc, from the fourth the value of d^3, &c.; and laſtly, from the compariſon of the middle terms, we may derive the values of f, expreſſed by an equation of which the number of dimenſions will be $\frac{m-1}{2}$. All the roots of which will be real, and will determine the values of f all real; which, being ſubſtituted in the trinomial $xx + fx + aa$, will ſupply us with ſo many trinomials, which, multiplied together, and alſo by $x \pm a$, will reſtore the propoſed formula $x^m \pm a^m$.

By this method we may find the following equations, which will ſerve for the reſolution of the binomial $x^m + a^m$, when m is an odd, integer, and poſitive number.

If

If $m = 3$, it will be $f + a = 0$.

If $m = 5$, then $ff + af - aa = 0$.

If $m = 7$, then $f^3 + aff - 2aaf - a^3 = 0$.

If $m = 9$, then $f^4 + af^3 - 3aaff - 2a^3f + a^4 = 0$.

If $m = 11$, then $f^5 + af^4 - 4aaf^3 - 3a^3f^2 + 3a^4f + a^5 = 0$.

If $m = 13$, then $f^6 + af^5 - 5a^2f^4 - 4a^3f^3 + 6a^4f^2 + 3a^5f - a^6 = 0$.

And thus we might proceed to find the other values of f, if m be an odd number.

If the propofed formula were $x^m - a^m$, and m were an odd integer affirmative number, dividing by $x - a$ as before, the fame equations would be had, only changing the figns in the fecond, fourth, and fixth term, and in all others in even places.

46. If, inftead of $x^m \pm a^m$, fuppofing m to be any odd affirmative integer, the formula were any other, but fuch, as that, dividing by $x \pm$ fome conftant quantity, that which refults fhould be a convertible formula; as $x^5 + bx^4 - aax^3 - aabx^2 + a^4x + a^4b$, which, being divided by $x + b$, will give $x^4 - aax^2 + a^4$; this laft being managed as ufual, and the values of f found and fubftituted in the trinomial $xx + fx + aa$, we fhould have fo many trinomials, which being multiplied together, and alfo by $x + b$, would reftore the propofed formula.

Let it be required, for example, to refolve $x^5 + a^5$ into it's real components, in which x may have no fractional exponents, and may not exceed the fecond dimenfion. The equation which is to give the values of f (according to what goes before) will be $ff + af - aa = 0$, from whence we derive thefe values of f, $f = \frac{-a \pm a\sqrt{5}}{2}$. Thefe being fubftituted, inftead of f, in the trinomial $xx + fx + aa$, we fhall have the two real trinomials $xx - \frac{1}{2}ax + \frac{1}{2}ax\sqrt{5} + aa$, and $xx - \frac{1}{2}ax - \frac{1}{2}ax\sqrt{5} + aa$, the product of which, together with $x + a$, will reftore the formula propofed.

Let it be required to refolve into real components the formula $x^5 + bx^4 - aax^3 - aabx^2 + a^4x + a^4b$, which, being divided by $x + b$, will give $x^4 - aax^2 + a^4$. The equation that gives us f will be $ff = 3aa$, and the values of f will be $f = \pm\sqrt{3aa}$. Thefe being fubftituted inftead of f in the trinomial $xx + fx + aa$, we fhall have thefe two real trinomials $xx + x\sqrt{3aa} + aa$, and $xx - x\sqrt{3aa} + aa$; the product of which, together with $x + b$, will reftore the formula propofed.

 47. From

47. From hence I conclude, that the integral of any differential formula whatever, the numerator of which is $\dot{x}$ into any conſtant quantity, and the denominator of a nature like to theſe here conſidered, will not depend on quadratures higher than thoſe of the circle and hyperbola, and which may be obtained by the rules here given.

48. But, becauſe in higher dimenſions the value of f cannot be obtained by actual ſeparation, from the equations before cited; in ſuch caſes it will be enough to have recourſe to the geometrical conſtruction of the ſame equations. Thus, to find the components of $x^7 + a^7$, and thence the integral of the formula $\frac{\dot{x}}{x^7 + a^7}$, the denominator being divided by $x + a$, the quotient will be $x^6 - ax^5 + aax^4 - a^3x^3 + a^4x^2 - a^5x + a^6$. The values of f for the reſolution of this formula muſt be furniſhed by the equation $f^3 + af^2 - 2aaf - a^3 = 0$. Wherefore, by the uſual methods of Algebra, by means of the interſections of two curves, or by any other way, having found the values of f affirmative and negative, which are to be all real; for example, let one be A, another — B, the other — C; the quantity $x^7 + a^7$ will be the product of $x + a$ into $xx + \mathrm{A}x + aa$ into $xx - \mathrm{B}x + aa$ into $xx - \mathrm{C}x + aa$; and the quantities A, B, C, will be real and given. Then we may proceed to the integration of the formula $\frac{\dot{x}}{x^7 + a^7}$, by the quadrature only of the circle and hyperbola.

49. By the ſame artifice by which we find the equations for the reſolution of the binomial $x^m \pm a^m$, we may find them for the reſolution of the trinomial $x^{2m} \pm 2aax^m + aa$, ſuppoſing $2m$ to be an even affirmative integral number. And thus, in general, as often as it is propoſed to reſolve a formula which is convertible, or is the product of a convertible into a linear quantity, and which has not a fraction in the exponents; they may always be reduced by the method here explained.

The caſe of the product of a convertible formula into a linear, we ſhall have when m is an odd number, and otherwiſe. Let this be an example, $x^8 + b^4x^4 - a^4x^4 - a^4b^4$, that is, $\overline{x^4 + b^4} \times \overline{x^4 - a^4}$, or $\overline{x^4 + b^4} \times \overline{xx + aa} \times \overline{xx - aa}$. Wherefore, the diviſor $x^4 + b^4$ being reſolved into it's real components of two dimenſions, which may be, for example, $xx + \mathrm{A}x + bb$, and $xx + \mathrm{B}x + bb$, it will be $\overline{x^4 + b^4} \times \overline{x^4 - a^4} = \overline{xx + \mathrm{A}x + bb} \times \overline{xx + \mathrm{B}x + bb} \times \overline{xx + aa} \times \overline{xx - aa}$. And if it had been $\overline{x^4 + b^4} \times \overline{x^4 + a^4}$, then, by the reſolution of $x^4 + a^4$ into $xx + \mathrm{C}x + aa$, and $xx + \mathrm{D}x + aa$, it would be $\overline{x^4 + b^4} \times \overline{x^4 + a^4} = \overline{xx + \mathrm{A}x + bb} \times \overline{xx + \mathrm{B}x + bb} \times \overline{xx + \mathrm{C}x + aa} \times \overline{xx + \mathrm{D}x + aa}$.

50. To

50. To have the integral of the formula $\frac{ma^m \dot{x}}{x^m \pm a^m}$, in which m denotes any affirmative integer number, let A, B, C, &c. represent the several values of f with their signs, which serve for the resolution of the denominator $x^m \pm a^m$. And it must be observed, that of these values one may sometimes be $= 0$, which will obtain as often as m is a term in this series 4, 8, 12, 16, &c. it being $x^m - a^m$ in the given formula. And as often as m is a term in this series 2, 6, 10, 14, 18, &c. when it is $x^m + a^m$. This being supposed, the integral required will be $\pm \frac{A}{a} l\sqrt{xx + Ax + aa} \pm \frac{B}{a} l\sqrt{xx + Bx + aa}$ $\pm \frac{C}{a} l\sqrt{xx + Cx + aa}$, &c. taking these logarithms from the logarithmic curve, the subtangent of which is $= a$; adding to, or subtracting from this aggregate of logarithmic terms, (according as the sign of the term a^m in the denominator shall be + or −,) twice the sum of so many arches of a circle, as are the values A, B, C, &c. of which arches these are the radii in order, $\sqrt{aa - \frac{1}{4}AA}$, $\sqrt{aa - \frac{1}{4}BB}$, $\sqrt{aa - \frac{1}{4}CC}$, &c. and the tangents are in the same order, $x + \frac{1}{2}A$, $x + \frac{1}{2}B$, $x + \frac{1}{2}C$, &c. Such will be the integral of the formula $\frac{ma^m \dot{x}}{x^m + a^m}$, if m shall be an even affirmative number. But in the same formula, if m shall be an odd affirmative number, it will be necessary to add to the whole the logarithm of $x + a$, because the denominator has also the real root $x + a$. And if the formula should be $\frac{ma^m \dot{x}}{x^m - a^m}$, m being an odd affirmative number; instead of the logarithm of $x + a$, that of $x - a$ must be added. And lastly, the formula being $\frac{ma^m \dot{x}}{x^m - a^m}$, and m being an even affirmative number, it will be necessary to add the logarithm of $x - a$, and to subtract that of $x + a$; still taking these logarithms from the logarithmic with subtangent $= a$.

51. But if in the proposed formula $\frac{\dot{x}}{x^m \pm a^m}$ the number m should be a negative number, that is, if it were $\frac{\dot{x}}{x^{-m} \pm a^{-m}}$, it would be expressed thus,

$\frac{\dot{x}}{\frac{1}{x^m} \pm \frac{1}{a^m}}$, which, reduced to a common denominator, is equivalent to this, $\frac{a^m x^m \dot{x}}{a^m \pm x^m}$; and dividing the numerator by the denominator till the greateſt power of the variable is leſs in this than in that, we ſhall have at laſt $\pm a^m \dot{x} \pm \frac{a^{2m} \dot{x}}{x^m \pm a^m}$, in which m will be a poſitive number. And what has been ſaid before will alſo take place, in the formula $\frac{\dot{x}}{x^m \pm a^m}$, when m is an integer negative number.

52. Moreover, if the fraction $\frac{\dot{x}}{x^m \pm a^m}$ be ſuppoſed to be multiplied by x^n, n being an integer number either affirmative or negative, the denominator being reſolved into it's real components, in which x does not exceed the ſecond dimenſion; this will be the caſe already conſidered by me at § 41, 42, and is therefore reducible to the quadrature of the circle and hyperbola.

53. But when n is negative, it may be reduced more expeditiouſly thus. Firſt, let n be leſs than m. The formula $\frac{\dot{x}}{\overline{x^m + a^m} \times x^n}$ may be thus expreſſed by equivalents, $\frac{\dot{x}}{a^m x^n} - \frac{x^{m-n} \dot{x}}{a^m \times \overline{x^m + a^m}}$. And likewiſe, the formula $\frac{\dot{x}}{\overline{x^m - a^m} \times x^n}$ by $-\frac{\dot{x}}{a^m x^n} + \frac{x^{m-n} \dot{x}}{a^m \times \overline{x^m - a^m}}$. Secondly, let n be greater than m. The formula $\frac{\dot{x}}{\overline{x^m + a^m} \times x^n}$ may be expreſſed by the equivalent ſeries $\frac{\dot{x}}{a^m x^n} - \frac{\dot{x}}{a^{2m} x^{n-m}} + \frac{\dot{x}}{a^{3m} x^{n-2m}} - \frac{\dot{x}}{a^{4m} x^{n-3m}}$, &c. till we come to that term, in which the exponent of x is but juſt greater than m; $\pm \frac{\dot{x}}{\overline{x^m + a^m} \times a^r x^t}$. Here the ſign muſt be $+$ or $-$, according as the alternate change of the ſigns ſhall require; and r is the ſame exponent of the quantity a, as in the antecedent term, and t is the remainder of the diviſion made of the number n by the number m, taken as often as it can be done.

Now

Now if it were $\frac{\dot{x}}{\overline{x^m - a^m} \times x^n}$, ſuppoſing n to be greater than m; all the terms of the ſeries ought to be affected by the negative ſign, and the term out of the ſeries, that is, $\frac{\dot{x}}{\overline{x^m - a^m} \times a^r x^t}$, ought always to have the affirmative ſign prefixed. Thus, if the formula were $\frac{\dot{x}}{\overline{x^3 + a^3} \times x^5}$, it would be equivalent to $\frac{\dot{x}}{a^3x^5} - \frac{\dot{x}}{\overline{x^3 + a^3} \times a^3x^2}$. But we know that $-\frac{\dot{x}}{\overline{x^3 + a^3} \times a^3x^2}$ is equal to $-\frac{\dot{x}}{a^6x^2}$ $+ \frac{x\dot{x}}{a^6 \times \overline{x^3 + a^3}}$. Therefore it will be $\frac{\dot{x}}{\overline{x^3 + a^3} \times x^5} = \frac{\dot{x}}{a^3x^5} - \frac{\dot{x}}{a^6x^2} + \frac{x\dot{x}}{a^6 \times \overline{x^3 + a^3}}$; all which are quantities that may be managed by the given rules.

54. But if m ſhall be a fraction either affirmative or negative, let t be the numerator of the fraction which is equal to m, and reduced to the ſimpleſt terms, and let p be the denominator of the ſame: ſo that the given formula may be thus expreſſed, $\frac{\dot{x}}{x^{\frac{t}{p}} \pm a^{\frac{t}{p}}}$. Put $x = y^p$, and $a = b^p$, and the formula will be converted into this, $\frac{py^{p-1}\dot{y}}{y^t \pm b^t}$, which has no fractions for it's exponents, and may therefore be reſolved by the given rules.

Let the formula be, for example, $\frac{\dot{x}}{x^{\frac{3}{2}} \pm a^{\frac{3}{2}}}$; make $x = yy$, $a = bb$, and it will be $\dot{x} = 2y\dot{y}$; and making the ſubſtitutions, the formula will be changed into $\frac{2y\dot{y}}{y^3 \pm b^3}$, which has no fractions for it's exponents.

55. Now if the given formula be $\frac{x^n\dot{x}}{x^m \pm a^m}$, in which m and n are broken numbers; making r the numerator of the fraction n, and p the denominator of the ſame; and thus making t the numerator of the fraction m, and q it's denominator, (ſuppoſing theſe fractions to be reduced to their ſmalleſt terms,) the formula will be $\frac{x^{\frac{r}{p}}\dot{x}}{x^{\frac{t}{q}} \pm a^{\frac{t}{q}}}$, in which r, p, q, t, will be integer numbers, poſitive or negative.

Now

Now let it be made $x = y^{pq}$, and $a = b^{pq}$; the formula will be converted into this, $\frac{pqy^{qr+pq-1}\dot{y}}{y^{pt} \pm b^{pt}}$, which has no fractions in it's exponents. Let it be, for example, the formula $\frac{x^{\frac{3}{2}}\dot{x}}{x^{\frac{4}{5}} \pm a^{\frac{4}{5}}}$; make $x = y^{10}$, $a = b^{10}$; it will be $\dot{x} = 10y^9\dot{y}$, $x^{\frac{3}{2}} = y^{15}$, $x^{\frac{4}{5}} = y^8$; and making the substitutions, the formula will be changed into $\frac{10y^{24}\dot{y}}{y^8 \pm b^8}$, which has no fractional exponents.

56. Lastly, if the formula shall be $\frac{x^n\dot{x}}{\overline{x^m \pm a^m}|^u}$, the exponents n, m, u, being positive integers, we may always have it's integral, supposing only the quadratures of the circle and hyperbola. And the integral will be composed of algebräical quantities, and of one fluential quantity; which will be done in the following manner.

Suppose the formula $\int \frac{x^n\dot{x}}{\overline{x^m \pm a^m}|^u} =$ $\frac{Bx^{n+um-2m+1} + Cx^{n+um-2m} + Dx^{n+um-2m-1}\ \&c}{\overline{x^m \pm a^m}|^{u-1}}$ as far as to a constant term, or to that term in which the exponent of x is o, and let this be K; then must be added $A\int \frac{x^n\dot{x}}{x^m \pm a^m}$; that is, it must be made $\int \frac{x^n\dot{x}}{\overline{x^m \pm a^m}|^u} =$ $\frac{Bx^{n+um-2m+1} + Cx^{n+um-2m} + Dx^{n+um-2m-1}\ \&c + K}{\overline{x^m \pm a^m}|^{u-1}} + A\int \frac{x^n\dot{x}}{x^m \pm a^m}$.

Difference the equation, make it = o, and set the terms in order. From making the first terms = o we shall find the value of the assumed quantity B. Making the second terms = o, we shall have the value of C. And so, one by one, the values of the others; which values being substituted instead of the capitals, as the fluent of $\frac{x^n\dot{x}}{x^m \pm a^m}$ will depend only on the quadratures of the circle and hyperbola, and the other terms in the *homogeneum comparationis* are purely algebräical, so the proposed formula will require no higher quadratures.

57. Sometimes it may happen, that some one of the co-efficients B, C, D, &c. may come out arbitrary, or to be determined at pleasure; but it will be

only when n is greater than $m - 1$. And it may alſo be obſerved, that as often as it is $m = n + 1$, the co-efficient A will be found $= 0$, and conſequently the integral of the propoſed formula will be algebräical.

58. But if, in the propoſed differential formula, the exponent n ſhould be a negative integer, ſo that it might be reduced to $\frac{\dot{x}}{x^n \times \overline{x^m \pm a^m}|^u}$; in which it is now poſitive; the integral would be

$$\frac{Bx^{um-2m} + Cx^{um-2m-1} + Dx^{um-2m-2}, \&c. + K}{x^{n-1} \times \overline{x^m \pm a^m}|^{u-1}} + A\int \frac{\dot{x}}{x^n \times \overline{x^m \pm a^m}}.$$ Which

co-efficients B, C, D, &c. will be determined in the ſame manner as before.

As, for example, $\frac{x\dot{x}}{\overline{x^3 + a^3}|^2}$; in which caſe we have $n = 1$, $m = 3$, $u = 2$. Wherefore it will be $\int \frac{x\dot{x}}{\overline{x^3+a^3}|^2} = \frac{Bx^2 + Cx + K}{x^3 + a^3} + A\int \frac{x\dot{x}}{x^3 + a^3}$. And taking the fluxions, $\frac{x\dot{x}}{\overline{x^3+a^3}|^2} = \frac{\overline{2Bx\dot{x} + C\dot{x}} \times \overline{x^3 + a^3} - 3x^2\dot{x} \times \overline{Bx^2 + Cx + K}}{\overline{x^3+a^3}|^2} + \frac{Ax\dot{x}}{x^3 + a^3}$. Then reducing to a common denominator, ſetting the equation in order, and making it equal to o, it will be

$$\left.\begin{array}{llll} 2Bx^4\dot{x} & + \ Cx^3\dot{x} & - 3Kx^2\dot{x} & + 2Ba^3x\dot{x} + Ca^3\dot{x} \\ - 3Bx^4\dot{x} & - 3Cx^3\dot{x} & & + Aa^3x\dot{x} \\ + Ax^4\dot{x} & & & - \quad x\dot{x} \end{array}\right\} = 0.$$

Now making the firſt, ſecond, third, &c. terms $= 0$ ſucceſſively, we ſhall find $A - B = 0$, or $B = A$; $C = 0$, $K = 0$; $2Ba^3 + Aa^3 - 1 = 0$, or $Aa^3 = 1 - 2Ba^3$; and putting A inſtead of B, it will be $A = \frac{1}{3a^3} = B$. Whence, laſtly, it is $\int \frac{x\dot{x}}{\overline{x^3+a^3}|^2} = \frac{xx}{3a^3 \times \overline{x^3+a^3}} + \frac{1}{3a^3} \times \int \frac{x\dot{x}}{x^3 + a^3}$. But $\int \frac{x\dot{x}}{x^3 + a^3}$ $= \frac{1}{3aa} l\sqrt{\overline{xx - ax + aa}} - l\overline{x + a}$; together with $\frac{2}{3aa}$ multiplied into the arch of a circle with radius $= \sqrt{\frac{3}{4}aa}$, and tangent $= x - \frac{1}{2}a$. So that it will be $\int \frac{x\dot{x}}{\overline{x^3+a^3}|^2} = \frac{xx}{3a^3 \times \overline{x^3+a^3}} + \frac{1}{9a^5} \times l\sqrt{\overline{xx - ax + aa}} - \frac{1}{9a^5} \times l\overline{x + a}$ $+ \frac{2}{9a^5} \times$ arch of a circle with radius $\sqrt{\frac{3}{4}aa}$, and tangent $= x - \frac{1}{2}a$: taking the logarithms from the logarithmic with ſubtangent $= a$.

59. But

59. But if the exponent m be negative, the formula muſt be changed into another that is equivalent to it, in which the exponent is poſitive; according to the manner ſhown at § 51 of this Book.

60. And if both m and n ſhould be fractions, the ſubſtitutions muſt be made according to § 55 of this Book.

61. Again, if the exponent u were not an integer, but a fraction either affirmative or negative, it will ſuffice that the formula be one of thoſe caſes conſidered at § 39. Foraſmuch as it may be tranſmuted into another form, which is capable of being managed by the given rules.

Thus the formula $\dfrac{x^n\dot{x}}{\overline{x^m \pm a^m}|^u}$, the exponents n, m, u, being poſitive or negative integers, or elſe rational fractions of any kind, with the ſigns + and — at pleaſure; it will be integrable, or, at leaſt, may be reduced to known quadratures, as often as the ſaid exponents ſhall have ſuch a relation to one another, that one of theſe two quantities compoſed of them, that is, $u - \frac{1}{m} - 1 - \frac{n}{m}$, or $\frac{1}{m} - 1 + \frac{n}{m}$, ſhall be equal to any integer number. If this integer number ſhall be poſitive, the formula will admit of an algebräical integration, except the caſes in which the power $x^{-1}\dot{x}$ ſhall intrude, which obliges us to recur to the logarithms. If this integer number ſhall be negative, the formula will be reduced to the quadrature of the circle, or of the hyperbola.

To obtain our purpoſe as to the firſt caſe, in which $u - \frac{1}{m} - \frac{n}{m} - 1$ is equal to an integer, make $x^m + a^m = zx^m$; then $x^m = \dfrac{a^m}{z-1}$, $x = \dfrac{a}{\overline{z-1}|^{\frac{1}{m}}}$, $x^n = \dfrac{a^n}{\overline{z-1}|^{\frac{n}{m}}}$, $x^{n+1} = \dfrac{a^{n+1}}{\overline{z-1}|^{\frac{1+n}{m}}}$; and therefore $x^n\dot{x} = -\dfrac{a^{n+1}\dot{z}}{m} \times \overline{z-1}|^{\frac{-n-1-m}{m}}$. But $x^m + a^m = zx^m = \dfrac{a^m z}{z-1}$, and $\overline{x^m + a^m}|^u = \dfrac{a^{mu}z^u}{\overline{z-1}|^u}$. Therefore, making the neceſſary ſubſtitutions in the propoſed formula, it will be $-\dfrac{a^{n+1-mu}\dot{z}z^{-u}}{m} \times \overline{z-1}|^{\frac{-n-1}{m}-1+u}$, which is plainly ſeen to be algebräically integrable, (except the excepted caſe,) when $\frac{-n-1}{m} - 1 + u$ is equal

to a poſitive integer number. And that if $\frac{-n-1}{m} - 1 + u$ is an integer number, but negative, by what is advanced in the foregoing articles, the integration of this formula will depend on no higher quadratures than thoſe of the circle and hyperbola.

I come now to the ſecond caſe, when $\frac{1}{m} - 1 + \frac{n}{m}$ is equal to an integer number. Make $x^m + a^m = z$, and then it will be $x^m = z - a^m$, $x = \overline{z - a^m}|^{\frac{1}{m}}$, $x^n = \overline{z - a^m}|^{\frac{n}{m}}$, $x^{n+1} = \overline{z - a^m}|^{\frac{1+n}{m}}$, $x^n \dot{x} = \frac{\dot{z}}{m} \times \overline{z - a^m}|^{\frac{n+1}{m} - 1}$. But $x^m + a^m = z$, and $\overline{x^m + a^m}|^u = z^u$; therefore, making the ſubſtitutions in the propoſed formula, it will become $\frac{\dot{z}}{m} \times \frac{\overline{z - a^m}|^{\frac{n+1}{m} - 1}}{z^u}$; or elſe $\frac{z^{-u}\dot{z}}{m} \times \overline{z - a^m}|^{\frac{n+1}{m} - 1}$, which is algebräically integrable, (excepting in the caſe excepted,) when $\frac{n+1}{m} - 1$ is equal to a poſitive integer, or a negative; for then the integration will depend on the known quadratures of the circle and hyperbola, as appears by the foregoing articles.

62. Now if the denominator of the propoſed fraction, raiſed to any integral power, ſhould not be a binomial, as has been conſidered hitherto, but ſhould be any multinomial whatever; provided it be reducible into it's real components, in which the variable does not exceed the ſecond dimenſion; either by means of convertible equations, or ſome other manner; the formula may always be reduced to known quadratures.

Let it be, for example, $\frac{\dot{x}}{\overline{xx + bx + aa}|^2 \times \overline{x + c}|^3}$; raiſing actually the powers of the denominator, make a fictitious equation thus:

$$\frac{\dot{x}}{\overline{xx+bx+aa}|^2 \times \overline{x+c}|^3} = \frac{Ax^3\dot{x} + Bx^2\dot{x} + Cx\dot{x} + D\dot{x}}{x^4 + 2bx^3 + 2aax^2 + bbx^2 + 2aabx + a^4} + \frac{Fx^2\dot{x} + Gx\dot{x} + H\dot{x}}{x^3 + 3cx^2 + 3ccx + c^3}.$$

Here are ſo many terms taken in general, as are the components of the denominator; and in theſe terms ſo many capitals, as is the higheſt power of the variable in it's reſpective denominator, multiplying alſothe firſt capital in each term by the higheſt power, leſſened by unity, of the variable in it's denominator, the ſecond capital by the ſame power diminiſhed by 2, and ſo on to the

laſt conſtant quantity. Theſe aſſumed conſtant quantities are to be determined in the uſual manner, and the firſt term will furniſh ſo many fractions divided by $\overline{xx+bx+aa}|^2$; in which denominator making the middle term to vaniſh, the fractions will be a particular caſe of the general canon $\frac{x^m \dot{x}}{\overline{x^n \pm a^n}|^u}$. And the ſecond term will give us ſo many fractions divided by $\overline{x+c}|^3$, which may be reduced to the uſual rule of denominators compounded of equal roots.

63. Moreover, if the numerator of the propoſed formula be multiplied by a poſitive or negative power of the variable; having found the values of the capitals, and operating as if the fraction had not been multiplied by any ſuch power; the reſulting terms may be multiplied by the ſaid power, and the reſt may be done as uſual.

64. I ſhall finiſh this Section by fulfilling my promiſe made to the reader, concerning the Method of Multinomials, of Sig. Count *James Riccati*, which is as follows.

By the name of Differential Multinomials I call ſuch fractions, as have for their numerators the fluxion $\dot{x}$, and for denominators an aggregate of powers, the exponents of which conſtitute an arithmetical progreſſion, which proceeds till it terminates in nothing. And till this condition is fulfilled, the abſent terms muſt be ſupplied, and their co-efficients made equal to nothing. Suppoſe we had this expreſſion $\frac{\dot{x}}{x^{\frac{1}{2}} + x^{\frac{1}{3}} + a}$. At firſt view it might ſeem to be a trinomial, but is really a quadrinomial, and is thus to be compleated: $\frac{\dot{x}}{x^{\frac{3}{6}} + x^{\frac{2}{6}} + 0x^{\frac{1}{6}} + a}$.

In any multinomial expreſſed by a fraction, the denominator of which is raiſed to the power p, being a poſitive integral number, there is a method which would be general, if it were not frequently made uſeleſs by the intervention of imaginary quantities. But there are ſome particular artifices, which often come opportunely to our aſſiſtance.

I begin with the trinomial $\frac{\dot{x}}{\overline{x^{2m} + ax^m + b}|^p} = \dot{y}$, becauſe to ſuch an expreſſion as this every trinomial may eaſily be reduced. Make $x^m = z + A$, where z is a new variable aſſumed, and A is a conſtant to be afterwards determined. The neceſſary computations being made, to arrive at the ſubſtitutions we ſhall have as follows,

x^{2m}

$$
\begin{aligned}
x^{2m} &= zz + 2Az + AA, \text{ and consequently} \\
ax^{m} &= az + aA \\
b &= b \\
\overline{x^{2m} + ax^{m} + b}\,\rceil^{p} &= \overline{zz + \overline{2A + a} \times z + AA + aA + b}\,\rceil^{p}.
\end{aligned}
$$

It ought to be contrived in ſuch manner, that the quantities $AA + aA + b$ may diſappear, by putting them $= 0$, and in caſes in which A is no imaginary quantity, this reduction ſucceeds very well. It is therefore $x^{m} = z + A$; and taking the fluxions, $mx^{m-1}\dot{x} = \dot{z}$, and $x = \overline{z + A}\,\rceil^{\frac{1}{m}}$. Then $\dot{x} = \dfrac{\dot{z}}{mx^{m-1}} = \dfrac{\dot{z}}{m \times \overline{z + A}\,\rceil^{\frac{m-1}{m}}}$.

In proceeding to the neceſſary ſubſtitutions, in our principal formula, inſtead of x and it's powers, are to be ſubſtituted the aſſumed variable z, with it's functions; and we ſhall find $\dfrac{\dot{x}}{\overline{x^{2m} + ax^{m} + b}\,\rceil^{p}} = \dfrac{\dot{z}}{m \times \overline{z + A}\,\rceil^{\frac{m-1}{m}} \times \overline{zz + \overline{2A + a} \times z}\,\rceil^{p}}$; and freeing it from the quantity z, which multiplies the binomial $z + \overline{2A + a}$ under the vinculum, it will be $\dfrac{\dot{x}}{\overline{x^{2m} + ax^{m} + b}\,\rceil^{p}} = \dfrac{z^{-p}\dot{z}}{m \times \overline{z + A}\,\rceil^{\frac{m-1}{m}} \times \overline{z + 2A + a}\,\rceil^{p}}$.

The moſt ſimple caſe is, when the exponent p is equal to unity, the other being when m is any number, integer or fraction, affirmative or negative; and, for brevity, making $2A + a = g$, the general expreſſion, [when $p = 1$,] will become this particular one, $\dfrac{z^{-1}\dot{z}}{g \times \overline{z + A}\,\rceil^{\frac{m-1}{m}} + z \times \overline{z + A}\,\rceil^{\frac{m-1}{m}}} = m\dot{y}$.

I make a firſt diviſion by dividing the numerator of the fraction by it's denominator, and the firſt quotient will be $\dfrac{z^{-1}\dot{z}}{g \times \overline{z + A}\,\rceil^{\frac{m-1}{m}}}$; and making the multiplication and the ſubtraction, according to the uſual method, the remainder will be $-\dfrac{\dot{z}}{g}$, to be divided by the denominator; and therefore

$$
\frac{z^{-1}\dot{z}}{g \times \overline{z + A}\,\rceil^{\frac{m-1}{m}} + z \times \overline{z + A}\,\rceil^{\frac{m-1}{m}}} = \frac{z^{-1}\dot{z}}{g \times \overline{z + A}\,\rceil^{\frac{m-1}{m}}} - \frac{\dot{z}}{gg \times \overline{z + A}\,\rceil^{\frac{m-1}{m}} + gz \times \overline{z + A}\,\rceil^{\frac{m-1}{m}}}.
$$

The

The firſt term of the ſecond member is already reduced to known quadratures, and the other term may eaſily be reduced, by making $z + A = u$, and performing the neceſſary ſubſtitutions. For then we ſhall have

$$\frac{-\dot{z}}{gg \times \overline{z+A}|^{\frac{m-1}{m}} + gz \times \overline{z+A}|^{\frac{m-1}{m}}} = \frac{u^{\frac{-m+1}{m}}\,\dot{u}}{gg - gA + gu}.$$

To purſue our inquiry, let the exponent p be equal to any poſitive and integer number; to obtain our deſire it will be ſufficient ſomething to produce the operation. Reſuming, then, the general formula $\frac{\dot{x}}{\overline{x^{2m} + ax^{m} + b}|^{p}} =$

$$\frac{z^{-p}\dot{z}}{m \times \overline{z+A}|^{\frac{m-1}{m}} \times \overline{z+g}|^{p}} = \dot{y}.$$ And, for example-ſake, making $p = 2$, this will be reduced to the following,

$$\frac{z^{-2}\dot{z}}{gg \times \overline{z+A}|^{\frac{m-1}{m}} + 2gz \times \overline{z+A}|^{\frac{m-1}{m}} + zz \times \overline{z+A}|^{\frac{m-1}{m}}} = m\dot{y}.$$

Then, as before, I divide the numerator of this fraction by it's denominator, and the firſt quotient will be $\frac{z^{-2}\dot{z}}{gg \times \overline{z+A}|^{\frac{m-1}{m}}}$; and, after the neceſſary operations, we ſhall have the remainder $-\frac{2z^{-1}\dot{z}}{g} - \frac{\dot{z}}{gg}$, to be again divided by the whole denominator. Then I make a ſecond diviſion with the fraction

$$\frac{-2z^{-1}\dot{z}}{g^{3} \times \overline{z+A}|^{\frac{m-1}{m}} + 2gg \times \overline{z+A}|^{\frac{m-1}{m}} + gzz \times \overline{z+A}|^{\frac{m-1}{m}}}.$$ Here, after the neceſſary operations, we ſhall have the remainder $\frac{4\dot{z}}{g} + \frac{2z\dot{z}}{gg}$, to be divided by the whole denominator. Whence there will ariſe the following equation,

$$\frac{z^{-2}\dot{z}}{\overline{z+A}|^{\frac{m-1}{m}} \times \overline{z+g}|^{2}} = \frac{z^{-2}\dot{z}}{gg \times \overline{z+A}|^{\frac{m-1}{m}}} - \frac{2z^{-1}\dot{z}}{g^{3} \times \overline{z+A}|^{\frac{m-1}{m}}} + \frac{3\dot{z}}{g^{2} \times \overline{z+A}|^{\frac{m-1}{m}} \times \overline{z+g}|^{2}}$$
$$+ \frac{2z\dot{z}}{g^{3} \times \overline{z+A}|^{\frac{m-1}{m}} \times \overline{z+g}|^{2}}.$$

The

The two firſt terms of the *homogeneum comparationis* are two binomials, and the other two may eaſily be reduced to the form of binomials, by making $z + A = u$, or $z + g = u$. In caſes more compounded, in which are made $p = 3$, or 4, or 5, &c. the tedioufneſs of calculation will indeed increaſe, but the method will ſtill be the ſame.

This method may be extended to all multinomials *in infinitum*, ſuppoſing p to be a poſitive integer; for, if it were a negative integer, the matter becomes ſo eaſy that there is no need to mention it. To apply the method, nothing elſe is required but to repeat the ſubſtitutions $x = z + A$, $z = u + B$, &c. always making thoſe terms to vaniſh, in which only conſtant quantities are found; by which means quadrinomials (for inſtance) may be reduced to trinomials, and theſe to binomials. It will alſo be needful, from time to time, to make uſe of a partial diviſion, that we may not be interrupted by negative exponents, which will often intrude in the numerator of the fraction. After all, the manner of operation will be better perceived by examples than by precepts.

Let us take the quadrinomial $\dfrac{\dot{x}}{\overline{x^{3m} + ax^{2m} + bx^{m} + c}|^{p}} = \dot{y}$. The conſtant quantities a, b, may be $= 0$. I ſuppoſe $x^{m} = z + A$; then we ſhall have

$$\begin{aligned} x^{3m} + ax^{2m} + bx^{m} + c = z^3 &+ 3Az^2 + 3AAz + A^3 \\ &+ az^2 + 2aAz + aA^2 \\ &+ bz + Ab + c. \end{aligned}$$

I make $A^3 + aA^2 + Ab + c = 0$, and thus I determine the value of the aſſumed conſtant quantity A. Then repeating the operations as in the trinomial, I find $\dfrac{z^{-p}\dot{z}}{\overline{z+A}|^{\frac{m-1}{m}} \times \overline{zz+gz+h}|^{p}}$. The letters g, h, denote conſtant quantities, which are ſubſtituted in the place of others more compounded. And, ſuppoſing p to be a poſitive integer, I raiſe the trinomial $zz + gz + h$ to the power p.

After this, I make uſe of as many diviſions as are neceſſary, to make the exponent of the variable in the numerator to be negative; and in the denominator, that no other quantity ſhall enter but the binomial $\overline{z + a}|^{\frac{m-1}{m}}$. And I ſet aſide ſuch fractions, as, neglecting the co-efficients, ſhall be analogous to this, $\dfrac{z^{-n}\dot{z}}{\overline{z+A}|^{\frac{m-1}{m}}}$; ſuppoſing n to be any poſitive integer. The other terms are repreſented

represented by the general formula $\dfrac{z^n\dot{z}}{\overline{z+A}|^{\frac{m-1}{m}} \times \overline{zz+gz+h}|^p}$. Then I repeat the operation, making $z = u + B$, making the last term to vanish as usual, and raising the binomial $u + B$ to any power $n + 1$, and substituting, instead of z and it's powers, their values expressed by the new variable u; all the parts will appear under the aspect expressed by the following formula,

$$\frac{u^{n-p}\dot{u}}{\overline{u+A+B}|^{\frac{m-1}{m}} \times \overline{u+k}|^p}.$$

When p is greater than n, so that the exponent $n - p$ is negative, then the divisions must be put in practice, and the formula thence arising will be $\dfrac{u^{-n}\dot{u}}{\overline{u+A+B}|^{\frac{m-1}{m}}}$; then $n - p$, being positive, we shall have $\dfrac{u^n\dot{u}}{\overline{u+A+B}|^{\frac{m-1}{m}} \times \overline{u+k}|^p}$. And lastly, making $u + k = \omega$, and, as well n as p being integer numbers, the binomials that will arise from the forementioned operations will always be reducible to more simple quadratures.

It is true, that, upon the account of imaginary quantities, this method remains limited; but very often the roots, either in the whole or in part, are real; and besides that, in many particular cases, these imaginary quantities may be eliminated. Nor ought we to despise the much we may have, because we cannot obtain all.

Let us take, for example, the trinomial $\dfrac{\dot{x}}{\overline{x+2\sqrt{x}+2}|^p}$. Make $x^{\frac{1}{2}} = z + A$, then $x + 2\sqrt{x} + 2 = zz + 2Az + 2z + AA + 2A + 2$. By making $AA + 2A + 2 = 0$, we find $A = \sqrt{-1} - 1$. Now here we have a magnitude made up of real and imaginary quantities; therefore, proceeding according to the method, we shall have $\dfrac{z^{-p}\dot{z}}{\overline{z+A}|^{-1} \times \overline{z+2A+2}|^p} = \dfrac{z^{1-p}\dot{z}}{\overline{z+2\sqrt{-1}}|^p} + \dfrac{Az^{-p}\dot{z}}{\overline{z+2\sqrt{-1}}|^p}$. Now, that the imaginary quantities may be avoided, let us change our manner, and in the magnitude $zz + \overline{2A+2} \times z + AA + 2A + 2$, let us bring it about, that the middle term $2Az + 2z$ may be destroyed, by putting it $= 0$; whence it is $A = -1$, and $AA + 2A + 2 = 1$. So that the formula will be as follows, $\dfrac{\dot{z}}{\overline{z-1}|^{-1} \times \overline{zz+1}|^p} = \dfrac{z\dot{z}}{\overline{zz+1}|^p} - \dfrac{\dot{z}}{\overline{zz+1}|^p}$. And now, in the two binomials of the *homogeneum comparationis*, which are equivalent to the two others already considered, we shall meet with no difficulty.

SECT. II.

Of the Rules of Integration, having recourſe to Infinite Series.

65. Now, to proceed to the other manner of Integration, or of finding fluents, which was mentioned at the beginning, that is, by means of infinite ſeries; it is neceſſary to premiſe theſe Rules following.

RULE I. To reduce a fraction to an infinite ſeries.

Divide the numerator by the denominator, according to the ordinary method of diviſion, and let the remainder be again divided, and thus from term to term *in infinitum*; and you will have a ſeries conſiſting of an infinite number of terms, which is equal to the propoſed fraction. Therefore it muſt be obſerved, to make that term the firſt which is the greateſt, and that as well in the numerator as in the denominator of the fraction propoſed. Wherefore, by operating after this manner, we ſhall have as follows:

$$\frac{f}{m+n} = \frac{f}{m} - \frac{fn}{m^2} + \frac{fn^2}{m^3} - \frac{fn^3}{m^4} + \frac{fn^4}{m^5}, \text{ \&c.}$$

$$\frac{f}{m-n} = \frac{f}{m} + \frac{fn}{m^2} + \frac{fn^2}{m^3} + \frac{fn^3}{m^4} + \frac{fn^4}{m^5}, \text{ \&c.}$$

$$\frac{af}{m^2 \pm n^2} = \frac{af}{m^2} \mp \frac{afn^2}{m^4} + \frac{afn^4}{m^6} \mp \frac{afn^6}{m^8} + \frac{afn^8}{m^{10}}, \text{ \&c.}$$

Here the ſigns of the ſeries muſt be alternately + and —, when the ſecond term of the denominator is poſitive; and all the ſigns muſt be poſitive when it has a negative ſign.

In like manner, it will be

$$\frac{f}{m^2 \pm mn} = \frac{f}{m^2} \mp \frac{fn}{m^3} + \frac{fn^2}{m^4} \mp \frac{fn^3}{m^5} + \frac{fn^4}{m^6}, \text{ \&c.}$$

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8, \text{ \&c.}$$

$$\frac{2x^{\frac{1}{2}} - x^{\frac{3}{2}}}{1 + x^{\frac{1}{2}} - 3x} = 2x^{\frac{1}{2}} - 2x + 7x^{\frac{3}{2}} - 13x^2 + 34x^{\frac{5}{2}}, \text{ \&c.}$$

$$\frac{f}{\overline{m \mp n}|^3} = \frac{f}{m^3} \pm \frac{3fn}{m^4} + \frac{6fn^2}{m^5} \pm \frac{10fn^3}{m^6} + \frac{15fn^4}{m^7}, \text{ \&c.}$$

Let

Let there be a fraction, of which the numerator and denominator are each an infinite ſeries; for example, this following:

$$\frac{1 + \frac{1}{2}ax^2 - \frac{1}{8}aax^4 + \frac{1}{16}a^3x^6 - \frac{5}{128}a^4x^8, \text{ \&c.}}{1 - \frac{1}{2}bx^2 - \frac{1}{8}bbx^4 - \frac{1}{16}b^3x^6 - \frac{5}{128}b^4x^8, \text{ \&c.}}$$

The quotient will be

$$\left.\begin{array}{l} 1 + \frac{1}{2}bx^2 + \frac{3}{8}b^2x^4 + \frac{5}{16}b^3x^6 + \frac{35}{128}b^4x^8 \\ \quad + \frac{1}{2}ax^2 + \frac{1}{4}abx^4 + \frac{3}{16}ab^2x^6 + \frac{5}{32}ab^3x^8 \\ \qquad\qquad - \frac{1}{8}a^2x^4 - \frac{1}{16}a^2bx^6 - \frac{3}{64}a^2b^2x^8 \\ \qquad\qquad\qquad\quad + \frac{1}{16}a^3x^6 + \frac{1}{32}a^3bx^8 \\ \qquad\qquad\qquad\qquad\quad - \frac{5}{128}a^4x^8 \end{array}\right\} \text{\&c.}$$

66. Rule II. To reduce a complicate radical quantity into an infinite ſeries.

Take, for example, $\sqrt{aa \pm xx}$; let the ſquare-root of the firſt term be extracted, and then let the operation be proſecuted *in infinitum*, in the uſual manner of the extraction of the ſquare-root, and we ſhall have

$$\sqrt{aa \pm xx} = a \pm \frac{x^2}{2a} - \frac{x^4}{8a^3} \pm \frac{x^6}{16a^5} - \frac{5x^8}{128a^7}, \text{ \&c.}$$

$$\sqrt{ax \pm xx} = a^{\frac{1}{2}}x^{\frac{1}{2}} \pm \frac{x^{\frac{3}{2}}}{2a^{\frac{1}{2}}} - \frac{x^{\frac{5}{2}}}{8a^{\frac{3}{2}}} \pm \frac{x^{\frac{7}{2}}}{16^{\frac{5}{2}}} - \frac{5x^{\frac{9}{2}}}{128a^{\frac{7}{2}}}, \text{ \&c.}$$

It may here be obſerved, that in each of theſe two ſeries, if the numerator and denominator of each term be multiplied by 3, beginning at the fourth, the numerical co-efficients of the numerators will be in order, 3, 3 × 5, 3 × 5 × 7, &c. ariſing from the continual multiplication of the odd numbers. Then in the denominators, beginning at the ſecond, they will be 2, 2 × 4, 2 × 4 × 6, 2 × 4 × 6 × 8, &c. ariſing from the continual multiplication of the even numbers.

67. Rule III. All this may be done more generally by the help of the following canon:

$$\overline{P + PQ}^{\frac{m}{n}} = P^{\frac{m}{n}} + \frac{m}{n}AQ + \frac{m-n}{2n}BQ + \frac{m-2n}{3n}CQ + \frac{m-3n}{4n}DQ, \text{ \&c.}$$

In which $P + PQ$ is the given quantity, $\frac{m}{n}$ is the numeral exponent, P repreſents the firſt term, Q is the quotient of all the other terms divided by the firſt, and every one of the capitals A, B, C, D, &c. ſignify the preceding terms

 reſpectively

respectively; so that by A is understood $P^{\frac{m}{n}}$, by B is meant $\frac{m}{n}$ AQ, by C, $\frac{m-n}{2n}$ BQ, and so on.

Let the formula $\sqrt{aa + xx}$ be proposed to be reduced into a series; then it will be $P = aa$, $Q = \frac{xx}{aa}$, $m = 1$, $n = 2$; therefore

$$\sqrt{aa + xx} = a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \frac{5x^8}{128a^7}, \text{ \&c.}$$

Let it be $\sqrt[5]{a^5 + a^4x - x^5}$, that is, $\overline{a^5 + a^4x - x^5}|^{\frac{1}{5}}$; it will be $P = a^5$, $Q = \frac{a^4x - x^5}{a^5}$, $m = 1$, $n = 5$; therefore $\overline{a^5 + a^4x - x^5}|^{\frac{1}{5}} = a + \frac{a^4x - x^5}{5a^4} - \frac{2a^8x^2 - 4a^4x^6 + 2x^{10}}{25a^9}$, &c.

Let it be $\frac{b}{\sqrt[3]{y^3 - aay}} = b \times \overline{y^3 - aay}|^{-\frac{1}{3}}$; it will be $P = y^3$, $Q = -\frac{aa}{yy}$, $m = -1$, $n = 3$; therefore

$$b \times \overline{y^3 - aay}|^{-\frac{1}{3}} = \frac{b}{y} + \frac{aab}{3y^3} + \frac{2a^4b}{9y^5} + \frac{14a^6b}{81y^7}, \text{ \&c.}$$

Let it be $\frac{b}{\sqrt[5]{\overline{a + x}|^3}}$, which would be expressed thus, $b \times \overline{a + x}|^{-\frac{3}{5}}$, and the rest would be done as before.

Let it be $b \times \overline{a + x}|^{-3}$; then $P = a$, $Q = \frac{x}{a}$, $m = -3$, $n = 1$; therefore $b \times \overline{a + x}|^{-3} = \frac{b}{a^3} - \frac{3bx}{a^4} + \frac{6bx^2}{a^5} - \frac{10bx^3}{a^6}$, &c.

68. Let us have a complicate quantity to raise to a given power, or let $a + x$ (for example) be raised to the power m. Then $P = a$, $Q = \frac{x}{a}$, $m = m$, $n = 1$; therefore

$$\overline{a + x}|^m = a^m + \frac{ma^{m-1}x}{1} + \frac{m \times \overline{m-1}\, a^{m-2}x^2}{1 \times 2} + \frac{m \times \overline{m-1} \times \overline{m-2}\, a^{m-3}x^3}{1 \times 2 \times 3}, \text{\&c.}$$

Let us have an infinite series to raise to a given power. For example, let $y + ay^2 + by^3 + cy^4 + dy^5$, &c. be raised to the power m. Then will $P = y$, $Q = ay + by^2 + cy^3 + dy^4$, &c. $m = m$, $n = 1$; wherefore

$$\overline{y + ay^2 + by^3 + cy^4 + dy^5, \&c.}\rceil^m = y^m + \frac{may^{m+1}}{1}$$

$$+ \frac{m \times \overline{m-1}\, a^2y^{m+2}}{1 \times 2} + \frac{m \times \overline{m-1} \times \overline{m-2}\, a^3y^{m+3}}{1 \times 2 \times 3}$$

$$+ \frac{mby^{m+2}}{1} + \frac{m \times \overline{m-1}\, aby^{m+3}}{1 \times 1}$$

$$+ \frac{mcy^{m+3}}{1}$$

$$+ \frac{m \times \overline{m-1} \times \overline{m-2} \times \overline{m-3}\, a^4y^{m+4}}{1 \times 2 \times 3 \times 4} \ \&c.$$

$$+ \frac{m \times \overline{m-1} \times \overline{m-2}\, a^2by^{m+4}}{1 \times 2 \times 1}$$

$$+ \frac{m \times \overline{m-1}\, acy^{m+4}}{1 \times 1}$$

$$+ \frac{m \times \overline{m-1}\, b^2y^{m+4}}{1 \times 2}$$

$$+ \frac{mdy^{m+4}}{1}$$

69. This being now ſuppoſed, let the differential formula $\frac{b\dot{x}}{a + x}$ be propoſed to be integrated. The fraction $\frac{b}{a + x}$ being reduced to a ſeries, and every numerator being multiplied by $\dot{x}$, we ſhall have $\frac{b\dot{x}}{a + x} = \frac{b\dot{x}}{a} - \frac{bx\dot{x}}{aa} + \frac{bx^2\dot{x}}{a^3} - \frac{bx^3\dot{x}}{a^4} + \frac{bx^4\dot{x}}{a^5}$, &c. And by integration,

$$\int \frac{b\dot{x}}{a + x} = \frac{bx}{a} - \frac{bx^2}{2aa} + \frac{bx^3}{3a^3} - \frac{bx^4}{4a^4} + \frac{bx^5}{5a^5}, \&c.$$

70. Let the formula be $\frac{a\dot{x}}{x}$. Making $x = b + z$, where b denotes any conſtant quantity at pleaſure, and z a new variable; it will be $\frac{a\dot{x}}{x} = \frac{a\dot{z}}{b + z}$.

The fraction $\frac{a}{b + z}$ being reduced to a ſeries, and multiplied by $\dot{z}$, it will be

$$\frac{a\dot{z}}{b+z} = \frac{a\dot{z}}{b} - \frac{az\dot{z}}{b^2} + \frac{az^2\dot{z}}{b^3} - \frac{az^3\dot{z}}{b^4} + \frac{az^4\dot{z}}{b^5}, \text{ \&c. And by integration,}$$

$$\int\frac{a\dot{z}}{b+z} = \frac{az}{b} - \frac{az^2}{2b^2} + \frac{az^3}{3b^3} - \frac{az^4}{4b^4} + \frac{az^5}{5b^5}, \text{ \&c.; that is,}$$

$$\int\frac{a\dot{x}}{x} = \frac{a \times \overline{x-b}}{b} - \frac{a \times \overline{x-b}\,\rangle^2}{2b^2} + \frac{a \times \overline{x-b}\,\rangle^3}{3b^3} - \frac{a \times \overline{x-b}\,\rangle^4}{4b^4}, \text{ \&c.}$$

71. Let the formula be $\frac{b\dot{x}}{\sqrt[5]{\overline{x+a}\,|^3}}$; this, reduced to a series, is $\frac{b\dot{x}}{\sqrt[5]{\overline{a+x}\,|^3}} = \frac{b\dot{x}}{a^{\frac{3}{5}}} - \frac{3bx\dot{x}}{5a^{\frac{8}{5}}} + \frac{12bx^2\dot{x}}{25a^{\frac{13}{5}}} - \frac{52bx^3\dot{x}}{125a^{\frac{18}{5}}}$, &c. And by integration, $\int\frac{b\dot{x}}{\sqrt[5]{\overline{a+x}\,\rangle^3}} = \frac{bx}{a^{\frac{3}{5}}} - \frac{3bx^2}{10a^{\frac{8}{5}}} + \frac{12bx^3}{75a^{\frac{13}{5}}} - \frac{52bx^4}{500a^{\frac{18}{5}}}$, &c. And the same may be done by any other proposed formula.

72. If the series thus found, which express the fluents of proposed differential formulæ, and which are composed of an infinite number of terms, shall be infinite in value; the fluents or integrals of the proposed fluxions will be infinite. And if these series shall be finite in value, and also summable, that is to say, if we know how to find the values of these series, though composed of terms infinite in number, and which very often may be done; we shall have them in a finite quantity, and therefore the algebräical integral of the proposed differential formulæ. But, if the series shall be finite in value, and yet not summable, the more terms shall be taken of the series, so much the nearer we shall approach to the true value of the formula; but we cannot arrive at the exact value, except we could take in the whole series.

73. In order to know what series are infinite in value, what are of a finite value, and which are summable; the treatise of Mr. *James Bernoulli de Seriebus infinitis*, may be consulted, and other authors who have written expresly on this subject.

74. But whenever the differential formula shall be composed of two terms only, we may, in general, and with expedition, make use of the following canon; in which the exponents m, n, t, may be integers or fractions, affirmative or negative; and which may be continued to as many terms as we please; for from these four terms set down, the law of continuation is sufficiently manifest.

$\int ay^{t-1}\dot{y} \times \overline{b+cy^n}|^m = \overline{b+cy^n}|^{m+1}$ into $\frac{ay^t}{tb} - \frac{t+mn+n}{t+n} \times \frac{ac}{tbb}y^{t+n} + \frac{t+mn+n}{t+n} \times \frac{t+mn+2n}{t+2n} \times \frac{ac^2}{tb^3}y^{t+2n} - \frac{t+mn+n}{t+n} \times \frac{t+mn+2n}{t+2n} \times \frac{t+mn+3n}{t+3n} \times \frac{ac^3}{tb^4}y^{t+3n}$, &c.

The manner of finding this canon is this. Take the fictitious equation $\int ay^{t-1}\dot{y} \times \overline{b+cy^n}|^m = \overline{b+cy^n}|^{m+1}$ into $Ay^t + By^{t+n} + Cy^{t+2n} + Dy^{t+3n} + Ey^{t+4n}$, &c.; in which the assumed quantities A, B, C, D, E, &c. are arbitrary and constant, to be determined afterwards as occasion may require. Then, by taking the fluxions of this fictitious equation, we shall have $ay^{t-1}\dot{y} \times \overline{b+cy^n}|^m = \overline{m+1} \times ncy^{n-1}\dot{y} \times \overline{b+cy^n}|^m$ into $Ay^t + By^{t+n} + Cy^{t+2n}$, &c. $+ \overline{b+cy^n}|^{m+1}$ into $tA\dot{y}y^{t-1} + \overline{t+n} \times B\dot{y}y^{t+n-1} + \overline{t+2n} \times C\dot{y}y^{t+2n-1}$, &c. Then dividing all by $\overline{b+cy^n}|^m$, and setting the terms in order, it will be

$$
\begin{aligned}
a\dot{y}y^{t-1} = tbA\dot{y}y^{t-1} &+ \overline{t+n} \times bB\dot{y}y^{t+n-1} + \overline{t+2n} \times bC\dot{y}y^{t+2n-1}, \text{ \&c.} \\
&+ \qquad tcA\dot{y}y^{t+n-1} + \overline{t+n} \times cB\dot{y}y^{t+2n-1}, \text{ \&c.} \\
&+ \overline{m+1} \times ncA\dot{y}y^{t+n-1} + \overline{m+1} \times ncB\dot{y}y^{t+2n-1}, \text{ \&c.}
\end{aligned}
$$

Here the term $a\dot{y}y^{t-1}$ might be transposed to the other side of the equation by which the whole will be equal to nothing, and therefore the co-efficients of each term will be equal to nothing, by which we should have as many equations as there are arbitrary quantities A, B, C, D, &c. by which they will be determined. Or, making the first terms on each side equal, it will be $tbA = a$, or $A = \frac{a}{tb}$. Then $\overline{t+n} \times bB + tcA + \overline{m+1} \times ncA = 0$, and substituting the value of A, it is $tbB + nbB + \frac{ac}{b} + \frac{mnac}{tb} + \frac{nac}{tb} = 0$, or $B = \frac{t+mn+n}{t+n} \times -\frac{ac}{tb^2}$. Again, $\overline{t+2n} \times bC + \overline{t+n} \times cB + \overline{m+1} \times ncB = 0$, or $C = \frac{\overline{t+n} \times -cB + \overline{m+1} \times -ncB}{b \times \overline{t+2n}}$, and substituting the value of B, it will be

C =

$C = \frac{\overline{t+mn+n} \times \overline{t+mn+2n} \times acc}{\overline{t+n} \times \overline{t+2n} \times tb^3}$. And thus from one to another, till we have the values of as many as we pleaſe of the ſeveral aſſumed conſtants; and theſe values, ſubſtituted in the fictitious equation, will ſupply us with the aforeſaid canon.

If the exponents m, n, t, of the propoſed formula ſhall be ſuch, that the canon or infinite ſeries will break off, or that any term ſhall become $= 0$, (in which caſe all the others that follow will alſo be $= 0$,) the ſeries becomes finite and terminated, or we ſhall have the algebräical integral of the propoſed differential formula. But it is neceſſary that the ſeries ſhould firſt break off in the numerator, or that the numerator ſhould become equal to nothing before the denominator. For, if the denominator be equal to nothing firſt, that term and all that follow after will be equal to infinite. Now, that the ſeries ſhould break off in the numerator, it is neceſſary that $-\frac{t}{n} - m$ ſhould be equal to ſome integer affirmative number.

But if the exponents t, m, n, of the propoſed formula ſhould be ſuch, that the ſeries never breaks off; then the expreſſion of the formula ſhould be changed into another equivalent to it. Thus, for example, the formula $a\dot{y}y^{t-1} \times \overline{b+cy^n}|^m$ ſhould be changed into this other, $a\dot{y}y^{t-1+mn} \times \overline{by^{-n}+c}|^m$, which is equivalent to the firſt, and it ſhould be tried whether or not this will anſwer our expectation. If not, the formula will not be algebräically integrable, at leaſt not by this canon. If the formula were $a\dot{y}y^{t-1} \times \overline{b-cy^n}|^m$, then all the terms of the canon would be poſitive.

Let it be $\frac{a^5\dot{x}\sqrt{bx+xx}}{x^5}$, that is, $a^5\dot{x}x^{-\frac{9}{2}} \times \overline{b+x}|^{\frac{1}{2}}$; it will be $t - 1 = -\frac{9}{2}$, $n = 1$, $m = \frac{1}{2}$, $c = 1$; whence the quantity $t + mn + 3n$ will be equal to nothing, and conſequently the fourth term $= 0$, and the others of the ſeries that follow. Therefore we ſhall have $\int \frac{a^5\dot{x} \times \sqrt{bx+xx}}{x^5} = \int a^5\dot{x}x^{-\frac{9}{2}} \times \overline{b+x}|^{\frac{1}{2}}$

$$= \overline{-\frac{2a^5x^{-\frac{7}{2}}}{7b} + \frac{2a^5}{7bb} \times \frac{4}{5}x^{-\frac{5}{2}} - \frac{2a^5}{7b^3} \times \frac{8}{15}x^{-\frac{3}{2}}} \times \overline{b+x}|^{\frac{3}{2}} =$$

$$\frac{-30a^5b^2 + 24a^5bx - 16a^5x^2}{105b^3x^{\frac{7}{2}}} \times \overline{b+x}|^{\frac{3}{2}}.$$

Let

Let it be $\frac{a\dot{y}}{yy\sqrt{aa+yy}}$; then $t = -1$, $n = 2$, $m = -\frac{1}{2}$, $c = 1$, $b = aa$; and therefore the second term of the series will be $= 0$. Hence

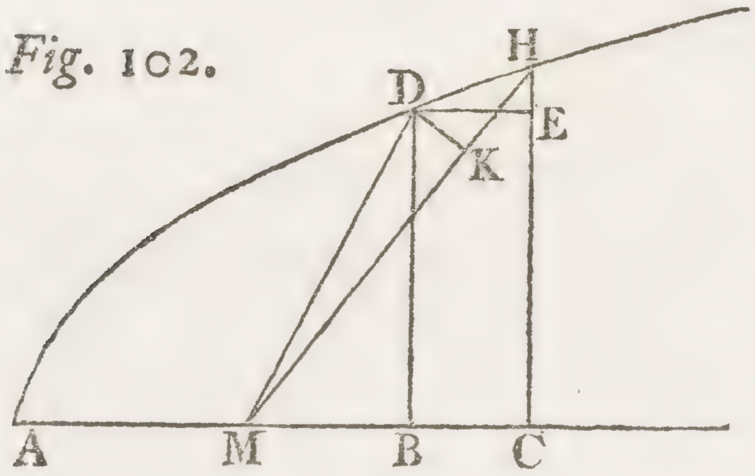
$$\int \frac{a\dot{y}}{yy\sqrt{aa+yy}} = \frac{ay^{-1}}{-aa} \times \overline{aa+yy}|^{\frac{1}{2}} = -\frac{\sqrt{aa+yy}}{ay}.$$

SECT. III.

The Rules of the foregoing Sections applied to the Rectification of Curve-lines, the Quadrature of Curvilinear Spaces, the Complanation of Curve Superficies, and the Cubature of their Solids.

75. To show the use of the foregoing Rules of the Integral Calculus, by applying it to the quadrature of spaces, to the rectification of curves, to the complanation or quadrature of superficies, and to the cubature of solids; let there be any curve ADH referred to an axis AB, with the ordinates parallel to each other, and at right angles to the axis. Draw CH parallel to the ordinate BD, and infinitely near to it, and also DE parallel to BC; the mixtilinear figure BDHC will be the fluxion, the differential, or the element of the space ABD; and because the space DEH is nothing in respect of the rectangle BDEC, we may take that rectangle for the element of the said space ABD. Therefore the sum of all these infinitesimal rectangles BDEC will be the space comprehended by the curve AD, and by the co-ordinates AB and BD. Wherefore, making $AB = x$, $BD = y$, it will be $BC = \dot{x}$, $EH = \dot{y}$, and the rectangle $BDEC = y\dot{x}$ will be the formula for such spaces. Therefore, in this formula, instead of y, if we substitute it's value given by x, and by the constant quantities of the equation of the curve; or, instead of $\dot{x}$, it's value

Fig. 102.



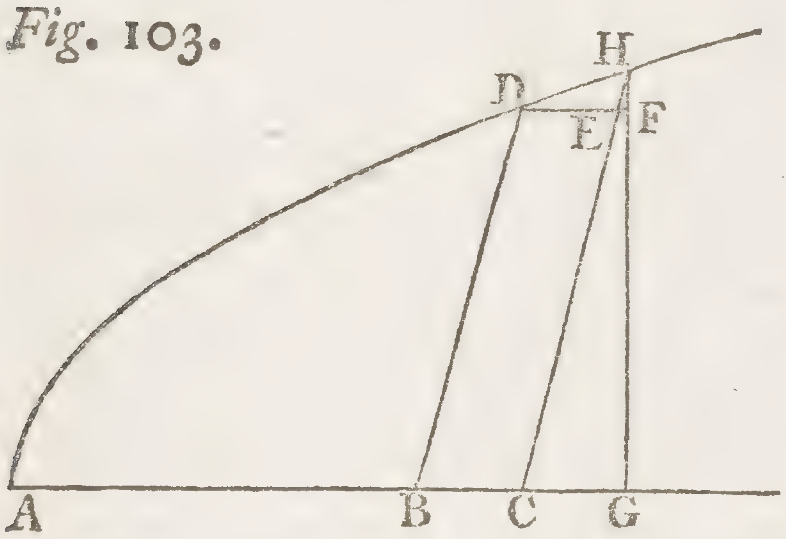
given by y and $\dot{y}$, and the conſtants, and then integrate the formula, this integral will be the required ſpace ABD.

Other expreſſions or formulæ may be had for the elements of ſpaces, by means of ſectors, or of trapezia, which, on certain occaſions, are ſometimes more convenient than rectangles; we ſhall hereafter ſee the uſe and manner of them in ſome examples.

76. For, if the curve be referred to a *focus*, that is, to a fixed point, ſuppoſe to M, from whence all the ordinates proceed; drawing MH infinitely near to the ordinate MD, the infiniteſimal ſpace MHD will be the element of the ſpace AMD. Then with centre M and radius MD, drawing the infinitely little arch DK, the little ſpace DKH will be nothing in reſpect of the ſpace MDK; and alſo, becauſe the little arch DK may be aſſumed for the tangent in D, or in K, it thence follows that the ſpace MDK ſhall be the element of the ſpace AMD.

Wherefore, calling $MD = y$, $KD = \dot{z}$, it will be $\frac{1}{2}y\dot{z}$ for the general formula of the ſpaces, in curves referred to a *focus*. And in this formula, inſtead of y, or of $\dot{z}$, if the reſpective values be ſubſtituted from the equation of the curve, the integral will be the ſpace required AMD.

Fig. 103.



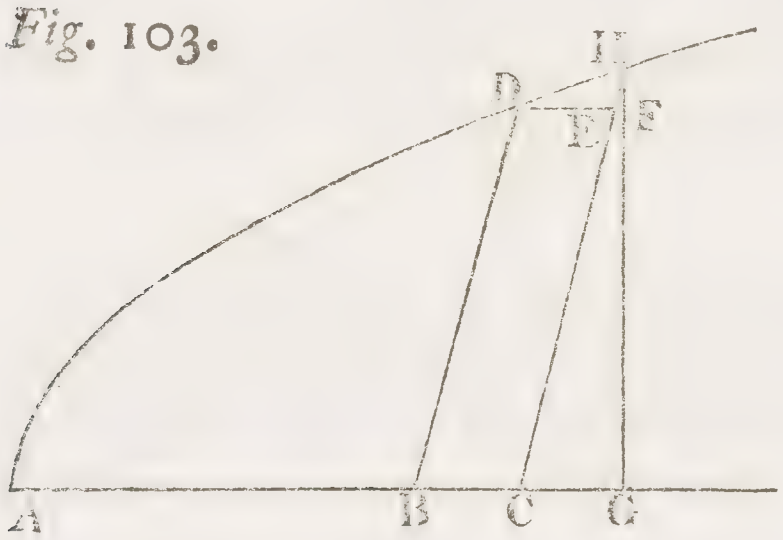
77. But if the curve ſhall be referred to a diameter, ſo that the ordinates ſhall not be at right angles to their abſciſſes; drawing HG perpendicular to AG, the product of HG, or of FG into BC, will be the little parallelogram BCED, and conſequently the element of the area ABD. Therefore the angle DBG being given, and conſequently the ratio of the whole ſine to the right ſine, which, for example, may be that of m to n; making, as uſual, $AB = x$, $BD = y$, then will HG or FG be $= \frac{ny}{m}$, and the parallelogram BCED will be $\frac{ny\dot{x}}{m}$, a general formula for this ſpace.

78. It is plain, that the ſum of all the infiniteſimal portions DH of the curve will form the curve itſelf, and therefore that DH will be it's element. Making, therefore, $AB = x$, (Fig. 102.) $BD = y$, and thence $BC = \dot{x}$, $EH = \dot{y}$; in ſuch curves as are referred to an axis, that is, with the co-ordinates at right angles, it will be $DH = \sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}$, a general formula for the rectification of theſe curves.

79. As

79. As to ſuch curves as are referred to a *focus*, making alſo MD $= y$, KD $= \dot{z}$, we ſhall have, in like manner, $\sqrt{\dot{y}\dot{y} + \dot{z}\dot{z}}$ for a general formula.

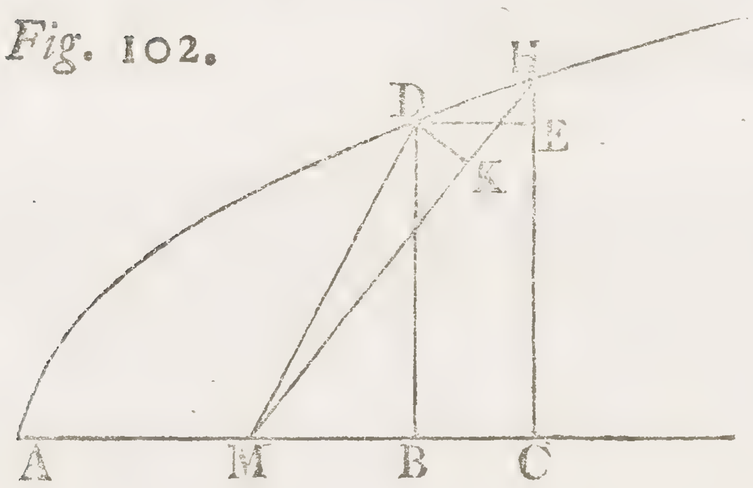
Fig. 103.



80. But as to the curves with the co-ordinates at oblique angles, the given angle being HCG, the ratio of the whole ſine to the ſine of the complement is given, which ſuppoſe is that of m to e; whence it will be CG $= \frac{e\dot{y}}{m}$, and EF $= \frac{e\dot{y}}{m}$, and therefore DH $= \sqrt{\dot{x}\dot{x} + \dot{y}\dot{y} + \frac{2e\dot{x}\dot{y}}{m}}$.

81. Now in each of theſe formulæ, inſtead of $\dot{y}$, or $\dot{x}$, or $\dot{z}$, ſubſtituting their reſpective values given by the other variable, and their differentials from the equation of the curve, and then making the integrations, we ſhall have the length of the curve required.

Fig. 102.



82. Let the plane AHC be conceived to move about the right line AC, the curve AH will deſcribe a ſuperficies, while the plane AHC deſcribes a ſolid. But the infiniteſimal portion DH will deſcribe an infiniteſimal zone, which will be the element of the ſuperficies deſcribed by the curve AH. And the infiniteſimal plane DBCH will deſcribe a ſolid alſo infiniteſimal, which will be the element of the ſolid deſcribed by the plane AHC. Now, as to curves referred to an axis with the ordinates at right angles; let the ratio of the radius to the circumference of a circle be that of r to c; the circumference deſcribed with radius BD $= y$ will be $\frac{cy}{r}$, and therefore $\frac{cy}{r}\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}$ will be the expreſſion of the infiniteſimal zone, and conſequently the general formula for the ſuperficies.

83. Alſo, $\frac{cyy}{2r}$ will be the area of the circle deſcribed with radius BD $= y$, and therefore $\frac{cyy\dot{x}}{2r}$ will be the expreſſion of the infinitely little cylinder deſcribed by the rectangle BCED. Now this does not differ from the ſolid generated by the plane BCHD, but by an infiniteſimal quantity of the ſecond order; therefore the general formula for theſe ſolids will be $\frac{cyy\dot{x}}{2r}$.

84. But

84. But as to the case of Fig. 103; that is, when the co-ordinates make a given oblique angle to each other; the radius of the circle, on which the little zone and the little cylinder insist, it is not CH $= y$, but indeed GH $= \frac{ny}{m}$; as likewise the element DH, which forms the zone, is not $\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}$, but $\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y} + \frac{2e\dot{x}\dot{y}}{m}}$; and the height of the little cylinder is not BC $= \dot{x}$, but FD $= \dot{x} + \frac{e\dot{y}}{m}$. Therefore the formula for the superficies, in this case, will be $\frac{cny}{rm}\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y} + \frac{2e\dot{x}\dot{y}}{m}}$.

85. The product of the circle with radius GH into the height FD, that is, $\frac{cnnyy}{2rmm} \times \overline{\dot{x} + \frac{e\dot{y}}{m}}$, is the element of the solid generated by the plane AGH. Therefore, from this subtracting the element of the solid generated by the triangle HCG, that is, $\frac{cnnyy}{2rmm} \times \frac{e\dot{y}}{m}$, what remains will be the element of the solid generated by the plane ABD, and therefore will be $\frac{cnnyy\dot{x}}{2rmm}$, the general formula for these solids.

86. As to the curves referred to a focus, because of the variable angle DMB, (Fig. 102.) and consequently because we cannot have the value of BD or CH, the radius of the circle, which must necessarily enter the formula of the quadrature of the superficies, and the cubature of the solid; it will be necessary, from the equation referred to the focus, to derive the equation of the same curve referred to an axis, and then we are to proceed in the manner before specified; observing that, in the cubature, it will be necessary to subtract from the integral the cone generated by the triangle MHC, to have the solid generated by the plane AMD.

87. From the differential equation of a curve to the focus, to obtain the equation of the same curve to an axis, the manner is this following.

Let the curve ADH (Fig. 102.) be considered, at the same time, both as related to the focus M, and also to the axis AMB. It is certain that the square of HD, the element of the curve, is equal as well to the two squares DK, KH, as to the two others DE, EH; and moreover, that the square of MD is equal to the two squares MB, BD. Making MB $= x$, BD $= y$, MD $= z$, and the little arch DK $= \dot{u}$, we shall have $\dot{z}\dot{z} + \dot{u}\dot{u} = \dot{x}\dot{x} + \dot{y}\dot{y}$, and $xx + yy = zz$.

Now the equation of the curve to the focus is expreffed, in general, by the formula $p\dot{z} = \dot{u}$, in which p is a known function or power of z; and it will be $\dot{z}\dot{z} + pp\dot{z}\dot{z} = \dot{x}\dot{x} + \dot{y}\dot{y}$. And putting, inftead of $\dot{y}$, it's value arifing from the equation $xx + yy = zz$, that is, $\dot{y} = \frac{z\dot{z} - x\dot{x}}{\sqrt{zz - xx}}$, we fhall find $\dot{z}\dot{z} + pp\dot{z}\dot{z} = \dot{x}\dot{x} + \frac{\overline{z\dot{z} - x\dot{x}}\,^2}{zz - xx}$, which may be reduced to this following, $pp\dot{z}\dot{z} \times \overline{zz - xx} = zz\dot{x}\dot{x} - 2xz\dot{x}\dot{z} + xx\dot{z}\dot{z}$; and extracting the fquare-root, it will be $p\dot{z} = \frac{z\dot{x} - x\dot{z}}{\sqrt{zz - xx}}$.

It is neceffary to clear again the foregoing equation, by freeing it from a mixture of unknown quantities, by making $x = \frac{zq}{b}$, and therefore $\dot{x} = \frac{z\dot{q} + q\dot{z}}{b}$. By the help of this affumed fubfidiary equation, make x and it's functions to vanifh, and we fhall have $\frac{p\dot{z}}{z} = \frac{\dot{q}}{\sqrt{bb - qq}}$. In this equation, if the value of p given by z fhall be fuch, that the quantity $\frac{p\dot{z}}{z}$ may be reduced to the differential of a circular arch by due fubftitutions; and that, making the neceffary integrations, the two circular arches fhall be to each other as number to number; then the curve fhall be algebräical, and we fhall find it's equation to the axis by a formula, after the manner of *Cartefius*. In every other cafe the curve will be tranfcendental.

EXAMPLE.

Let the equation of a curve referred to a focus be $\frac{z\dot{z}}{\sqrt{cc - 2bz - zz}} = \dot{u}$. We fhall have, in this cafe, $p = \frac{z}{\sqrt{cc - 2bz - zz}}$; and in the equation $\frac{p\dot{z}}{z} = \frac{\dot{q}}{\sqrt{bb - qq}}$ fubftituting the value of p, it will be $\frac{\dot{z}}{\sqrt{cc - 2bz - zz}} = \frac{\dot{q}}{\sqrt{bb - qq}}$. Make $b + z = t$, then $bb + 2bz + zz = tt$, and $bb - tt = -2bz - zz$; wherefore, making the fubftitution, it will be $\frac{\dot{t}}{\sqrt{cc + bb - tt}} = \frac{\dot{q}}{\sqrt{bb - qq}}$.

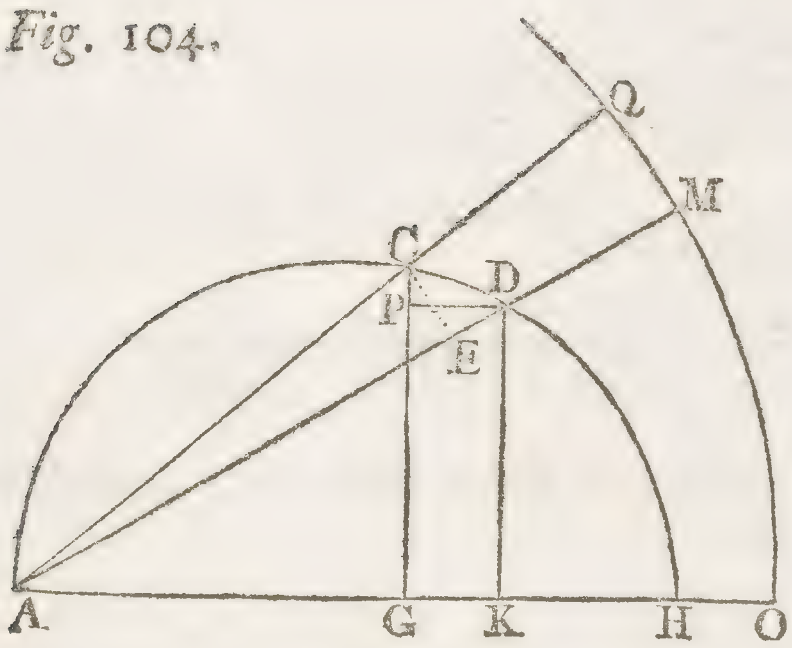
For

For a particular caſe, let it be $cc + bb = hh$, on which ſuppoſition it will be $t = q$, that is, $b + z = q = \frac{hx}{z}$. Therefore $bz + zz = hx$, and, inſtead of z, ſubſtituting it's value, the equation of the curve will be $b\sqrt{xx + yy} + xx + yy = hx$.

88. The aſſigned canon alſo teaches us the manner of paſſing from the differential equation of a curve to the axis to that of the focus, in the way following.

EXAMPLE I.

Fig. 104.



Let it be propoſed to find the equation to a focus in a circle, taking the focus in a point of the circumference A.

Make AH $= b$, AG $= x$, AC $= z = \sqrt{bx}$. Reſume the formula $\frac{p\dot{z}}{z} = \frac{\dot{q}}{\sqrt{bb - qq}}$, where is taken $q = \frac{bx}{z}$. Becauſe, by the local equation of the circle, it is $bx = zz$, it will be $q = z$. Then making q to vaniſh, by ſubſtituting it's value z, it will be $\frac{p\dot{z}}{z} = \frac{\dot{z}}{\sqrt{bb - zz}}$, or $p = \frac{z}{\sqrt{bb - zz}}$. Therefore, in the formula $p\dot{z} = \dot{u}$, if, inſtead of p, we ſhould ſubſtitute it's value now found, it will be $\frac{z\dot{z}}{\sqrt{bb - zz}} = \dot{u}$, an equation of the circle to the focus, which is taken in A, a point of the circumference.

EXAMPLE II.

89. Let it be propoſed to find the equation of a conic ſection, referred to it's umbilicus M, that is, to it's focus (Fig. 102.)

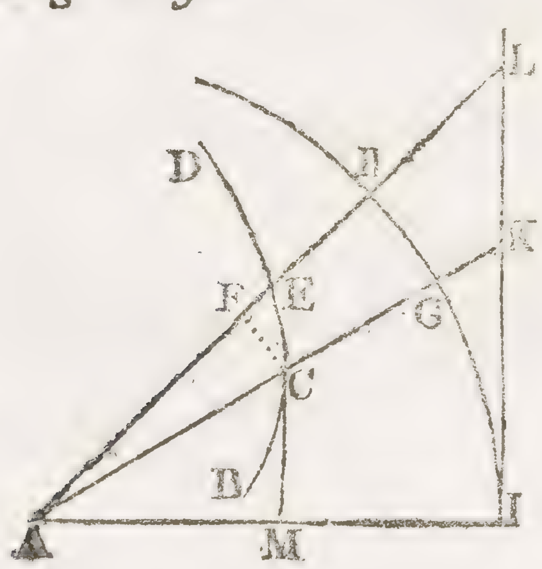
Make

Make MB $= x$, BD $= y$; the general equation, which comprehends all the fections of a cone, will be this; $a \pm \frac{cx}{b} = \sqrt{xx + yy}$; to the parabola with the parameter $2a$, when $c = b$; to the ellipfis with tranfverfe axis $= \frac{2abb}{bb - cc}$, if b be greater than c; to the hyperbola with tranfverfe axis $= \frac{2abb}{cc - bb}$, with conjugate axis $= \frac{2ab}{\sqrt{cc - bb}}$, diftance of the vertex from the focus $= \frac{ab}{b + c}$, if b be lefs than c. If $c = 0$, it will be to the circle with diameter $= 2a$. Put $\sqrt{xx + yy} = z$; therefore $a \pm \frac{cx}{b} = z$. And befides, $bx = zq$; then $a \pm \frac{czq}{bb} = z$, or $\pm \frac{bb}{c} \mp \frac{abb}{cz} = q$. And taking the fluxions, $\pm \frac{abcb\dot{z}}{cczz} = \dot{q}$, and $qq = \frac{bbbb}{cc} - \frac{2abbbb}{ccz} + \frac{aabbbb}{cczz}$, and $bb - qq = bb - \frac{bbbb}{cc} + \frac{2abbbb}{ccz} - \frac{aabbbb}{cczz}$. Hence $\frac{\dot{q}}{\sqrt{bb - qq}} = \frac{\pm abcb\dot{z}}{cz\sqrt{bbcczz - bbbbzz + 2abbbbz - aabbbb}} = \frac{p\dot{z}}{z}$, and therefore $p = \frac{\pm abb}{\sqrt{bbcczz - bbbbzz + 2abbbbz - aabbbb}}$. And as it is $p\dot{z} = \dot{u}$, we fhall have the equation required, $\dot{u} = \frac{\pm abb\dot{z}}{\sqrt{bbcczz - bbbbz^2 + 2abbbbz - aabbbb}}$.

The negative fign ferves when the abfciffes are taken from the focus towards the vertex, and the pofitive are the contrary way.

90. I faid we ought to reduce the equation of the curve to the focus, to another referred to the axis; not becaufe this is abfolutely neceffary for the complanation of fuperficies, or for the cubature of folids; for the whole may be obtained by means of this known theorem: The periphery of the curve, drawn into the line defcribed by the centre of gravity of that periphery, is equal to the fuperficies of the folid which is generated by it's rotation. And the area of the curve, drawn into the line defcribed by the centre of gravity of the faid area, is equal to the faid folid. But here we muft not fuppofe our readers fo fkillful as to be acquainted with the theory of Centres of Gravity.

Fig. 105.



Now, to have a competent notion of curves referred to a focus, I fhall make an attempt at finding out their conftruction. Let BCD be one of thefe; the co-ordinates infinitely near are AC, AE, which proceed from the point A, and may be called z, their difference FE $= \dot{z}$, and the little arch CF, defcribed with centre A, may be $= \dot{u}$. The nature of the curve is commonly expreffed by the differential equation

tion $p\dot{z} = \dot{u}$, in which p is any how given by z. Wherefore it muſt be obſerved, that the firſt member $p\dot{z}$, having the variable z, all which take their origin from the pole A, is integrable either algebraically or tranſcendentally. But the other member $\dot{u}$ cannot be integrated without falling into a parallogiſm, as not being yet the complete fluxion of the arch u. For that element $\dot{u}$ increaſes or decreaſes in a double reſpect, that is, in itſelf, and alſo by the increaſing or diminiſhing of the ordinates AC, AE. To proceed, therefore, with accuracy, with any radius at pleaſure, $AI = r$, let a circle IGH be deſcribed, and in the periphery let any determinate point I be taken, from which, as from a fixed point, the increaſing arches IG, IH, have their origin. And producing, if neceſſary, the variables AC, AE, to G and H, the ſectors ACF, AGH, will be ſimilar, and therefore it is $z \,.\, \dot{u} :: r \,.\, \dot{GH}$, which may be called $\dot{q}$. Then $\frac{z\dot{q}}{r} = \dot{u}$. But, by the general equation of the curve, it is $p\dot{z} = \dot{u}$; then $\frac{z\dot{q}}{r} = p\dot{z}$, and therefore $\frac{rp\dot{z}}{z} = \dot{q}$. Now, by finding the fluent, it will be $\int \frac{rp\dot{z}}{z} = q = IG$. The adding or taking away of the conſtants in the integration, will have no other effect, but to diverſify the ſituation of the point I.

EXAMPLE I.

Let the logarithmic ſpiral be to be conſtructed, the equation of which is $\frac{a\dot{z}}{b} = \dot{u}$. But $\dot{u} = \frac{z\dot{q}}{r}$, therefore $\frac{a\dot{z}}{b} = \frac{z\dot{q}}{r}$. Or, becauſe the radius AI is aſſumed at pleaſure, making $b = r$, and taking a as unity, it will be $\frac{\dot{z}}{z} = \dot{q}$. And by integration, $lz = q$, the geometrical conſtruction of which is tranſcendental, but yet is very ſimple.

EXAMPLE II.

Let it be the hyperbolical ſpiral, with the conſtant ſubtangent $= a$, and therefore the equation is $\frac{a\dot{z}}{z} = \dot{u}$. But $\dot{u} = \frac{z\dot{q}}{r}$, therefore $\frac{ar\dot{z}}{zz} = \dot{q}$; and by integrating, it will be $b - \frac{ar}{z} = q$.

In

In ſuch conſtructions we have always the circular arch IG, which forms the *homogeneum comparationis*; the other member $\int \frac{rp\dot{z}}{z}$ may be analytically integrable, as in the ſecond example, or tranſcendentally, by means of the quadrature of the hyperbola, as in the firſt, or by any other method more compounded. Whence, in one caſe only, our curves may be algebräical, and that is, when the quantity $\int \frac{rp\dot{z}}{z}$ may be reduced to the rectification of an arch of a circle, which to it's correſpondent IG is as number to number. If the proportion happen to be ſurd, then the curve will indeed be mechanic, as BCED, but not dependent on the quadrature of the circle, being reduced to a different problem, conſiſting in the dividing circular arches in any given ratio; which may be obtained by means of the helix or ſpiral of *Archimedes*, or of the quadratrix of *Dinoſtratus*.

The things afore-mentioned furniſh us with another manner of paſſing from expreſſions of curves to a focus, to thoſe which are referred to an axis, or on the contrary. For, becauſe $\frac{rp\dot{z}}{z} = \dot{q} = \frac{rr\dot{t}}{rr + tt}$, making the tangent IK $= t$, (§ 26.) this tangent t will be given analytically or tranſcendentally by z. But AI $= r$, AK $= \sqrt{rr + tt}$, AM $= x$, MC $= y$. Therefore $\frac{rz}{x} = \sqrt{rr + tt}$, and, after due reductions, $\frac{r\sqrt{zz - xx}}{x} = t = \frac{ry}{x}$. But t is given by z, and $z = \sqrt{xx + yy}$; ſo that we are arrived at the curve in reſpect to the axis, which may ſoon be reduced to the uſual co-ordinates x and y. By going the ſame ſteps backwards, we may paſs from the equation to the axis, to that in reſpect of the focus.

I reſume the example of § 87; that is, the curve $\frac{z\dot{z}}{\sqrt{cc - 2bz - zz}} = \dot{u}$ referred to a focus, to reduce it to the axis. Now, if $p\dot{z} = \dot{u}$ be taken for a general equation of curves referred to a focus, it will be, in this particular caſe, $p = \frac{z}{\sqrt{cc - 2bz - zz}}$. So that, ſubſtituting this value, inſtead of p, in the equation $\frac{rp\dot{z}}{z} = \dot{q} = \frac{rr\dot{t}}{rr + tt}$, it will be $\frac{r\dot{z}}{\sqrt{cc - 2bz - zz}} = \frac{rr\dot{t}}{rr + tt}$. Make $b + z = s$, $\dot{z} = \dot{s}$, $bb + 2bz + zz = ss$; whence $- 2bz - zz = bb - ss$. And ſubſtituting theſe values, it will be $\frac{r\dot{s}}{\sqrt{cc + bb - ss}} = \frac{rr\dot{t}}{rr + tt}$. Making $cc + bb = hh$, it will be $\frac{r\dot{s}}{\sqrt{hh - ss}} = \frac{rh\dot{s}}{h\sqrt{hh - ss}} = \frac{rr\dot{t}}{rr + tt}$. But the integral of the

the firſt member will be the arch of a circle, the radius of which is b, and s is the ſine of the complement (§ 37.) multiplied by the conſtant fraction $\frac{r}{b}$; and the integral of the ſecond is an arch of a circle with radius $= r$, and tangent equal to t. Wherefore the firſt arch will be to the ſecond as b to r, or they will be to each other as their radii reſpectively; then they will be ſimilar, and therefore their tangents alſo will be in the ſame ratio as their radii. Therefore the tangent of the firſt arch is $\frac{b}{s}\sqrt{bb - ss}$; and it will be $\frac{b}{s}\sqrt{bb - ss} \,.\, t :: b \,.\, r$, or $t = \frac{r}{s}\sqrt{bb - ss}$. So that, reſtoring the value of s, and putting $\frac{ry}{x}$ inſtead of t, we ſhall have $\frac{ry}{x} = \frac{r\sqrt{bb - bb - 2bz - zz}}{b + z}$, which is an equation reduced to the axis, and which may be expreſſed by the co-ordinates x and y only, by putting, inſtead of zz, it's value $xx + yy$. Then the equation will be $by + y\sqrt{xx+yy} = x\sqrt{bb - bb - 2b\sqrt{xx+yy} - xx - yy}$, which is the ſame as that found at § 87, as before cited.

To paſs from equations to an axis, to thoſe belonging to a focus, I take Example I. at § 88, the equation of which to the circle is $z = \sqrt{bx}$ (Fig. 104.) The tangent given by z of the arch OQ, deſcribed with centre A, and radius r, was found to be $\frac{r\sqrt{bb - zz}}{z} = t$. Then, in the canonical equation $\dot{q} = \frac{rr\dot{t}}{rr + tt}$, inſtead of t and $\dot{t}$, ſubſtituting their reſpective values, we ſhall have $-\dot{q} = -\frac{r\dot{z}}{\sqrt{bb - zz}}$. I put it $-\dot{q}$, becauſe, as AC $= z$ increaſes, the arch OQ $= q$ will diminiſh. But $\dot{q} = \frac{r\dot{u}}{z}$; wherefore $\frac{r\dot{u}}{z} = \frac{r\dot{z}}{\sqrt{bb - zz}}$; that is, $\frac{z\dot{z}}{\sqrt{bb - zz}} = \dot{u}$, which is the ſame equation as that found at § 88.

91. The particular formulæ, which are found in the caſe of curves having their co-ordinates at oblique angles, are not leſs uſeful, becauſe ſuch equations may always be changed into others, which have their co-ordinates at right angles; and after that we may make uſe of the ordinary formulæ.

To ſhow this, make HG $= p$, (Fig. 103.) AG $= q$; then it is $p = \frac{ny}{m}$, $q = x + \frac{ey}{m}$, naming, as before, AB $= x$, BD $= y$, and the ratio of the whole ſine to the right ſine that of m to e. Therefore it will be $y = \frac{mp}{n}$, and $x = q - \frac{ey}{m} = q - \frac{ep}{n}$. Wherefore, inſtead of x and y, ſubſti-

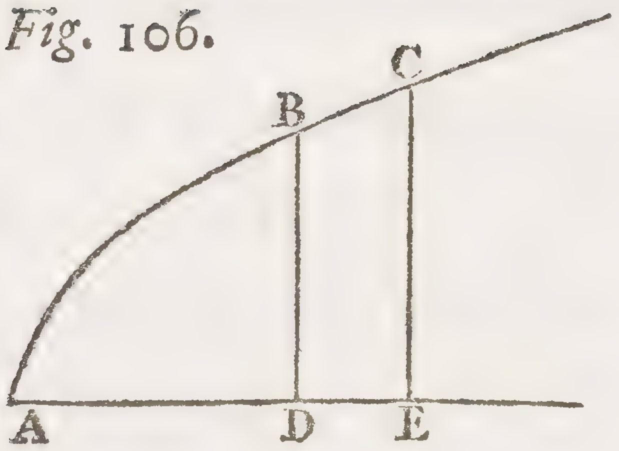
tuting, in the propofed equation, thefe values given by p and q, we fhall have the equation of the curve with the ordinates at right angles to each other. But it will often happen that the primitive equation will be fimple; and yet, by transforming it, it may become fufficiently compound. Alfo, though the variables are feparate in the propofed equation, they may not be fo in the transformed equation; and what may increafe the difficulty, they cannot be feparated by the ordinary rules of Divifion, Extraction of Roots, &c. However, in many particular cafes, perhaps it may not be amifs to try each method, that we may make choice of that which, in the given cafe, fhall be moft convenient.

But now it will be time to proceed to Examples, in which it is always underftood, except when warning is given to the contrary, that the co-ordinates are at right angles to each other.

EXAMPLE I.

The quadrature of curvilinear fpaces.

Fig. 106.



92. Let ABC be an *Apollonian* parabola, with the equation $ax = yy$, any abfcifs AD $= x$, it's ordinate DB $= y$, and the fpace ADB is to be fquared. Therefore it will be $y = \sqrt{ax}$; and this value, being fubftituted, inftead of y, in the general formula for fpaces $y\dot{x}$, it will be $\dot{x}\sqrt{ax}$; and by integration, it will be $\frac{2}{3}x\sqrt{ax} + b$. The quantity b is the ufual conftant, which, in the integration, ought to be added, and which now ought to be determined. In the point A, that is, when $x = 0$, the fpace is nothing, and therefore the integral $\frac{2}{3}x\sqrt{ax} + b$, which expreffes this fpace, ought alfo to be nothing. Therefore, making $x = 0$, it will be $\frac{2}{3}0 \times \sqrt{a} \times 0 + b = 0$, that is, $b = 0$; which is as much as to fay that, in this cafe, no conftant quantity is to be joined to the integral. Therefore the fpace ABD $= \frac{2}{3}x\sqrt{ax}$. But $\sqrt{ax} = y$. Whence ABD $= \frac{2}{3}xy$, that is, is equal to two third parts of the rectangle of the abfcifs into the ordinate.

Now, if we fhould require the fpace comprehended by an affigned and determinate abfcifs and ordinate, for example, when it is $x = 2a$; as, by the equation of the curve, it is in this cafe $y = \sqrt{2aa}$, this fpace will be $= \frac{4}{3}aa\sqrt{2}$. If the abfciffes of the parabola fhould not begin at the vertex A, but at fome given point D; making, for example, AD $= a$, any line DE $= x$, the parameter $= f$, the equation will be $af + fx = yy$, and $y = \sqrt{af + fx}$. Subftituting

tuting this value in the formula $y\dot{x}$, it will be $\dot{x}\sqrt{af + fx}$, and by integrating, $\frac{2}{3} \times \overline{a + x} \times \sqrt{af + fx} + b$ will be equal to the ſpace DECB. But, to determine the conſtant quantity b, it muſt be conſidered, that at the point D, where $x = 0$, the ſpace will alſo be $= 0$; ſo that, in the integral, making $x = 0$, it will be $\frac{2}{3}a\sqrt{af} + b = 0$, and therefore the conſtant $b = -\frac{2}{3}a\sqrt{af}$. So that, to have the integral complete, inſtead of adding b, we muſt ſubtract $\frac{2}{3}a\sqrt{af}$, and therefore the ſpace required will be DECB $= \frac{2}{3} \times \overline{a + x} \times \sqrt{af + fx} - \frac{2}{3}a\sqrt{af}$.

Let AE $= a$, and let x begin at E towards A, and take any line ED $= x$; the equation will be $af - fx = yy$, and $y = \sqrt{af - fx}$. Whence $y\dot{x} = \dot{x}\sqrt{af - fx}$, and by integration, it will be $-\frac{2}{3} \times \overline{a - x} \times \sqrt{af - fx} + b$. But when $x = 0$, the ſpace alſo $= 0$. Therefore, in the integral, making $x = 0$, it will become $-\frac{2}{3}a\sqrt{af} + b = 0$, or $b = \frac{2}{3}a\sqrt{af}$. Therefore the ſpace EDBC $= \frac{2}{3}a\sqrt{af} - \frac{2}{3}\overline{a - x}\sqrt{af - fx}$.

It may be obſerved, that, in general, the parabolical ſpace AEC $= \frac{2}{3}$AE $\times$ EC; wherefore the ſpace ADB $= \frac{2}{3}$AD $\times$ DB; ſo that the ſpace DECB will be $= \frac{2}{3}$AE $\times$ EC $- \frac{2}{3}$AD $\times$ DB; which agrees with the calculus in both caſes, when the origin of x is in the point D towards E, and in the point E towards D.

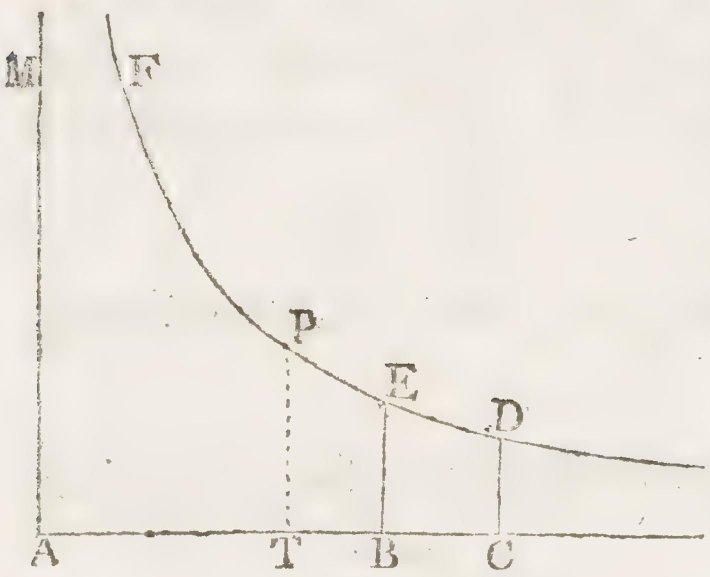
I take the general equation to all parabolas, of what degree ſoever, $a^m x^n = y^r$; whence it will be $y = a^{\frac{m}{r}} x^{\frac{n}{r}}$, and therefore the formula $y\dot{x} = a^{\frac{m}{r}} x^{\frac{n}{r}} \dot{x}$; and, by integration, the ſpace will be $= \frac{ra^{\frac{m}{r}} x^{\frac{n+r}{r}}}{n + r} + b$. But, taking $x = 0$, it is found that $b = 0$; ſo that there is no conſtant quantity to be annexed to it, but the integral before found is complete. Now, putting y inſtead of $a^{\frac{m}{r}} x^{\frac{n}{r}}$, it will be $\frac{rxy}{n + r} =$ to the ſpace required.

EXAMPLE II.

93. Let the curve be $y = \sqrt[m]{x + a}$; therefore it will be $y\dot{x} = \dot{x}\sqrt[m]{x + a}$; and, by integration, the ſpace will be $\frac{m}{m+1} \times \overline{x + a} \times \overline{x + a}|^{\frac{1}{m}} + b$. But, making $x = 0$, it will be $b = -\frac{m}{m+1} \times a\sqrt[m]{a}$. Therefore the complete integral or ſpace required $= \frac{m}{m+1} \times \overline{x + a} \times \sqrt[m]{x + a} - \frac{m}{m+1} \times a\sqrt[m]{a}$.

EXAMPLE III.

Fig. 107.



94. Let FED be the hyperbola between the aſymptotes, and make AB $= x$, BE $= y$, and the equation is $xy = aa$. Then $y = \frac{aa}{x}$, and therefore $y\dot{x} = \frac{aa\dot{x}}{x}$; and, by integration, the ſpace will be $= alx + b$, taking the logarithm from the logarithmic curve with ſubtangent $= a$. But, putting $x = 0$, the logarithm of 0 is an infinite negative quantity, and therefore the ſpace is infinite which is contained by the curve EF continued *in infinitum*, by the aſymptote, and by the co-ordinates AB, BE.

Let there be a hyperboloid of this equation $a^3 = xyy$; then $y = \sqrt{\frac{a^3}{x}}$, and therefore $y\dot{x} = \dot{x}\sqrt{\frac{a^3}{x}}$; and, by integration, the ſpace will be $= 2\sqrt{a^3x} + b$. Now, putting $x = 0$, it is $b = 0$; therefore no conſtant quantity need be added to complete the integral. So that the ſpace ABEF, infinitely produced upwards, will be the finite quantity $2\sqrt{a^3x}$, or from the equation of the curve $= 2xy$.

Let there be a hyperboloid of this equation, $a^3 = xxy$; then $y = \frac{a^3}{x^2}$, and $y\dot{x} = \frac{a^3\dot{x}}{xx}$; and, by integration, the ſpace will be $= -\frac{a^3}{x} + b$. But, putting

$x = 0$,

$x = 0$, it will be $\frac{a^3}{0}$, an infinite quantity, and therefore b is infinite. Wherefore, to have the integral complete, an infinite quantity ought to be added to it, and therefore the ſpace itſelf is infinite.

Let the equation be $a^{m+n} = x^n y^m$, which is to all hyperboloids in general; then $y = a^{\frac{m+n}{m}} x^{-\frac{n}{m}}$, and therefore $\int y\dot{x} = \frac{ma^{\frac{m+n}{m}} x^{\frac{m-n}{m}}}{m-n} + b$. Now, if $m = 1$, $n = 1$, that is, $xy = aa$, we ſhould have $\int y\dot{x} = \frac{aa}{0} + b$, an infinite quantity; whence the ſpace will be infinite, as was ſeen before.

If $n = 1$, $m = 2$, that is, $a^3 = xyy$, then $\int y\dot{x} = 2\sqrt{a^3x} + b$. But, putting $x = 0$, it will be alſo $b = 0$; therefore the complete integral, or the ſpace required, will be $= 2\sqrt{a^3x} = 2xy$, by the equation of the curve; which is therefore finite, though infinitely produced upwards towards F.

If $n = 2$, $m = 1$, that is, $a^3 = xxy$, it will be $\int y\dot{x} = -\frac{a^3}{x} + b$. But, making $x = 0$, b will be infinite; ſo that an infinite quantity is to be added to the integral, and the ſpace itſelf will be infinite.

If $n = 1$, $m = 3$, that is, $a^4 = xy^3$; it will be $\int y\dot{x} = \frac{3}{2}a^{\frac{4}{3}}x^{\frac{2}{3}} + b$. But, making $x = 0$, it will be $b = 0$, and therefore the integral is complete. That is, the ſpace will be $= \sqrt[3]{a^4xx} = \frac{3}{2}xy$, a finite quantity, however infinitely produced upwards.

If $n = 3$, $m = 1$, that is, $a^4 = x^3y$; it will be $\int y\dot{x} = -\frac{a^4}{2xx} + b$. But, making $x = 0$, b will be infinite, and therefore the ſpace is infinite.

If $n = 1$, $m = 4$, that is, $a^5 = xy^4$; it will be $\int y\dot{x} = \frac{4}{3}\sqrt[4]{a^5x^3} + b$. But, making $x = 0$, it will be $b = 0$; ſo that the integral is complete, and the whole ſpace $= \frac{4}{3}\sqrt[4]{a^5x^3} = \frac{4}{3}xy$, a finite quantity.

If $n = 4$, $m = 1$, that is, $a^5 = x^4y$; it will be $\int y\dot{x} = -\frac{a^5}{3x^3} + b$. Now making $x = 0$, b will be infinite, and therefore the ſpace is infinite. In the ſame manner we might proceed to other caſes, as far as we pleaſe.

Now let us take the abſciſſes from the point B, to find the ſpace BCDE. Make AB $= b$, BC $= x$, CD $= y$, and let it be the ſame *Apollonian* hyperbola, whoſe equation is $by + xy = aa$. Then it will be $y = \frac{aa}{b+x}$, and

therefore $y\dot{x} = \frac{aa\dot{x}}{b+x}$. Then, by integration, $\int y\dot{x} = al\overline{b+x} + f$, taking the logarithm from the logarithmic with ſubtangent $= a$. But, to determine the conſtant quantity f, making $x = 0$, it ought to be $f = -alb$; ſo that the complete integral or ſpace BCDE will be $al\overline{b+x} - alb$.

If we take x negative $=$ BA $= -b$, then $al\overline{b+x}$ is equal to a multiplied into the logarithm of 0. But the logarithm of 0 is an infinite negative quantity; ſo that, in this caſe, the ſpace is negative, that is, towards M, and alſo infinite, as has been ſeen above; and therefore the ſpace between the *Apollonian* hyperbola and it's aſymptotes is infinite, being infinitely produced both ways.

Let it be the cubical hyperboloid whoſe equation is $byy + xyy = a^3$. It will be $y = \sqrt{\frac{a^3}{b+x}}$, whence $y\dot{x} = \dot{x}\sqrt{\frac{a^3}{b+x}}$, and by integration, $\int y\dot{x} = 2\sqrt{a^3b + a^3x} + f$. But, making $x = 0$, it will be $f = -2\sqrt{a^3b}$; ſo that the complete integral or ſpace EBCD will be $= 2\sqrt{a^3b + a^3x} - 2\sqrt{a^3b}$; and taking x infinite, the ſpace EBCD, infinitely produced towards C, will be infinite alſo.

Taking x negative $=$ BA $= -b$, the integral will be $-2\sqrt{a^3b}$, ſo that the ſpace will be negative; that is, it will be FEBAM, and will be finite, however infinitely produced towards M; as is alſo ſeen before.

Let it be the hyperboloid of this equation $\overline{b+x}|^2 \times y = a^3$. It will be $y = \frac{a^3}{\overline{b+x}|^2}$, whence $y\dot{x} = \frac{a^3\dot{x}}{\overline{b+x}|^2}$. And, by integrating, $\int y\dot{x} = -\frac{a^3}{b+x} + f$. Now, putting $x = 0$, it will be $f = \frac{a^3}{b}$, and therefore the complete integral, or the ſpace EBCD, will be $\frac{a^3}{b} - \frac{a^3}{b+x}$. Taking x infinite, the term $-\frac{a^3}{b+x}$ will be $= 0$; ſo that the ſpace will be finite, though infinitely produced towards C. Let x be negative $=$ BA $= -b$; the integral will be $\frac{a^3}{b} - \frac{a^3}{0}$. But $-\frac{a^3}{0}$ is infinite and negative, and therefore the ſpace towards M will be infinite. By proceeding in this manner, we may find that the ſpace between the *Apollonian* hyperbola and it's aſymptotes, produced both ways infinitely, will be infinite; between the firſt cubical hyperboloid and it's aſymptotes, it will be finite towards M, and infinite towards C; between the ſecond cubical hyperboloid and it's aſymptotes, it will be infinite towards M, and finite towards C; between the firſt hyperboloid of the fourth kind and it's aſymptotes,

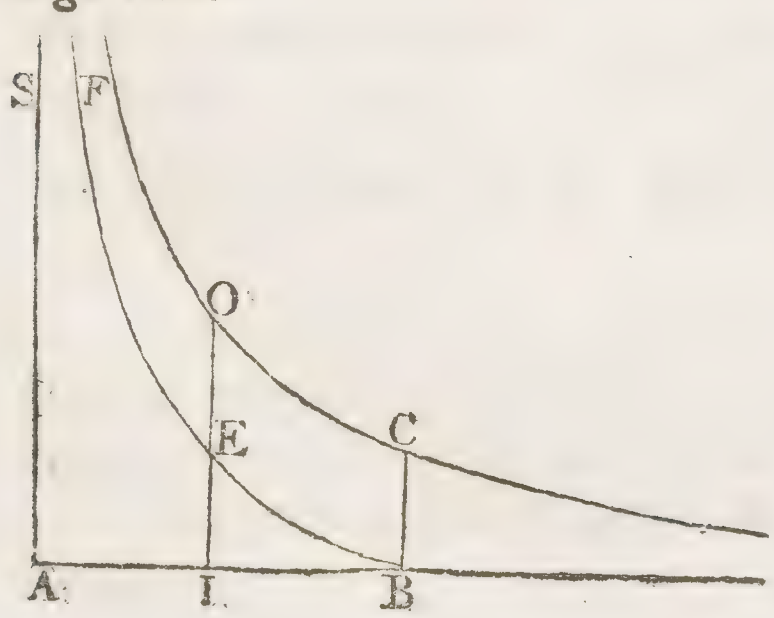
asymptotes, it will be finite towards M, and infinite towards C; between the second hyperboloid and it's asymptotes, it will be finite towards C, and infinite towards M. And so on.

Now, to have recourse to infinite series: I take the expression of the space BCDE, of the aforesaid *Apollonian* hyperbola, that is, $\frac{aa\dot{x}}{b+x}$. This, reduced into a series, will be $= \frac{a^2\dot{x}}{b} - \frac{a^2x\dot{x}}{bb} + \frac{a^2x^2\dot{x}}{b^3} - \frac{ax^3\dot{x}}{b^4}$, &c. And, by integration, $\frac{a^2x}{b} - \frac{a^2x^2}{2b^2} + \frac{a^2x^3}{3b^3} - \frac{ax^4}{4b^3}$, &c.; which series, infinitely continued, will be accurately equal to the space BCDE. And if it were summable, it would give us the space required in finite terms, that is, algebräically: and this would be the true quadrature of the hyperbola. But as this is not summable, the more terms we take of it, beginning with the first, the nearer approach we shall make to the just value of this space.

Now I take the abscis BT on the negative side, and the equation of the curve will be $by - xy = aa$, and therefore $y\dot{x} = \frac{aa\dot{x}}{b-x}$; and, reducing to a series, it will be $y\dot{x} = \frac{a^2\dot{x}}{b} + \frac{a^2x\dot{x}}{b^2} + \frac{a^2x^2\dot{x}}{b^3} + \frac{a^2x^3\dot{x}}{b^4} + \frac{a^2x^4\dot{x}}{b^5}$, &c. And by integration, $\int y\dot{x} = \frac{a^2x}{b} + \frac{a^2x^2}{2b^2} + \frac{a^2x^3}{3b^3} + \frac{a^2x^4}{4b^4} + \frac{a^2x^5}{5b^5}$, &c. which is equal to the space BTPE. Taking BT = BA, the space FEBAM, infinitely produced towards M, will be $= aa + \frac{1}{2}aa + \frac{1}{3}aa + \frac{1}{4}aa + \frac{1}{5}aa$, &c.; the value of which series being infinite, the space it denotes will be infinite also.

EXAMPLE IV.

Fig. 108.

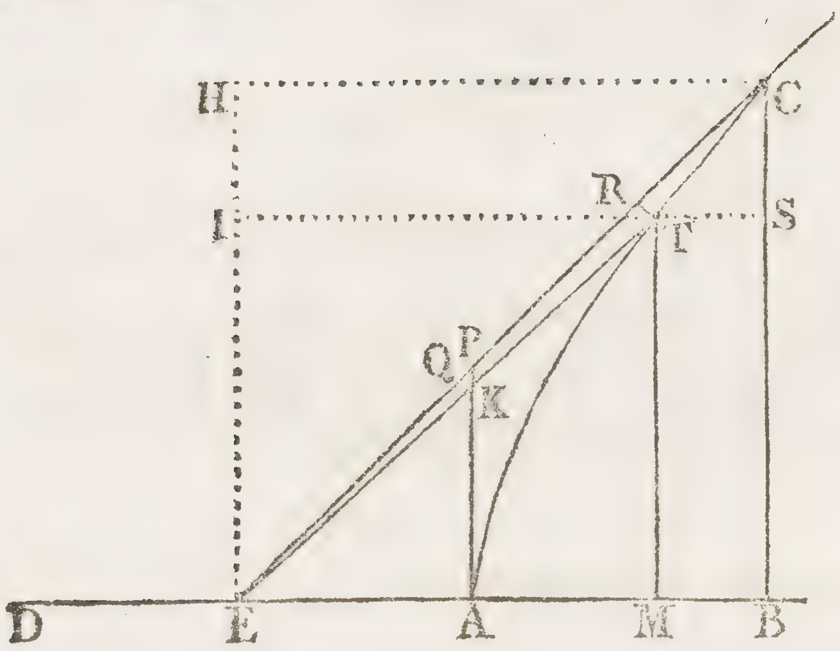


95. Let OC be an equilateral hyperbola between the asymptotes AS, AB, and make AB = BC = a, BI = $-x$. Let the mechanical curve BEF be conceived to be described, such, that the rectangle of AB into any ordinate IE may be equal to the corresponding hyperbolical space BCOI. The indeterminate space SABEF is required. Make the ordinate IE = z. It has been found already, that the space BCOI is equal to

to the ſeries $ax + \frac{1}{2}x^2 + \frac{x^3}{3a} + \frac{x^4}{4a^2} + \frac{x^5}{5a^3}$, &c. making a and b equal. Then, by the property of the curve, it will be $z = x + \frac{x^2}{2a} + \frac{x^3}{3a^2} + \frac{x^4}{4a^3}$, &c. and therefore $z\dot{x} = x\dot{x} + \frac{x^2\dot{x}}{2a} + \frac{x^3\dot{x}}{3a^2} + \frac{x^4\dot{x}}{4a^3}$, &c. And finally, by integration, the ſpace BIE will be $= \frac{xx}{2} + \frac{x^3}{6a} + \frac{x^4}{12a^2} + \frac{x^5}{20a^3} + \frac{x^6}{30a^4}$, &c. Now, taking $x = a = $ BA, as to the whole ſpace SABEF infinitely produced, it will be $= \frac{1}{2}aa + \frac{1}{6}aa + \frac{1}{12}aa + \frac{1}{20}aa + \frac{1}{30}aa$, &c. which ſeries is ſummable, and is $= aa$; ſo that it is algebräically quadrable, and the ſpace SABEF, infinitely produced, is equal to the ſquare of BA.

EXAMPLE V.

Fig. 109.



96. Let ATC be a hyperbola, it's tranſverſe axis AD $= 2a$, the parameter $= p$, EB $= x$, BC $= y$, and therefore the equation $xx - aa = \frac{2ayy}{p}$, and let the ſpace ABC be required. It will be therefore $y = \sqrt{\frac{px^2 - pa^2}{2a}}$, and the formula will be $y\dot{x} = \dot{x}\sqrt{\frac{pxx - paa}{2a}}$. Now, if we proceed to integration, we ſhould find, after the uſual manner, that the integral is partly algebräical, and partly logarithmical; ſo that the ſpace ABC of the hyperbola depends on the deſcription of the logarithmic curve.

If we would have the ſpace ACHE; making MT infinitely near to BC, it's element will be the infiniteſimal ſpace ITCH; and therefore the formula will be $x\dot{y}$, in which, inſtead of x, ſubſtituting it's value given by y from the equation, it will be $x\dot{y} = \dot{y}\sqrt{\frac{2ayy + aap}{p}}$, the integral of which, in the ſame manner, depends upon the logarithmic curve.

And, as well in the formula $y\dot{x}$ of the firſt ſpace, as in $x\dot{y}$ of the ſecond, if, inſtead of $\dot{x}$ in that, or of $\dot{y}$ in this, we ſhould ſubſtitute their reſpective values given from the equation; we ſhould likewiſe find integrals of the ſame nature.

Now,

Now, to return to infinite ſeries. I take the formula of the ſpace ACHEA, that is, $x\dot{y}$. Then $x\dot{y} = \dot{y}\sqrt{\frac{2ayy + aap}{p}}$; and, for greater facility, making $2a = p$, (for the conſtants make no alteration in the method,) that is, ſuppoſing the hyperbola to be equilateral, it will be $x\dot{y} = \dot{y}\sqrt{yy + aa}$; and, reducing the radical to an infinite ſeries, it will be $x\dot{y} = a\dot{y} + \frac{y^2\dot{y}}{2a} - \frac{y^4\dot{y}}{8a^3} + \frac{y^6\dot{y}}{16a^5} - \frac{5y^8\dot{y}}{128a^7}$, &c. And by integration, $\int x\dot{y}$, or the ſpace ACHEA, $= ay + \frac{y^3}{6a} - \frac{y^5}{40a^3} + \frac{y^7}{7 \times 16a^5} - \frac{5y^9}{9 \times 128a^7}$, &c. a ſeries, the ſummation of which is unknown. And ſubtracting this ſeries from the rectangle xy, we ſhould have the ſpace ABC.

From the centre E let the lines ET, EC, be drawn infinitely near, and let AKP be a tangent at the vertex. With centre E let the little circular arches KQ, TR, be drawn. It will be $AK = \frac{ay}{x}$, $KP = \frac{ax\dot{y} - ay\dot{x}}{xx}$, $ET = \sqrt{xx + yy}$, $EK = \frac{a\sqrt{xx + yy}}{x}$. And, becauſe of ſimilar triangles PKQ, KEA, or TEM, it will be $KQ = \frac{ax\dot{y} - ay\dot{x}}{x\sqrt{xx + yy}}$. And, becauſe of ſimilar ſectors EKQ, ETR, it will be $TR = \frac{x\dot{y} - y\dot{x}}{\sqrt{xx + yy}}$; and therefore it will be $\frac{1}{2}ET \times TR = \frac{x\dot{y} - y\dot{x}}{2}$, the element of the ſector ETA. And, inſtead of y and $\dot{y}$, ſubſtituting their values given from the equation of the curve $y = \sqrt{xx - aa}$, (ſuppoſing the hyperbola to be equilateral,) it will be $\frac{aa\dot{x}}{2\sqrt{xx - aa}}$; and by integration, $\int \frac{aa\dot{x}}{2\sqrt{xx - aa}}$, that is, the ſector ETA, will be equal to $- \frac{1}{2}al\,x - \overline{\sqrt{xx - aa}}$ in the logarithmic with ſubtangent $= a$; which ſpace is therefore expreſſed by a negative quantity, becauſe it is aſſumed on the negative ſide.

By reducing the formula into a ſeries, we ſhall find $\frac{aa\dot{x}}{2\sqrt{xx - aa}} = \frac{aa\dot{x}}{2x} + \frac{a^4\dot{x}}{4x^3} + \frac{3a^6\dot{x}}{16x^5} + \frac{5a^8\dot{x}}{32x^7} + \frac{35a^{10}\dot{x}}{256x^9}$, &c.

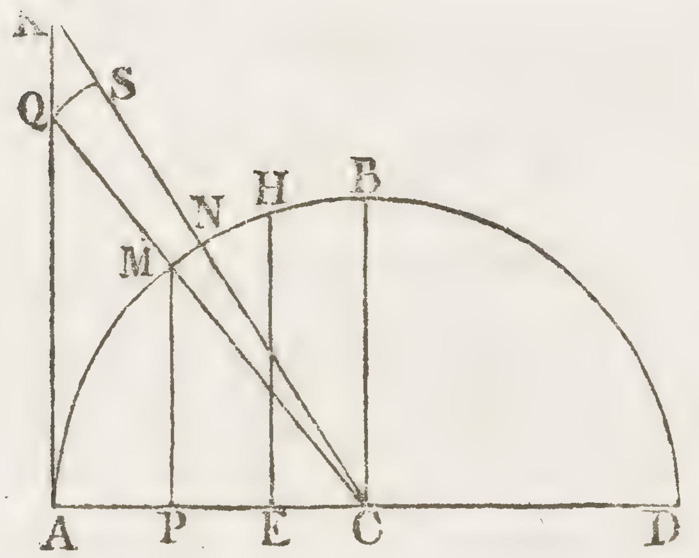
Now, to integrate the firſt term of the ſeries, there would be occaſion, firſt, to reduce it to an infinite ſeries. Therefore it would be better to do it more expeditiouſly after the following manner. Make $EM = x$, $MT = y$, $AK = z$, then $KP = \dot{z}$. Make $KE = p$, $AE = a$, the tranſverſe ſemiaxis, and the

8 ſemi-

ſemi-conjugate $= b$. Therefore it will be KQ $= \frac{a\dot{z}}{p}$, ET $= \frac{px}{a}$, TR $= \frac{x\dot{z}}{p}$, and therefore $\frac{1}{2}$ET $\times$ TR $= \frac{xx\dot{z}}{2a}$. But, by the equation of the curve, it is $y = \frac{b}{a}\sqrt{xx - aa}$; and by ſimilar triangles EAK, EMT, it will be $y = \frac{xz}{a}$. Therefore $zx = b\sqrt{xx - aa}$, and $xx = \frac{aabb}{bb - zz}$, and conſequently the formula will be $\frac{\frac{1}{2}abb\dot{z}}{bb - zz}$, which, reduced to a ſeries, will be $\frac{a\dot{z}}{2} + \frac{az^2\dot{z}}{2b^2} + \frac{az^4\dot{z}}{2b^4} + \frac{az^6\dot{z}}{2b^6} + \frac{az^8\dot{z}}{2b^8}$, &c.; and by integration, $\int \frac{\frac{1}{2}abb\dot{z}}{bb - zz}$, that is, the ſpace ETA, will be $= \frac{az}{2} + \frac{az^3}{6b^2} + \frac{az^5}{10b^4} + \frac{az^7}{14b^6} + \frac{az^9}{18b^3}$, &c.

EXAMPLE VI.

Fig. 110.



97. Let ABD be a circle deſcribed with diameter AD $= a$, and let the area of any half ſegment AHE be required. Make AE $= x$, EH $= y$; the equation will be $y = \sqrt{ax - xx}$, and therefore $y\dot{x} = \dot{x}\sqrt{ax - xx}$. Here it would be to no purpoſe to free the formula from it's radical, or to try any other methods, in order to change it into ſome other formula, which may admit of an algebräical integration, or by means of the logarithms. For this would be an uſeleſs trouble, becauſe we ſhould ſtill be brought to a formula of quadrature or rectification of the circle; as has been obſerved at § 37. And therefore we ſhall thus proceed by way of infinite ſeries.

Reſolving the formula into a ſeries, it will be $\dot{x}\sqrt{ax - xx} = a^{\frac{1}{2}}x^{\frac{1}{2}}\dot{x} - \frac{x^{\frac{3}{2}}\dot{x}}{2a^{\frac{1}{2}}} - \frac{x^{\frac{5}{2}}\dot{x}}{8a^{\frac{3}{2}}} - \frac{x^{\frac{7}{2}}\dot{x}}{16a^{\frac{5}{2}}}$, &c. And by integration, $\int y\dot{x}$, or the ſpace AEH $= \frac{2}{3}a^{\frac{1}{2}}x^{\frac{3}{2}} - \frac{x^{\frac{5}{2}}}{5a^{\frac{1}{2}}} - \frac{x^{\frac{7}{2}}}{28a^{\frac{3}{2}}} - \frac{x^{\frac{9}{2}}}{72a^{\frac{5}{2}}}$, &c.

Now

Now make the radius CA $= a$, and let CE $= x$, EH $= y$, and the equation will be $y = \sqrt{aa - xx}$. Therefore $y\dot{x} = \dot{x}\sqrt{aa - xx}$; and reducing this to a feries, $y\dot{x} = a\dot{x} - \frac{x^2\dot{x}}{2a} - \frac{x^4\dot{x}}{8a^3} - \frac{x^6\dot{x}}{16a^5} - \frac{5x^8\dot{x}}{128a^7}$, &c. And by integration, $\int y\dot{x}$, that is, the fpace CEHB $= ax - \frac{x^3}{6a} - \frac{x^5}{40a^3} - \frac{x^7}{112a^5} - \frac{5x^9}{1152a^7}$, &c.

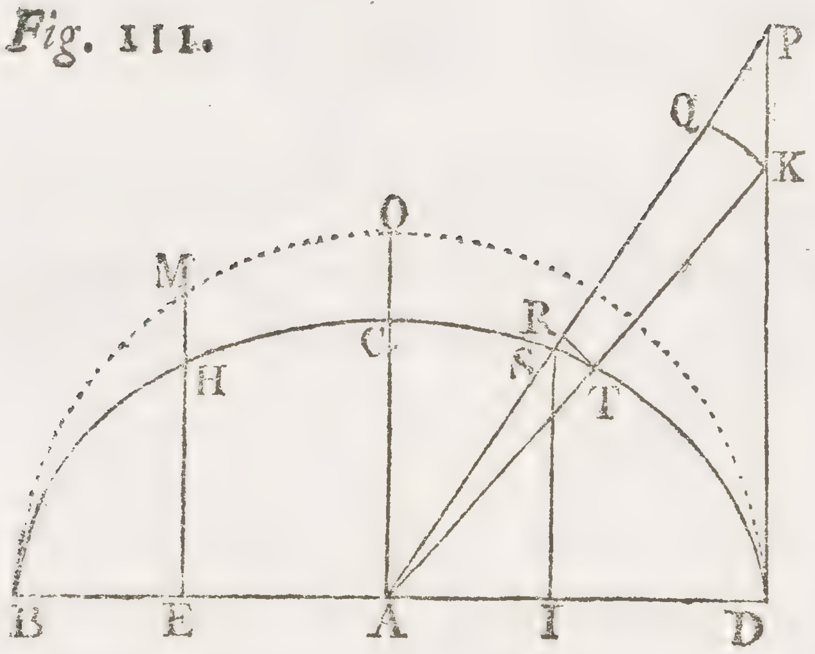
And making $x = a$, in refpect of the whole quadrant, it will be $aa - \frac{1}{6}aa - \frac{1}{40}aa - \frac{1}{112}aa - \frac{5}{1152}aa$, &c. the quadruple of which feries will be the area of the whole circle.

Now, by means of a fector. Make CA $= a$, AQ $= x$, and drawing CK infinitely near to CQ, it will be QK $= \dot{x}$, CQ $= \sqrt{aa + xx}$; and with centre C defcribing the infinitefimal arch QS; becaufe of fimilar triangles KSQ, QAC, it will be QS $= \frac{a\dot{x}}{\sqrt{aa + xx}}$, and therefore MN $= \frac{aa\dot{x}}{aa + xx}$. Whence the little fector CMN, the element of the fector CAM, will be $= \frac{a^3\dot{x}}{2 \times \overline{aa + xx}}$, which, reduced into a feries, will be $= \frac{a^3\dot{x}}{2a^2} - \frac{a^3x^2\dot{x}}{2a^4} + \frac{a^3x^4\dot{x}}{2a^6} - \frac{a^3x^6\dot{x}}{2a^8}$, &c. And by integration, it will be $\int \frac{\frac{1}{2}a^3\dot{x}}{aa + xx}$, or the fector CMA $= \frac{ax}{2} - \frac{x^3}{6a} + \frac{x^5}{10a^3} - \frac{x^7}{14a^5} + \frac{x^9}{18a^7}$, &c.; and fuppofing the arch AM to be half the quadrant, that is, taking $x = a$, the feries is $\frac{aa}{2} - \frac{aa}{6} + \frac{aa}{10} - \frac{aa}{14}$, &c.; and the double of this, or $aa - \frac{1}{3}aa + \frac{1}{5}aa - \frac{1}{7}aa$, &c. will be the quadrant ABC.

Inftead of taking the radius CA $= a$, if I had taken it $= \sqrt{\frac{1}{8}aa}$, the quadrant would have been ABC $= \frac{aa}{8} - \frac{aa}{3 \times 8} + \frac{aa}{5 \times 8} - \frac{aa}{7 \times 8}$, &c.; and actually fubtracting every negative term from the pofitive term before it, [and multiplying the refult by 4,] it would be $\frac{aa}{3} + \frac{aa}{35} + \frac{aa}{99} + \frac{aa}{195}$, &c. [= the area of the whole circle;] which is the fame feries as is inferted by Mr. *Leibnitz* in the *Leipfic* Acts, for the year 1682.

EXAMPLE VII.

Fig. 111.



98. Let BCD be an ellipsis, the transverse semiaxis AB $= a$, the semi-conjugate AC $= b$, AE $= x$, EH $= y$; whence the equation will be $\frac{bb}{aa} \times \overline{aa - xx} = yy$, and therefore $y\dot{x} = \frac{b\dot{x}}{a}\sqrt{aa - xx}$, the element of the area AEHC. But $\dot{x}\sqrt{aa - xx}$ is the formula for squaring the circle BOD, the diameter of which is the transverse axis of the ellipsis; so that the quadrature of the ellipsis will depend on that of the circle. And because $\int\frac{b\dot{x}}{a}\sqrt{aa - xx} = $ EHCA, and $\int\dot{x}\sqrt{aa - xx} = $ EMOA, any space of the ellipsis to the correspondent space of the circle on the diameter DB, will be as b to a, that is, as the conjugate semiaxis to the transverse semiaxis; and consequently the whole ellipsis to the whole circle will be in the same ratio. But, as circles are to each other as the squares of their diameters or radii, if we take a circle the radius of which is $= \sqrt{ab}$, that is, a mean proportional between the two semiaxes of the ellipsis BCD, this circle will be to the circle BOD as $ab \,.\, aa :: b \,.\, a$. But the area of the ellipsis BCD is to the same circle BOD, in this very ratio. Therefore the area of the ellipsis will be equal to the area of the circle, the radius of which is a mean proportional between the two semiaxes of the ellipsis.

Now, by the help of series. The formula $\frac{b\dot{x}}{a}\sqrt{aa - xx}$, being reduced to a series, will be $= \frac{b\dot{x}}{a}$ into $a - \frac{x^2}{2a} - \frac{x^4}{8a^3} - \frac{x^6}{16a^5} - \frac{5x^8}{128a^7}$, &c. And by integration, $\int\frac{b\dot{x}}{a}\sqrt{aa - xx}$, or area ACHE, $= bx - \frac{bx^3}{6a^2} - \frac{bx^5}{40a^4} - \frac{bx^7}{112a^6} - \frac{5bx^9}{1152a^8}$, &c. And making $x = a$, the area ACB, or a fourth part of the ellipsis, will be $= ab$ into $1 - \frac{1}{6} - \frac{1}{40} - \frac{1}{112} - \frac{5}{1152}$, &c.

In the same ellipsis, taking any arch DS, let DP be a tangent in D, AI $= x$, IS $= y$; and through the point S drawing AP infinitely near AK, which cuts the ellipsis in T. With centre A let the little arches of a circle KQ, TR, be described.

defcribed. Then it will be AS $= \sqrt{xx + yy} =$ AT, DP $= \frac{a y}{x}$, AK$=$AP $= \frac{a\sqrt{xx + yy}}{x}$, KP $= \frac{-ax\dot{y} + ay\dot{x}}{xx}$, PK being a negative difference. And by the fimilitude of the triangles PQK, PAD, it will be KQ $= \frac{-ax\dot{y} + ay\dot{x}}{x\sqrt{xx + yy}}$. And by the fimilitude of the fectors ATR, AKQ, it will be TR $= \frac{-x\dot{y} + y\dot{x}}{\sqrt{xx + yy}}$, and therefore $\frac{1}{2}$TR $\times$ AT, that is, $\frac{-x\dot{y} + y\dot{x}}{2}$, will be the formula for the fpace ACT. This will be finally $\frac{ab\dot{x}}{2\sqrt{aa - xx}}$, by fubftituting, inftead of y and $\dot{y}$, their values given from the equation of the curve.

But $\int \frac{a\dot{x}}{\sqrt{aa - xx}}$ is the rectification of the circle, as was feen at § 37, and as will be here feen alfo. Therefore the quadrature of elliptical fectors will depend on the rectification or quadrature of the circle. It would fignify nothing to take pains to free the formula from it's radical, becaufe, notwithftanding this, we fhould ftill fall upon a formula, which would depend on the fame circle.

Now, by the means of feries, we fhould find it to be $\frac{ab\dot{x}}{2\sqrt{aa - xx}} = \frac{b\dot{x}}{2} + \frac{bx^2\dot{x}}{4aa} + \frac{3bx^4\dot{x}}{16a^4} + \frac{5bx^6\dot{x}}{32a^6} + \frac{35bx^8\dot{x}}{256a^8}$, &c. And by integration, the fpace ATC $= \frac{bx}{2} + \frac{bx^3}{12a^2} + \frac{3bx^5}{80a^4} + \frac{5bx^7}{224a^6} + \frac{35bx^9}{2304a^8}$, &c. And making $x = a$, in refpect to the whole fpace ADC, a fourth part of the entire elliptical fpace, it will be $= \frac{ab}{2} + \frac{ab}{12} + \frac{3ab}{80} + \frac{5ab}{224} + \frac{35ab}{2304}$, &c.

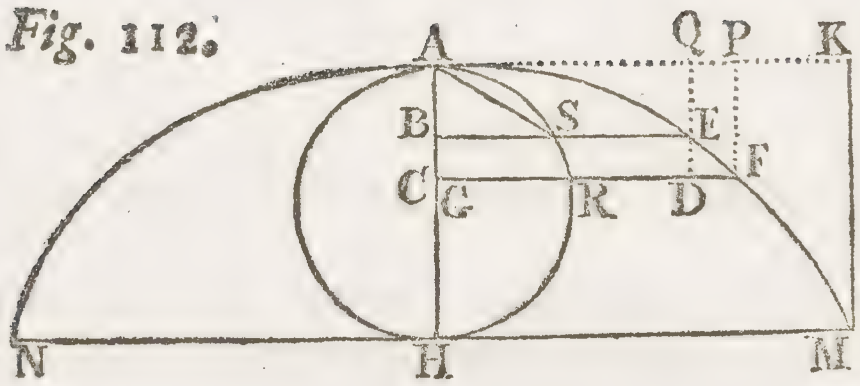
If we would free the formula from the radical vinculum, making the fubftitution of $\sqrt{aa - xx} = a - \frac{xz}{a}$, it would be changed into this other, $\frac{aab\dot{z}}{aa + zz}$, which, being reduced into a feries, would be found to be $b\dot{z} - \frac{bz^2\dot{z}}{a^2} + \frac{bz^4\dot{z}}{a^4} - \frac{bz^6\dot{z}}{a^6} + \frac{bz^8\dot{z}}{a^8}$, &c. And by integration, $bz - \frac{bz^3}{3a^2} + \frac{bz^5}{5a^4} - \frac{bz^7}{7a^6} + \frac{bz^9}{9a^8}$ &c.; and making $x = a$, in which cafe it is alfo $z = a$, it will be $ab - \frac{1}{3}ab + \frac{1}{5}ab - \frac{1}{7}ab + \frac{1}{9}ab$, &c. in refpect of a quadrant of the ellipfis.

And

And if we ſuppoſe $a = b$, the ellipſis becomes a circle with radius $= a$, and the ſeries will be as at § 97, which will expreſs the quadrant. And therefore, from hence it may alſo be ſeen, that the area of the ellipſis is to the area of the circle, the diameter of which is equal to the tranſverſe axis of the ellipſis, as the conjugate axis is to the tranſverſe axis of the ſame ellipſis.

EXAMPLE VIII.

Fig. 112.



99. Let NAM be a cycloid, it's generating circle ARH, and make $AH = a$, $AB = x$, $BC = \dot x$, $BE = y$, $DF = \dot y$. The equation will be $\dot y = \frac{a\dot x - x\dot x}{\sqrt{ax - xx}} = \frac{\dot x\sqrt{a - x}}{\sqrt x}$. But the little ſpace QEFP is the element of the ſpace AEQ, and therefore FP × PQ, that is, $\frac{x\dot x\sqrt{a - x}}{\sqrt x} = \dot x\sqrt{ax - xx}$ will be it's formula. But $\int\dot x\sqrt{ax - xx}$ is the circular ſegment ASB; therefore the cycloidal ſpace AEQ will be equal to the correſpondent circular ſpace ASB, and the whole ſpace AMK will be equal to the ſemicircle. But the rectangle AHMK is quadruple of the ſemicircle, becauſe it is the product of the ſemiperiphery into the diameter. Therefore the ſpace AMH will be triple of the ſemicircle, and therefore the whole cycloidal ſpace will be triple of the generating circle.

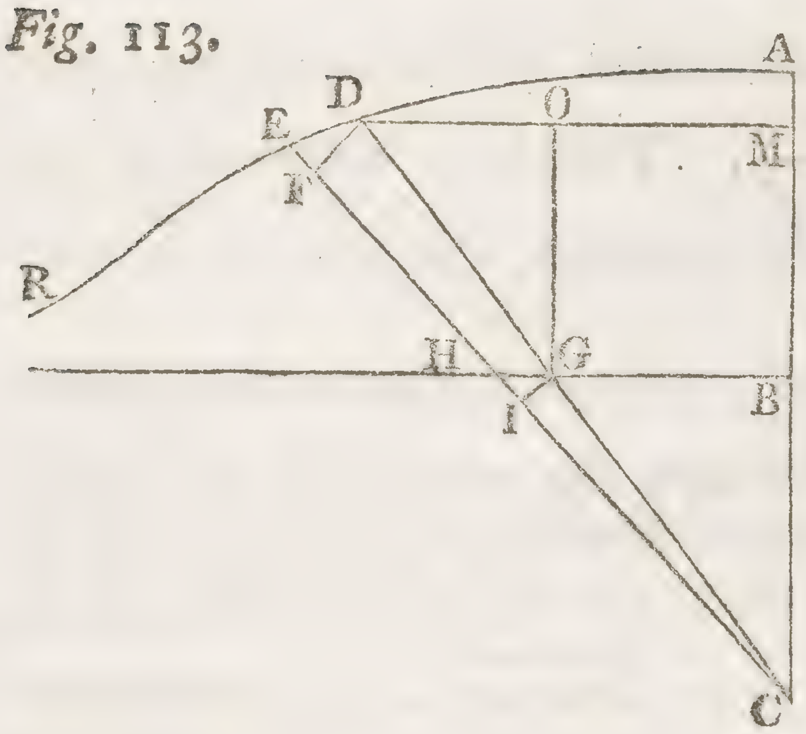
If we would have the ſpace AFC immediately; as the little ſpace FCBE, that is, $y\dot x$, is it's element, and from the equation of the curve we have $\dot y = \frac{\dot x\sqrt{a - x}}{\sqrt x}$; let the *homogeneum comparationis* be reduced into a ſeries, firſt multiplying the numerator and denominator by $\sqrt x$; whence it would be $\frac{\dot x\sqrt{ax - xx}}{x}$ $= \frac{a^{\frac{1}{2}}\dot x}{x^{\frac{1}{2}}} - \frac{x^{\frac{1}{2}}\dot x}{2a^{\frac{1}{2}}} - \frac{x^{\frac{3}{2}}\dot x}{8a^{\frac{3}{2}}} - \frac{x^{\frac{5}{2}}\dot x}{16a^{\frac{5}{2}}}$, &c.; and therefore, by integration, $\int\frac{\dot x\sqrt{ax - xx}}{x} = y = 2a^{\frac{1}{2}}x^{\frac{1}{2}} - \frac{x^{\frac{3}{2}}}{3a^{\frac{1}{2}}} - \frac{x^{\frac{5}{2}}}{20a^{\frac{3}{2}}} - \frac{x^{\frac{7}{2}}}{56a^{\frac{5}{2}}}$, &c. Whence $y\dot x =$

$2a^{\frac{1}{2}}x^{\frac{1}{2}}\dot{x} - \frac{x^{\frac{3}{2}}\dot{x}}{3a^{\frac{1}{2}}} - \frac{x^{\frac{5}{2}}\dot{x}}{20a^{\frac{3}{2}}} - \frac{x^{\frac{7}{2}}\dot{x}}{56a^{\frac{5}{2}}}$, &c. And laſtly, by integration, $\int y\dot{x} =$

$$ABE = \frac{4a^{\frac{1}{2}}x^{\frac{3}{2}}}{3} - \frac{2x^{\frac{5}{2}}}{15a^{\frac{1}{2}}} - \frac{x^{\frac{7}{2}}}{70a^{\frac{3}{2}}} - \frac{x^{\frac{9}{2}}}{252a^{\frac{5}{2}}}, \text{ \&c.}$$

EXAMPLE IX.

Fig. 113.



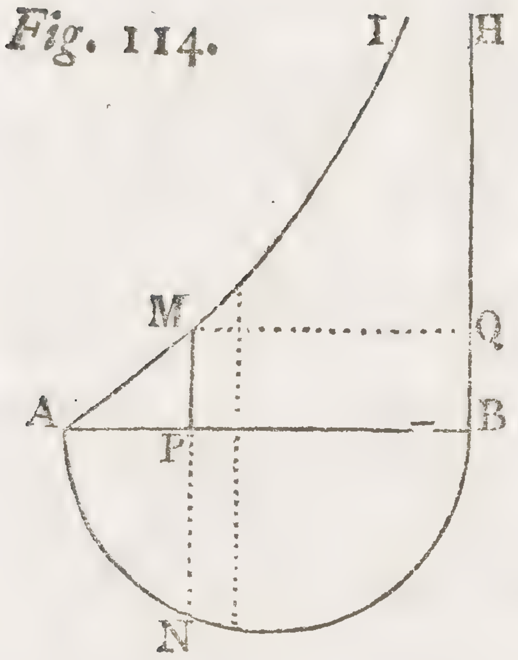
100. Let ADR be the conchoid, CB $=$ BA $= a$, CM $= x$, MD $= y$, and let the ſpace ADGB be required. Make CG $= z$, which will always be given by x and y of the propoſed curve, as is plain enough. Let CE be infinitely near to CD, and with centre C, intervals CG, CD, let the two little arches GI, DF, be deſcribed. It will be HI $= \dot{z}$, and the trapezium FDGI will be the element of the ſpace required. By the ſimilar triangles HIG, BGC, it will be GI $= \frac{a\dot{z}}{\sqrt{zz - aa}}$; and by the ſimilar ſectors CGI, CDF, it will be DF $= \frac{az\dot{z} + aa\dot{z}}{z\sqrt{zz - aa}}$. But the trapezium FDGI $=$ $\overline{DF + GI} \times \frac{1}{2}GD = \frac{2a^2z\dot{z} + a^3\dot{z}}{2z\sqrt{zz - aa}}$. Therefore $\int \frac{2a^2z\dot{z} + a^3\dot{z}}{2z\sqrt{zz - aa}}$, that is, $[al\,\overline{z + \sqrt{zz - aa}} - ala + \frac{1}{2}a \times$ arch of a circle of which the radius $=$ and the tangent $= \sqrt{zz - aa}$,] (taking the logarithm in the logarithmic with ſubtangent $= a$,) will be equal to the ſpace required.

Alſo, the whole ſpace may be had of the ſame conchoid, and likewiſe the parts, by conſidering the curve in relation to it's axis. In the ſame Figure make AB $=$ DG $=$ BC $= a$, BM $= x$, MD $= y$; and from the point G let there be drawn GO perpendicular to the ordinate MD; it will be DO $= \sqrt{aa - xx}$, becauſe of the right angle GOD; and by the ſimilitude of the triangles CBG, GOD, it will be BG $= \frac{a\sqrt{aa - xx}}{x} =$ MO. Therefore MD

$= \sqrt{aa - xx} + \frac{a\sqrt{aa - xx}}{x} = y$. Whence $y\dot{x}$, that is, the element of the space, will be $\dot{x}\sqrt{aa - xx} + \frac{a\dot{x}\sqrt{aa - xx}}{x}$. The fluent of the first term depends on the quadrature of the circle, and of the second on that of the hyperbola.

EXAMPLE X.

Fig. 114.



101. Let AMI be the cissoid of *Diocles*, the equation of which is $yy = \frac{x^3}{a - x}$. Therefore, substituting the value of y given by the equation, the formula will be $\frac{x^{\frac{3}{2}}\dot{x}}{\sqrt{a - x}}$, the integral of which depends on the quadrature of the circle. To have the relation of the whole space of the cissoid to that of the generating circle, it must be considered, that, the equation being $yy = \frac{x^3}{a - x}$, it will be also $yy \times \overline{ax - xx} = x^4$, and therefore $y\sqrt{ax - xx} = xx$. This supposed, by differencing the proposed equation $ayy - xyy = x^3$, there arises $2ay\dot{y} - 2xy\dot{y} - yy\dot{x} = 3xx\dot{x}$, that is, $2\dot{y} \times \overline{a - x} - y\dot{x} = \frac{3xx\dot{x}}{y}$. And, because $xx = y\sqrt{ax - xx}$, therefore $2\dot{y} \times \overline{a - x} - y\dot{x} = 3\dot{x}\sqrt{ax - xx}$. But $\dot{y} \times \overline{a - x}$ is the element of the space AMQB, and $y\dot{x}$ is the element of the space AMP; then, by integrating, as to the whole space, it is $\int \dot{y} \times \overline{a - x} = \int y\dot{x}$. Then, in this circumstance, it will be $2\int \dot{y} \times \overline{a - x} - \int y\dot{x} = \int \dot{y} \times \overline{a - x}$, and therefore $\int \dot{y} \times \overline{a - x} = 3\int \dot{x}\sqrt{ax - xx}$; and because, in the case of the total space of the cissoid, $\int \dot{x}\sqrt{ax - xx}$ is the area of the semicircle ABN; thence the space of the cissoid, infinitely produced, will be triple of the generating circle.

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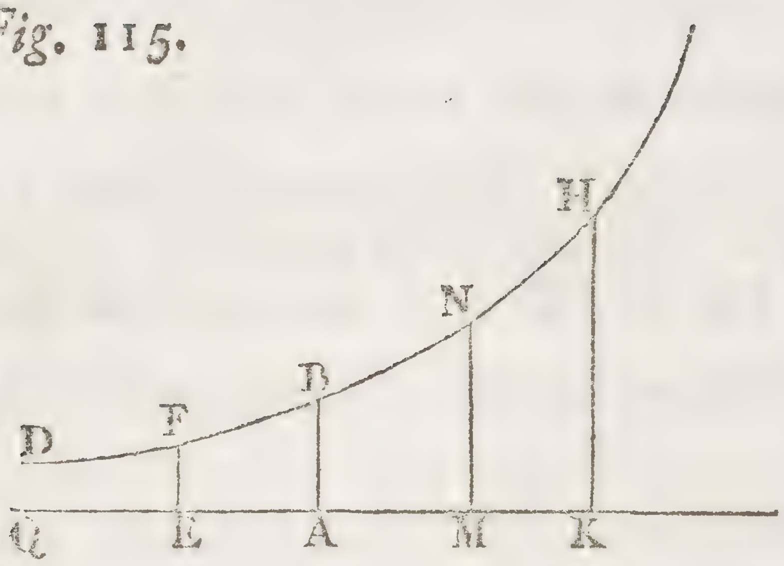
EXAMPLE XI.

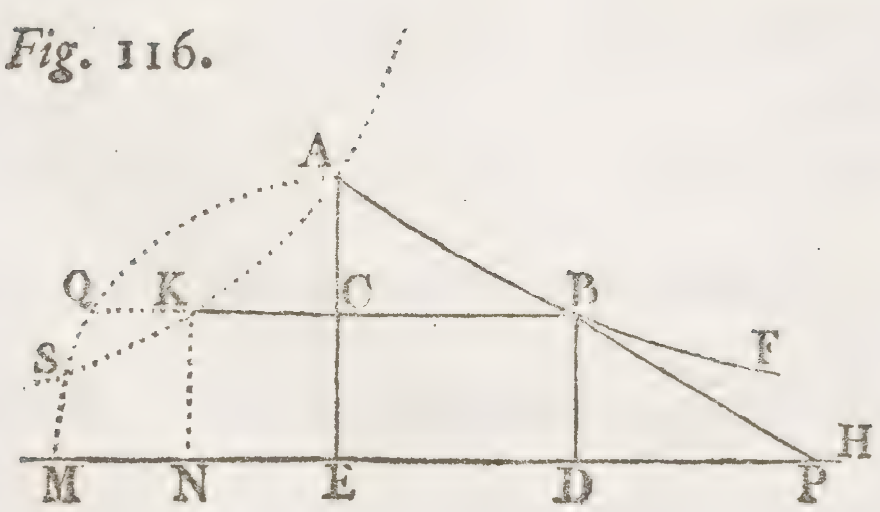
Fig. 115.



102. Let HBD be the logarithmic to the aſymptote MQ, and let AB $= a =$ ſubtangent, KH $= y$, AK $= x$, and the equation $\frac{a\dot{y}}{y} = \dot{x}$. Then the formula will be $y\dot{x} = a\dot{y}$, and by integration, $\int y\dot{x} = ay + bb$. But, ſuppoſing $y = a$, it will be $bb = -aa$; ſo that the integral complete, or ſpace AKHB $= ay - aa$. Taking any other ordinate MN $= z$, it will be alſo AMNB $= az - aa$, ſo that MKHN $= ay - az$. Let there be an ordinate EF leſs than AB, and equal to y, AE $= -x$; in the ſame manner, the equation will be $\frac{a\dot{y}}{y} = \dot{x}$, becauſe, x being negative, it's difference will be negative alſo. But the abſciſs x increaſing, the ordinate y decreaſes, and therefore $\dot{y}$ muſt be negative. Now, becauſe the element of the ſpace will alſo be negative, this element will be $-y\dot{x}$, that is, $-a\dot{y}$; and by integrating, $-ay + bb$. But when $y = a$, it will be $bb = aa$; therefore the complete integral, that is, the ſpace AEFB, will be $= aa - ay$. And making $y = 0$, that is, when it is infinitely produced towards Q, it will be $= aa$. And conſequently the ſame ſpace, infinitely produced towards Q, but which begins from any ordinate EF $= y$, will be $= ay$.

EXAMPLE XII.

Fig. 116.

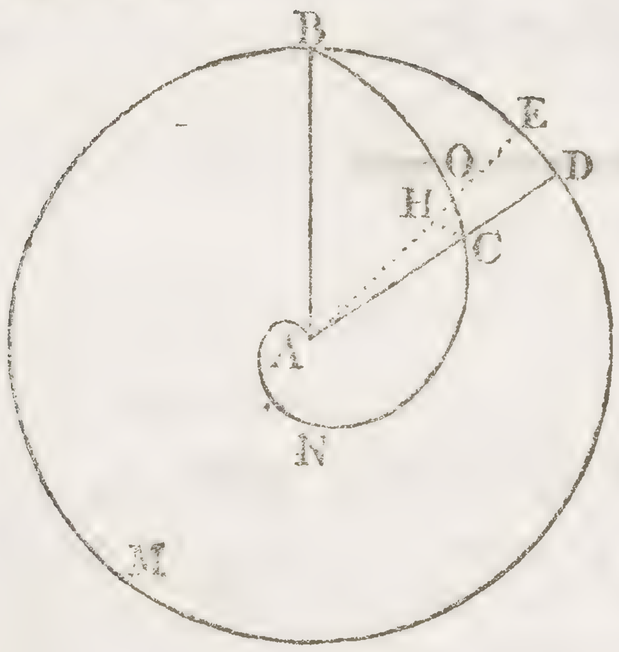


103. Let the curve ABF be the *tractrix*, the primary property of which is this, that the tangent BP, at any point B, is always equal to a conſtant right line given. Make any abſciſs ED $= x$, the ordinate DB $= y$, the arch of the curve AB $= u$, and the given right line $= a$. Becauſe, as the abſciſs ED increaſes, the ordinate DB diminiſhes, it's element will be negative,

tive, that is, $-\dot{y}$. Whence, from the property of the curve, we ſhall have the equation $-\frac{y\dot{u}}{\dot{y}} = a$; and, inſtead of $\dot{u}$, putting it's value $\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}$, it is $\dot{x} = \frac{-\dot{y}\sqrt{aa - yy}}{y}$. This being done, in the formula for areas $y\dot{x}$, inſtead of $\dot{x}$, putting it's value given by the equation of the curve, we ſhall have $-\dot{y}\sqrt{aa - yy}$ for the element of any ſpace ABDE. But, ſuppoſing the firſt of the ordinates AE $= a$, and with radius EA deſcribing the quadrant AQM, and drawing BQ parallel to MH; becauſe DB $=$ EC $= y$, and, by the property of the circle, CQ $= \sqrt{aa - yy}$, the element of the circular ſpace CQA will alſo be $-\dot{y}\sqrt{aa - yy}$. Whence the ſpace CQA will be equal to the ſpace ABDE; and ſo of others. And conſequently the ſpace infinitely produced, comprehended by the *tractrix* ABF, by the aſymptote EH, and by the right line AE, will be equal to the quadrant AME.

EXAMPLE XIII.

Fig. 117.



104. Let ACB be a ſpiral, and AB $= a$ the radius of the circle BMD, the periphery of which $= b$, any arch BD $= x$, AC $= y$; the equation will be $by = ax$. Drawing AE infinitely near to AD, it will be ED $= \dot{x}$; and with centre A deſcribe the infiniteſimal arch CH. Becauſe of ſimilar ſectors ACH, ADE, it will be CH $= \frac{y\dot{x}}{a}$, and therefore the ſector ACH, the element of the ſpace ANCA, will be $= \frac{yy\dot{x}}{2a}$. But, by the equation of the curve, it is $y = \frac{ax}{b}$; therefore that element will be $= \frac{axx\dot{x}}{2bb}$, and by integration, and omitting the conſtant quantity as ſuperfluous, the ſpace ACN will be $\frac{ax^3}{6bb}$; and making $x = b$, in reſpect of the whole ſpace ANB, which will be $= \frac{1}{6}ab$.

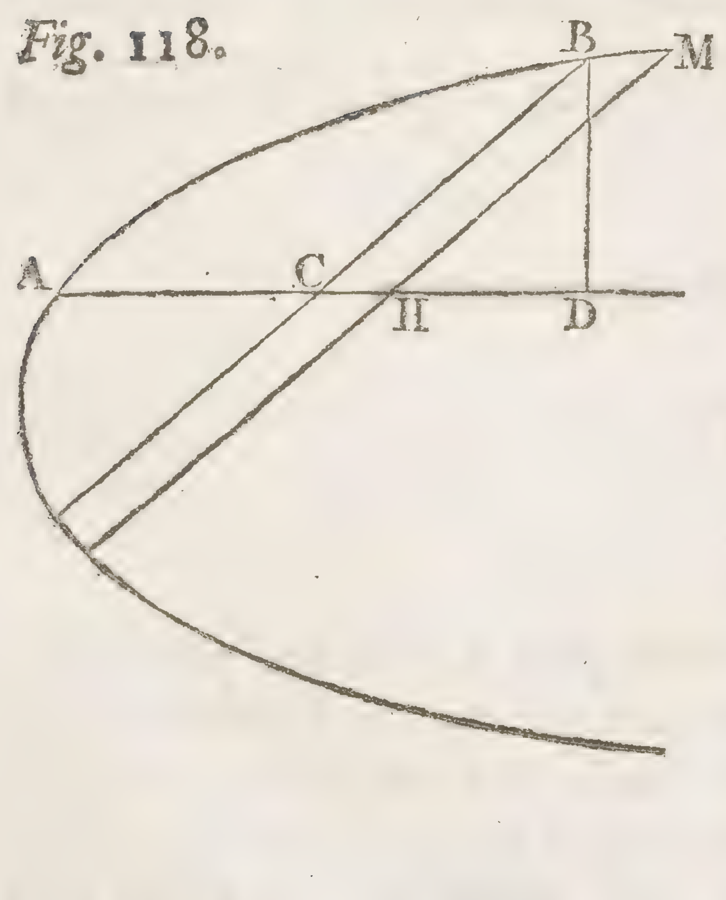
Let

Let the equation be general to infinite ſpirals $a^m x^n = b^n y^m$; then it will be $yy = \frac{aax^{\frac{2n}{m}}}{b^{\frac{2n}{m}}}$, and the formula of the ſpace will be $\frac{ax^{\frac{2n}{m}}\dot{x}}{2b^{\frac{2n}{m}}}$, and by integration, $\frac{max^{\frac{2n+m}{m}}}{\overline{4n+2m} \times b^{\frac{2n}{m}}}$; and making $x = b$, the whole ſpace will be $= \frac{mab}{4n + 2m}$.

It is eaſy to perceive, that the ſpace ABMDCNA, terminated by the radius AB, the circular arch BMD, and the portion of the ſpiral ANC, will be $\frac{ax}{2} - \frac{ax^3}{6bb}$; becauſe it is equal to the ſector ABMDA, diminiſhed by the ſpace ACN. But if we would have it by means of the differential formula, it is enough to obſerve, that it's element will be the infiniteſimal trapezium ECHD, which is known to be $= \overline{DE + CH} \times \frac{1}{2}CD$, that is, $\overline{\dot{x} + \frac{y\dot{x}}{a}} \times \frac{a-y}{2} = \frac{aa\dot{x} - yy\dot{x}}{2a}$. And, inſtead of yy, putting it's value $\frac{aax\dot{x}}{bb}$ given by the equation, it will be $\frac{a\dot{x}}{2} - \frac{ax x\dot{x}}{2bb}$; and by integration, $\frac{ax}{2} - \frac{ax^3}{6bb}$, omitting the ſuperfluous conſtant quantity.

EXAMPLE XIV.

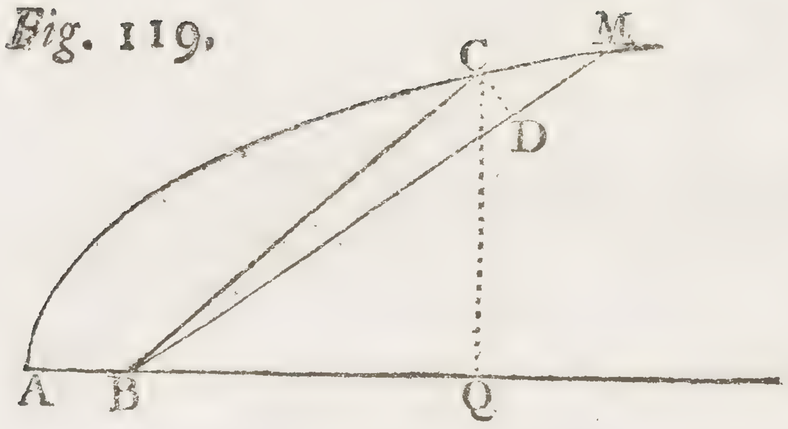
Fig. 118.



105. Let ABM be the parabola, whoſe equation is $ax = yy$, and make AC $= x$, CB $= y$, and let the ratio of the whole ſine to the right ſine of the angle BCD be that of a to b; to the ſine of the complement be that of a to f; then it will be BD $= \frac{by}{a}$, and CD $= \frac{fy}{a}$. Let CH $= \dot{x}$, then CH $\times$ DB $=$ CHMB, the element of the ſpace ACB, and therefore the formula will be $\frac{by\dot{x}}{a}$. And, inſtead of y, putting it's value given from the equation, that is, $\sqrt{ax}$, it will be $\frac{b\dot{x}\sqrt{ax}}{a}$; and by integration, $\frac{2bx\sqrt{ax}}{3a}$, or $\frac{2bxy}{3a} = \frac{2}{3}AC \times BD$.

EXAMPLE XV.

Fig. 119.

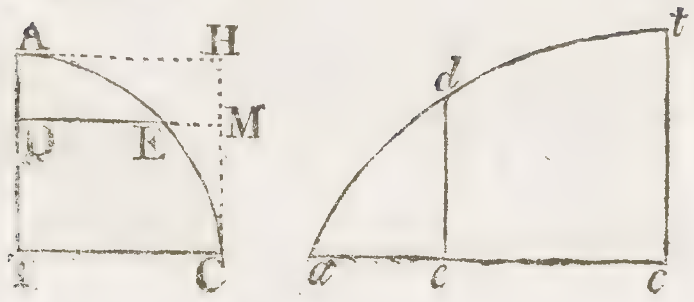


106. Let ACM be a parabola referred to the focus B, the equation of which will be $\frac{a\dot{z}}{\sqrt{2az - aa}} = \dot{u}$, making BC $= z$, CD $= \dot{u}$, an infinitely little arch of a circle, and the parameter $= 2a$. Then the infinitesimal sector BMC, or BDC, will be the element of the space ABC, and therefore $\frac{1}{2}z\dot{u}$, or $\frac{az\dot{z}}{2\sqrt{2az - aa}}$, will be the formula; the integral of which will be found to be $\frac{z + a}{6}\sqrt{2az - aa} + mm$. Now, taking $z =$ BA $= \frac{1}{2}a$, in which case the space ought to be nothing, it will be $mm = 0$, and therefore the complete integral, that is, the space ABC, is $\frac{z + a}{6}\sqrt{2az - aa}$.

And in fact, from the point C letting fall CQ perpendicular to AQ, the space BCA is equal to the space QCA lessened by the triangle BQC. But, making BQ $= x$, QC $= y$, it will be QCA $-$ QCB $= \frac{2}{3} \times \overline{\frac{1}{2}a + x} \times y - \frac{1}{2}xy = \frac{2a + x}{6} \times y$. Therefore BCA $= \frac{2a + x}{6} \times y$. But, by the property of the parabola, BC $=$ AQ $+$ AB $= x + a$, that is, $z = x + a$, and $y = \sqrt{aa + 2ax} = \sqrt{2az - aa}$. Therefore, these values being substituted instead of x and y, we shall find BCA $= \frac{2a + x}{6} \times y = \frac{a + z}{6}\sqrt{2az - aa}$, as above.

EXAMPLE XVI.

Fig. 120.



107. If the fourth part AC of the periphery of a circle be conceived to be stretched out into a right line (ac), and taking any portion (ae) equal to the arch AE, let there be raised the perpendicular (ed) equal to the right sine DE;

DE; the curve (at) which paſſes through all the points (d) ſo determined, is called the line of right ſines. Producing (ac) till it be equal to the ſemicircumference of the circle, the curve will have another branch beyond (ct), ſimilar and equal to the firſt.

Let the radius be $= r$, any arch AE $= x = (ae)$, the correſponding ſine DE $= y = (ed)$; becauſe the fluxion or differential of the arch, expreſſed by means of the ſine, is found to be $\frac{r\dot{y}}{\sqrt{rr - yy}}$, we ſhall have $\dot{x} = \frac{r\dot{y}}{\sqrt{rr-yy}}$, which is the equation of our curve. Therefore the formula $y\dot{x}$, by ſubſtituting the value of $\dot{x}$, will be $\frac{ry\dot{y}}{\sqrt{rr - yy}}$; and by integration, $- r\sqrt{rr - yy} + n$. But, putting $y = 0$, it is $n = rr$. Therefore the complete integral is $rr - r\sqrt{rr - yy}$ $=$ ſpace (ade); and making $y = r$, it will be $rr =$ to the whole ſpace (atc). Whence, making TH the ſquare of the radius, and producing the ſine DE to M, the ſpace (ade) will be equal to the rectangle DH, and the whole ſpace (atc) equal to the ſquare TH.

108. The Examples now produced may ſuffice to ſhow the uſe of the method. It only remains to obſerve, that often the equations of the curves, the areas of which are to be ſquared, (and this is alſo to be underſtood in reſpect to rectifications, quadratures of ſuperficies, and cubatures,) may be ſuch, that they have not the variable quantities ſeparate, nor can they be ſeparated by diviſion only, and conſequently are not reducible to the formulas required. Such would be the curve, whoſe equation is $x^3 + y^3 = axy$, for example.

In theſe caſes there is occaſion to take the advantage of ſome proper ſubſtitution, by means of which the equation may be transformed into another, in which the variable quantities are ſeparate, or at leaſt are ſeparable. But it cannot be determined, in general, what thoſe ſubſtitutions ought to be. There is need of practice, and perhaps many trials, to know when this may be ſucceſsfully performed.

As to the propoſed equation $x^3 + y^3 = axy$, make $y = \frac{axx}{zz}$; and making the ſubſtitution, the equation will be $x^3 + \frac{a^3x^6}{z^6} = \frac{a^2x^3}{zz}$, that is, $x^3 = \frac{aaz^4 - z^6}{a^3}$. Then, by differencing, $x^2\dot{x} = \frac{4a^2z^3\dot{z} - 6z^5\dot{z}}{3a^3}$. Wherefore, taking the formula for ſpaces, which is $y\dot{x}$, becauſe, by ſubſtitution, it is $y = \frac{axx}{zz}$, this formula will be $\frac{axx\dot{x}}{zz}$; and ſubſtituting, inſtead of $xx\dot{x}$, it's value now found, $\frac{4aaz^3\dot{z} - 6z^5\dot{z}}{3a^3}$, it will be $y\dot{x} = \frac{4aaz\dot{z} - 6z^3\dot{z}}{3a^2}$; and by integration, $\int y\dot{x} = \frac{2}{3}zz$ —

$-\frac{z^4}{2aa}$; and, inftead of zz, reftoring it's value $\frac{axx}{y}$, it will be finally $\int y\dot{x} = \frac{2axx}{3y} - \frac{x^4}{2yy}$.

EXAMPLE XVII.

109. Let the curve be $a^5x^2y^2 - x^9 = a^6y^3$, whofe area is required. Put $y = \frac{xx}{z}$, and the equation will be transformed into this other, $a^5z - x^3z^3 = a^6$, from whence we have $x = \frac{a\sqrt[3]{aaz - a^3}}{z}$; and therefore $\dot{x} = \frac{a^3\dot{z}}{3z \times \overline{aaz - a^3}|^{\frac{2}{3}}} - \frac{a\dot{z} \times \overline{aaz - a^3}|^{\frac{1}{3}}}{zz}$, and $y = \frac{aa \times \overline{aaz - a^3}|^{\frac{2}{3}}}{z^3}$. Hence we fhall have the element of the area $y\dot{x} = \frac{a^5\dot{z}}{3z^4} - \frac{a^3\dot{z}}{z^5} \times \overline{aaz - a^3} = \frac{a^6\dot{z}}{z^5} - \frac{2a^5\dot{z}}{3z^4}$; and therefore, by integration, $\int y\dot{x} = \frac{-a^6}{4z^4} + \frac{2a^5}{9z^3}$. And, inftead of z, reftoring it's value $\frac{xx}{y}$, the area will be $-\frac{a^6y^4}{4x^8} + \frac{2a^5y^3}{9x^6}$.

To this purpofe may be feen the Method of Mr. *Craig*, in his Book *De Calculo Fluentium.*

EXAMPLE XVIII.

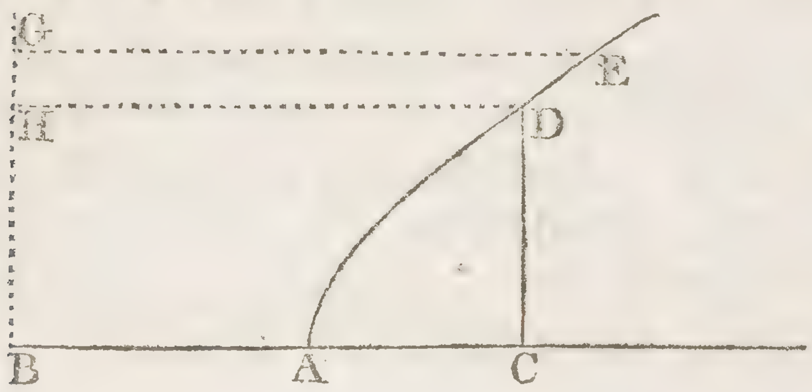
The rectification of curves.

110. Let the *Apollonian* parabola be given to be rectified; that is, to find a right line equal to any arch of the fame parabola, the equation of which is $ax = yy$. It's fluxion will be $a\dot{x} = 2y\dot{y}$, and $\dot{x}\dot{x} = \frac{4yy\dot{y}\dot{y}}{aa}$. Now the formula for rectification is $\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}$; fo that, fubftituting here, inftead of $\dot{x}\dot{x}$, it's value given from the fluxional equation, it will be $\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}} = \frac{\sqrt{4yy\dot{y}\dot{y} + aa\dot{y}\dot{y}}}{a} = \frac{\dot{y}}{a}\sqrt{4yy + aa}$, the element of the *Apollonian* parabola

$ax =$

$ax = yy$. Proceeding to the integration; by making the ſubſtitution of $\sqrt{4yy + aa} = 2y + z$, in order to take away the radical, we ſhall find it to be $\frac{\dot{y}}{a}\sqrt{4yy + aa} = -\frac{a^3\dot{z}}{8z^3} - \frac{a\dot{z}}{4z} - \frac{z\dot{z}}{8a}$, the integral of which we may ſee is partly algebräical, and partly logarithmical; and therefore the rectification of the parabola depends on the quadrature of the hyperbola; which truth may be diſcovered after this other manner. Let ADE be an equilateral hyperbola, with ſemiaxis $= a$, BC $= x$ from the centre, CD $= 2y$, the equation of which will be $xx - aa = 4yy$. Drawing GE infinitely near to HD, then HGED will be the element of the ſpace ADHB. But we know HGED to be $2\dot{y}\sqrt{4yy + aa}$, which is the ſame formula as that for the rectification of the parabola, excepting the conſtant denominator $2a$. Therefore, &c.

Fig. 121.

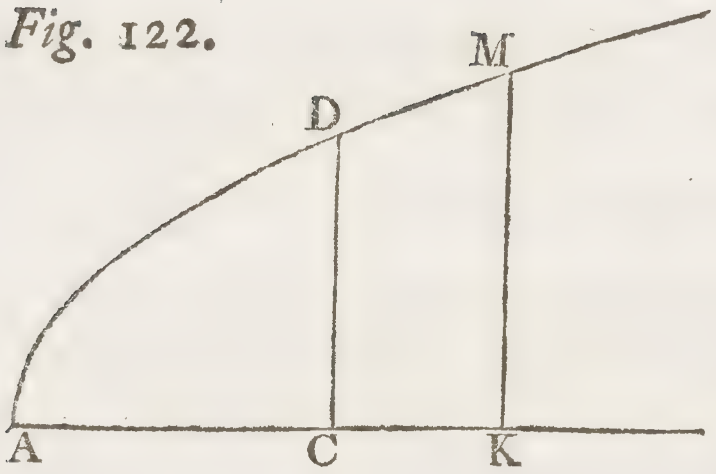


By the help of infinite ſeries. I take the above-written formula for the rectification of the parabola, that is, $\frac{\dot{y}}{a}\sqrt{4yy + aa}$, which, being reduced to a ſeries, will be $\dot{y} + \frac{2y^2\dot{y}}{aa} - \frac{2y^4\dot{y}}{a^4} + \frac{4y^6\dot{y}}{a^6} - \frac{10y^8\dot{y}}{a^8}$, &c. And, by integration, $\int\frac{\dot{y}}{a}\sqrt{4yy + aa} = y + \frac{2y^3}{3a^2} - \frac{2y^5}{5a^4} + \frac{4y^7}{7a^6} - \frac{10y^9}{9a^8}$, &c. will be any arch whatever.

In the general formula $\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}$, inſtead of ſubſtituting, in the place of $\dot{x}$, it's value given by $\dot{y}$ from the equation of the curve; if we ſhould ſubſtitute, in the place of $\dot{y}$, it's value given by $\dot{x}$, it would be $\frac{\dot{x}\sqrt{4ax + aa}}{\sqrt{4ax}}$, or $\frac{\dot{x}\sqrt{4xx + ax}}{2x}$, which is not indeed more manageable than the other.

If the parabola was not that of *Apollonius*, but the ſecond cubic, the equation of which is $axx = y^3$; by taking the difference, it would be $\dot{x}\dot{x} = \frac{9y\dot{y}\dot{y}}{4a}$, and therefore the formula $\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}} = \dot{y}\sqrt{\frac{9y + 4a}{4a}}$, the integral of which is $\frac{9ay + 4aa}{27aa}\sqrt{9ay + 4aa} + m$. But, putting $y = 0$, it will be $m = -\frac{8}{27}a$; therefore the complete integral, or the length of the arch, will be $\frac{9ay + 4aa}{27aa}\sqrt{9ay + 4aa} - \frac{8}{27}a$.

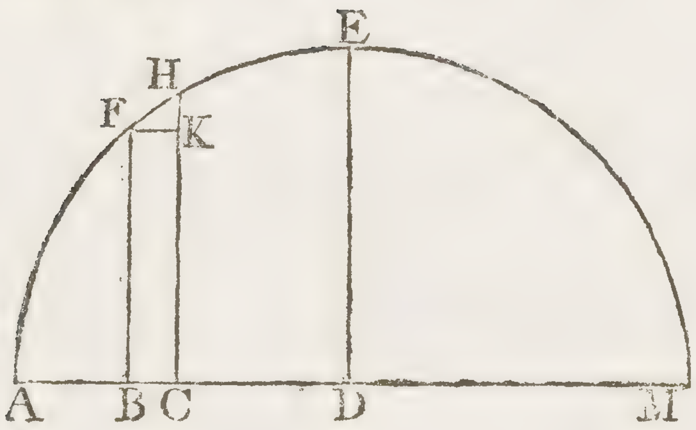
Fig. 122.



In the *Apollonian* parabola ADM, if it shall be AC $= \frac{4}{9}a$, and taking any line CK $= y$, the parameter $= \frac{9}{4}a$; it will be AK $= \frac{4}{9}a + y$, KM $= \sqrt{\frac{4aa + 9ay}{4}}$. Whence the element of the area MKCD will be $\dot y\sqrt{\frac{4aa + 9ay}{4}}$, which is the same with the element of the length of the second cubical parabola, except the constant quantity a. And therefore the rectification of this, and the quadrature of that, is the same thing. Whence, because the quadrature of that may be found algebraically, this is also algebraically rectifiable. And hence, in general, if the expression of the element of any given curve, divided by the difference of the unknown quantity, be put for the ordinate, and the unknown quantity be put for the abscis; a new curve will thence arise, the quadrature of which will give the rectification of the given curve.

EXAMPLE XIX.

Fig. 123.



111. Let AEM be a circle, it's diameter AM $= a$, AB $= x$; it will be BF $= y$ $= \sqrt{ax - xx}$. Then $\dot y = \frac{\frac{1}{2}a\dot x - x\dot x}{\sqrt{ax - xx}}$, $\dot y\dot y = \frac{\frac{1}{4}aa\dot x\dot x - ax\dot x\dot x + xx\dot x\dot x}{ax - xx}$. And therefore the element of the curve FH $= \sqrt{\dot x\dot x + \dot y\dot y} = \frac{a\dot x}{2\sqrt{ax - xx}}$, and reducing it to a series, it will be

$$\frac{1}{2}a^{\frac{1}{2}}x^{-\frac{1}{2}}\dot x + \frac{x^{\frac{1}{2}}\dot x}{2 \times 2a^{\frac{1}{2}}} + \frac{3x^{\frac{3}{2}}\dot x}{2 \times 2 \times 4a^{\frac{3}{2}}} + \frac{15x^{\frac{5}{2}}\dot x}{2 \times 2 \times 4 \times 6a^{\frac{5}{2}}} + \frac{105x^{\frac{7}{2}}\dot x}{2 \times 2 \times 4 \times 6 \times 8a^{\frac{7}{2}}}, \text{ \&c.}$$

And by integration, it will be $a^{\frac{1}{2}}x^{\frac{1}{2}} + \frac{x^{\frac{3}{2}}}{2 \times 3a^{\frac{1}{2}}} + \frac{3x^{\frac{5}{2}}}{2 \times 4 \times 5a^{\frac{3}{2}}} + \frac{15x^{\frac{7}{2}}}{2 \times 4 \times 6 \times 7a^{\frac{5}{2}}} + \frac{105x^{\frac{9}{2}}}{2 \times 4 \times 6 \times 8 \times 9a^{\frac{7}{2}}}$, &c. Or, because it is $\dot x\dot x = \frac{\dot y\dot y \times \overline{ax - xx}}{\frac{1}{4}aa - ax + xx}$, that is, by substituting yy instead of $ax - xx$, $\dot x\dot x = \frac{yy\dot y\dot y}{\frac{1}{4}aa - yy}$; then putting this value, instead

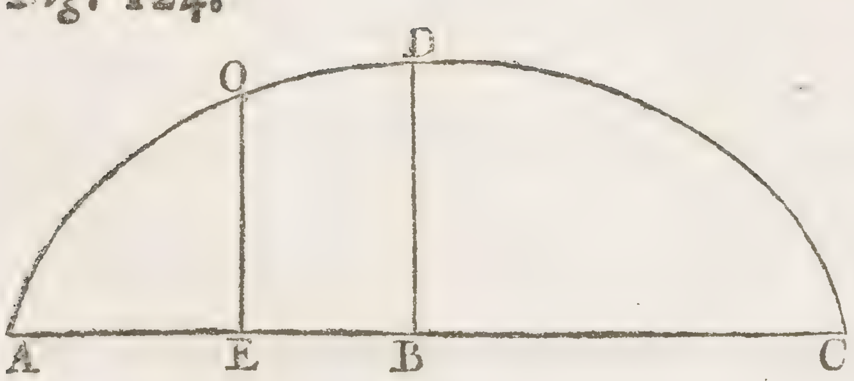
stead of $\dot{x}\dot{x}$ in the general formula, it will be $\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}} = \frac{a\dot{y}}{2\sqrt{\frac{1}{4}aa - yy}}$; which, being reduced to a series, will be found to be $= \dot{y} + \frac{2yy\dot{y}}{aa} + \frac{6y^4\dot{y}}{a^4} + \frac{20y^6\dot{y}}{a^6} + \frac{70y^8\dot{y}}{a^8}$, &c. And by integration, it will be finally the arch FA $= y + \frac{2y^3}{3a^2} + \frac{6y^5}{5a^4} + \frac{20y^7}{7a^6} + \frac{70y^9}{9a^8}$, &c.

But if the radius were made $= a$, the series would be $y + \frac{y^3}{2 \times 3a^2} + \frac{3y^5}{2 \times 4 \times 5a^4} + \frac{15y^7}{2 \times 4 \times 6 \times 7a^6} + \frac{105y^9}{2 \times 4 \times 6 \times 8 \times 9a^8}$, &c.

Lastly, if it were DB $= x$, the radius DA $= a$, it would be $y = \sqrt{aa - xx}$, and $\dot{y} = \frac{-x\dot{x}}{\sqrt{aa - xx}}$; therefore $\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}} = \frac{a\dot{x}}{\sqrt{aa - xx}}$; and, reducing to a series, it will be $\frac{a\dot{x}}{\sqrt{aa - xx}} = \dot{x} + \frac{x^2\dot{x}}{2aa} + \frac{3x^4\dot{x}}{2 \times 4a^4} + \frac{15x^6\dot{x}}{2 \times 4 \times 6a^6} + \frac{105x^8\dot{x}}{2 \times 4 \times 6 \times 8a^8}$, &c. Whence the arch EF $= x + \frac{x^3}{2 \times 3a^2} + \frac{3x^5}{2 \times 4 \times 5a^4} + \frac{15x^7}{2 \times 4 \times 6 \times 7a^6} + \frac{105x^9}{2 \times 4 \times 6 \times 8 \times 9a^8}$, &c.

EXAMPLE XX.

Fig. 124.



112. Let ADC be an ellipsis with transverse semiaxis BA $= a$, and conjugate semiaxis BD $= b$, BE $= x$, EO $= y$; the equation will be $\frac{aayy}{bb} = aa - xx$, and therefore $y\dot{y} = \frac{-bbx\dot{x}}{aa}$, and $\dot{y}\dot{y} = \frac{bbxx\dot{x}\dot{x}}{aa \times aa - xx}$, and the general formula $\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}} = \sqrt{\dot{x}\dot{x} + \frac{bbxx\dot{x}\dot{x}}{a^4 - aaxx}} = \frac{\dot{x}\sqrt{a^4 - a^2x^2 + b^2x^2}}{a\sqrt{aa - xx}}$.

Instead

Inftead of fubftituting the value of $\dot{y}$ given by x from the equation, if we fhould fubftitute the value of $\dot{x}$, it would be $\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}} = \frac{\dot{y}\sqrt{aayy - bbyy + b^4}}{b\sqrt{bb - yy}}$.

But both of the expreffions fo found would want one of the conditions of § 38, without which it may be feen, that thefe formulas cannot be freed from radical figns, and fo prepared for integration. Then to proceed to feries, I take one of the two formulas, for inftance $\frac{\dot{x}\sqrt{a^4 - a^2x^2 + b^2x^2}}{a\sqrt{aa - xx}}$, which alfo may be thus expreffed, $\dot{x}\sqrt{1 + \frac{bbxx}{a^4 - aaxx}}$; and this being reduced to a feries, will be $=$

$$\dot{x} + \frac{\frac{1}{2}bbxx\dot{x}}{aa \times \overline{aa - xx}} - \frac{\frac{1}{8}b^4x^4\dot{x}}{a^4 \times \overline{aa - xx}|^2} + \frac{\frac{1}{16}b^6x^6\dot{x}}{a^6 \times \overline{aa - xx}|^3} - \frac{\frac{5}{128}b^8x^8\dot{x}}{a^8 \times \overline{aa - xx}|^4}, \text{ \&c.}$$

And again, reducing every term of this into a feries, beginning at the fecond, it will be $\dot{x}\sqrt{1 + \frac{bbxx}{a^4 \times \overline{aa - xx}}} = \dot{x} + \frac{\frac{1}{2}bbxx\dot{x}}{aa}$ into $\frac{1}{aa} + \frac{x^2}{a^4} + \frac{x^4}{a^6} + \frac{x^6}{a^8}$, &c.

$$- \frac{\frac{1}{8}b^4x^4\dot{x}}{a^4} \text{ into } \frac{1}{a^4} + \frac{2x^2}{a^6} + \frac{3x^4}{a^8} + \frac{4x^6}{a^{10}}, \text{ \&c.}$$

$$+ \frac{\frac{1}{16}b^6x^6\dot{x}}{a^6} \text{ into } \frac{1}{a^6} + \frac{3x^2}{a^8} + \frac{6x^4}{a^{10}} + \frac{10x^6}{a^{12}}, \text{ \&c.}$$

$$- \frac{\frac{5}{128}b^8x^8\dot{x}}{a^8} \text{ into } \frac{1}{a^8} + \frac{4x^2}{a^{10}} + \frac{10x^4}{a^{12}} + \frac{20x^6}{a^{14}}, \text{ \&c.}$$

And by integration, the arch DO will be $\int\dot{x}\sqrt{1 + \frac{bbxx}{a^4 - a^2x^2}} =$

$$x + \frac{bb}{2aa} \text{ into } \frac{x^3}{3a^2} + \frac{x^5}{5a^4} + \frac{x^7}{7a^6} + \frac{x^9}{9a^8}, \text{ \&c.}$$

$$- \frac{b^4}{8a^4} \text{ into } \frac{x^5}{5a^4} + \frac{2x^7}{7a^6} + \frac{3x^9}{9a^8} + \frac{4x^{11}}{11a^{10}}, \text{ \&c.}$$

$$+ \frac{b^6}{16a^6} \text{ into } \frac{x^7}{7a^6} + \frac{3x^9}{9a^8} + \frac{6x^{11}}{11a^{10}}, \text{ \&c.}$$

$$- \frac{5b^8}{128a^8} \text{ into } \frac{x^9}{9a^8} + \frac{4x^{11}}{11a^{10}}, \text{ \&c.}$$

And laftly, reducing the homogeneous terms into the fame denomination, we fhall find DO $=$

$$x + \frac{b^2x^3}{6a^4} + \frac{4a^2b^2 - b^4}{40a^8}x^5 + \frac{8a^4b^2 - 4a^2b^4 + b^6}{112a^{12}}x^7$$

$$+ \frac{64a^6b^2 - 48a^4b^4 + 24a^2b^6 - 5b^8}{9 \times 128a^{16}}x^9, \text{ \&c.}$$

Now, if we ſhould ſuppoſe $a = b$, in which caſe the ellipſis would become a circle, we ſhall have the arch DO $= x + \frac{x^3}{6a^2} + \frac{3x^5}{40a^4} + \frac{5x^7}{112a^6} + \frac{35x^9}{9 \times 128a^8}$, &c. juſt as was found before, at § 111.

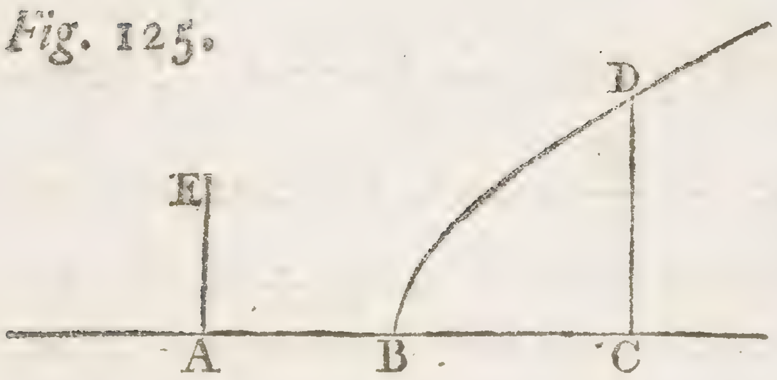
After another manner, thus. In the formula $\frac{\dot{x}\sqrt{a^4 - a^2x^2 + b^2x^2}}{a\sqrt{aa - xx}}$, if we make $bb - aa = -cc$, ſo that it may be $\frac{\dot{x}\sqrt{a^4 - ccxx}}{a\sqrt{aa - xx}}$, the two radicals being reſolved into ſeries, it will be $\frac{\dot{x}\sqrt{a^4 - c^2x^2}}{a\sqrt{aa - xx}} =$

$$\frac{\frac{\dot{x}}{a} \text{ into } a^2 - \frac{c^2x^2}{2a^2} - \frac{c^4x^4}{8a^6} - \frac{c^6x^6}{16a^{10}} - \frac{5c^8x^8}{128a^{14}}, \text{ \&c.}}{a - \frac{x^2}{2a} - \frac{x^4}{8a^3} - \frac{x^6}{16a^5} - \frac{5x^8}{128a^7}, \text{ \&c.}}$$

; and actually making the diviſion of the numerator by the denominator, after a very long calculation we ſhall find another ſeries, which, being integrated, and the value of cc reſtored in it's place, will give us the ſame ſeries as above, which expreſſes the value of the arch DO.

EXAMPLE XXI.

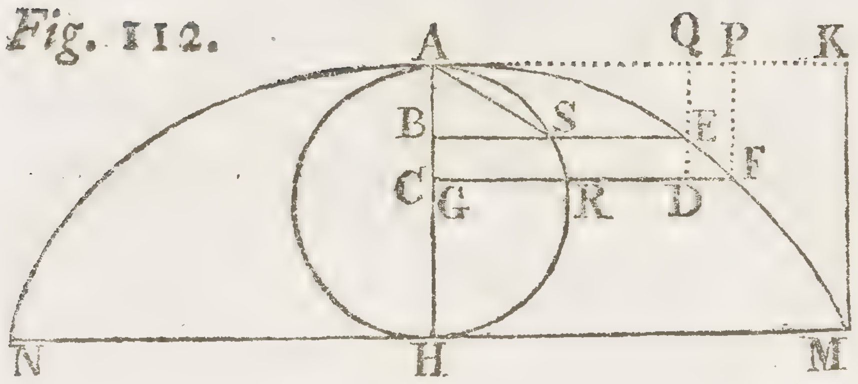
Fig. 125.



113. Let BD be an hyperbola with tranſverſe ſemiaxis AB $= a$, conjugate ſemiaxis AE $= b$, CD $= y$, AC $= x$; the equation will be $xx - aa = \frac{aayy}{bb}$. Then, by taking the fluxions, it will be $\dot{x} = \frac{ay\dot{y}}{b\sqrt{bb + yy}}$, whence $\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}} = \dot{y}\sqrt{1 + \frac{aayy}{b^4 + b^2y^2}} = \frac{\dot{y}}{b} \times \frac{\sqrt{bbyy + aayy + b^4}}{\sqrt{bb + yy}}$. Therefore, this being reduced into a ſeries, after either of the ways before made uſe of for the ellipſis, we ſhall find it's integral, or the arch BD $= y + \frac{a^2y^3}{6b^4} - \frac{4a^2b^2 + a^4}{40b^8}y^5 + \frac{8a^2b^4 + 4a^4b^2 + a^6}{112b^{12}}y^7 - \frac{64a^2b^6 + 48a^4b^4 + 24a^6b^2 + 5a^8}{9 \times 128b^{16}}y^9$, &c. which is the ſame ſeries as that for the ellipſis, excepting the ſigns, and the change of the conſtants a, b.

EXAMPLE XXII.

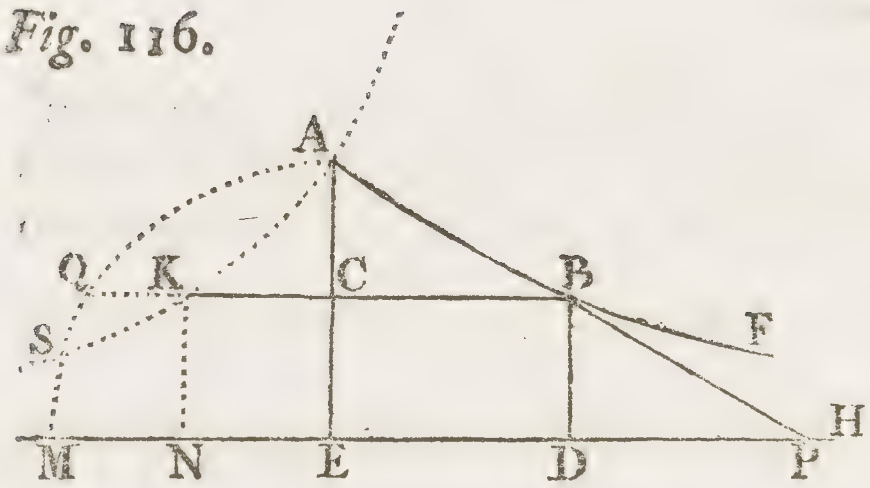
Fig. 112.



114. Let it be the cycloid of Example VIII. of Quadratures, the equation of which we know to be $\dot{y} = \dot{x}\sqrt{\frac{a - x}{x}}$; therefore the formula will be $\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}} = \dot{x}\sqrt{\frac{a}{x}}$, and therefore, by integration, it will be the arch EA $= 2\sqrt{ax}$, or the double of the chord AS of the corresponding circular arch AS. And putting $x = a$, AM will be the double of the diameter of the generating circle, and therefore the whole cycloid will be quadruple.

EXAMPLE XXIII.

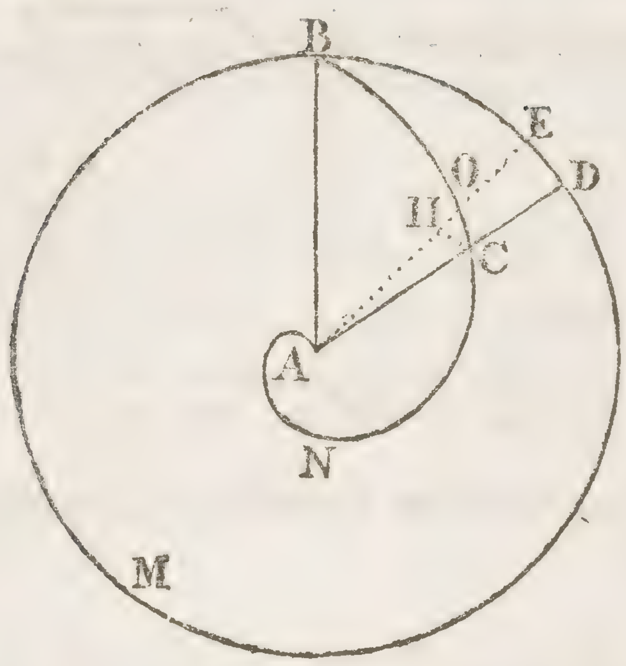
Fig. 116.



115. Let ABF be the *tractrix*, whose equation is (§ 103.) $-\frac{y\dot{u}}{\dot{y}} = a$. Therefore $\dot{u} = -\frac{a\dot{y}}{y}$, and, by integration, any arch AB $= u = -ly \pm n$, in the logarithmic curve with subtangent $= a$. But, making $u = 0$, it is $y = a$, and $ly = 0$; therefore $n = 0$, and the complete integral will be $u = -ly$. Therefore, if the logarithmic AKS be described through the point A, with the subtangent AE, to the asymptote MH; taking any point B in the *tractrix*, and drawing to the logarithmic BK parallel to the asymptote, and letting fall the perpendicular KN, the intercepted line NE will be equal to the arch AB.

EXAMPLE XXIV.

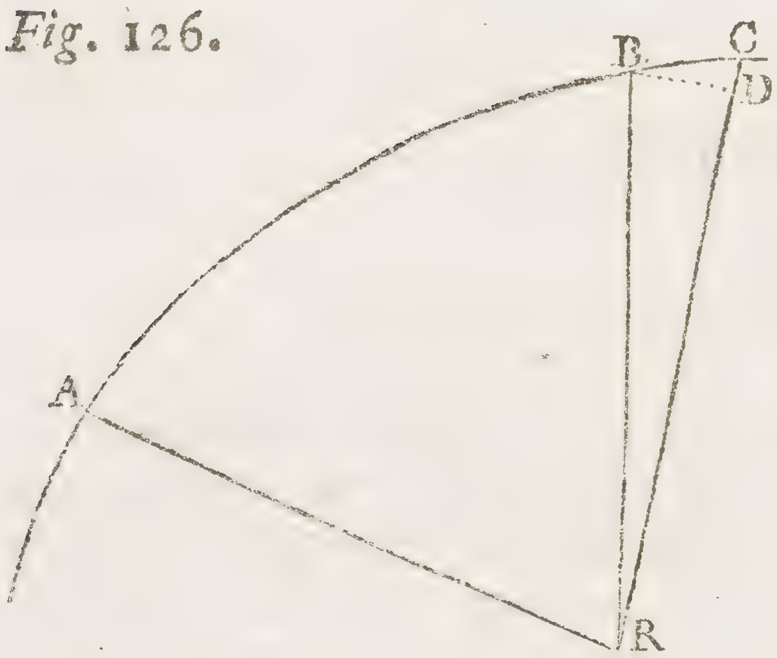
Fig. 117.



116. Let ACB be the ſpiral of *Archimedes* of § 104. the radius of the circle $= a$, the circumference $= b$, the arch BMD $= x$, and AC $= y$. Let AE be infinitely near to AD, and therefore DE $= \dot{x}$. With centre A let the arch CH be deſcribed; then it will be CH $= \frac{y\dot{x}}{a}$, and OH $= \dot{y}$. Therefore CO, the element of the curve, is equal to $\frac{\sqrt{yy\dot{x}\dot{x} + aa\dot{y}\dot{y}}}{a}$. But the equation of the curve is $ax = by$, and therefore $\dot{x}\dot{x} = \frac{bb\dot{y}\dot{y}}{aa}$; whence, making the ſubſtitution, it will be CO $= \frac{\dot{y}}{aa}\sqrt{a^4 + bbyy}$. The integral of this, after a long calculation, which, to avoid being tedious, I ſhall omit, will be found to depend on the logarithms, or, which is the ſame, on the quadrature of the hyperbola.

Now, by infinite ſeries. Firſt, I make $a^4 = bbmm$; whence the formula will be this, $\frac{b\dot{y}}{aa}\sqrt{mm + yy}$, which, being reduced to a ſeries, will be $\frac{b\dot{y}}{aa}$ into $m + \frac{yy}{2m} - \frac{y^4}{8m^3} + \frac{y^6}{16m^5} - \frac{5y^8}{128m^7}$, &c.; and therefore, by integration, the arch AC $= \frac{bmy}{aa} + \frac{by^3}{6a^2m} - \frac{by^5}{40a^2m^3} + \frac{by^7}{112a^2m^5} - \frac{5by^9}{9 \times 128a^2m^7}$, &c. And making $y = a$, the whole curve ACB $= \frac{bm}{a} + \frac{ab}{6m} - \frac{a^3b}{40m^3} + \frac{a^5b}{112m^5} - \frac{5a^7b}{9 \times 128m^7}$, &c. Then, inſtead of m, reſtoring it's value $\frac{aa}{b}$, it will be ACB $= a + \frac{bb}{6a} - \frac{b^4}{40a^3} + \frac{b^6}{112a^5} - \frac{5b^8}{9 \times 128a^7}$, &c.

Fig. 126.



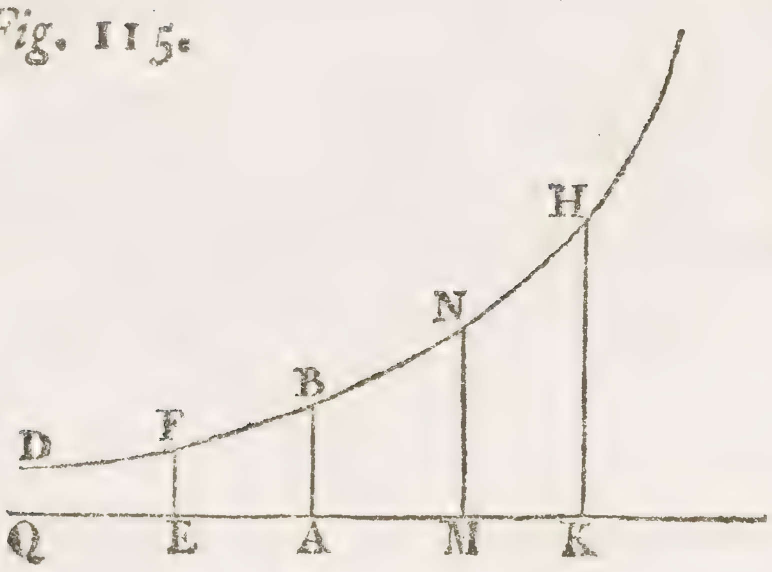
If the curve ABC were the logarithmic ſpiral, whoſe equation is $a\dot{y} = b\dot{x}$; making RB $= y$, and the infinitely little arch BD $= \dot{x}$; putting, in the general formula $\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}$, the value of $\dot{x}$ given from the equation, it will be $\frac{\dot{y}\sqrt{aa + bb}}{b}$, and by integration, the curve AB $= \frac{y}{b}\sqrt{aa + bb}$.

Let the curve ABC be the hyperbolical ſpiral, in which the ſubtangent is always conſtant; and therefore, retaining the ſame names as above, the equation will be $y\dot{x} = a\dot{y}$. Therefore it will be $\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}} = \frac{\dot{y}}{y}\sqrt{aa + yy}$; the integral of which formula, freed from the radical ſign, will be found to depend on the logarithmic.

By means of ſeries we ſhall find $\frac{\dot{y}}{y}\sqrt{aa + yy} = \dot{y}$ into $\frac{a}{y} + \frac{y}{2a} - \frac{y^3}{8a^3} + \frac{y^5}{16a^5} - \frac{5y^7}{128^7}$, &c. But if we would proceed to the integration, the firſt term cannot be integrated, but by the help of another infinite ſeries. Wherefore, the ſum of the ſaid ſeries being integrated, all but the firſt term, together with the integral of the ſeries expreſſing that firſt term, will form a ſeries which will be the value of the curve propoſed.

EXAMPLE XXV.

Fig. 115.



117. Let HBD be the logarithmic, AB the ſubtangent $= a$, AK $= x$, KH $= y$, and the equation $\frac{a\dot{y}}{y} = \dot{x}$. The value of $\dot{x}$ being ſubſtituted in the general formula, it will be $\frac{\dot{y}}{y}\sqrt{aa + yy}$, of which the integral depends on the ſame logarithmic. I ſhall forbear to apply infinite ſeries, becauſe their uſe may be ſufficiently ſeen in the former Examples.

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EXAMPLE XXVI.

118. Let the curve be the *Apollonian* parabola, with it's co-ordinates at any oblique angle, and whose equation is $ax = yy$. This being differenced, and substituted in the general formula for rectifications, when the ordinates are at oblique angles; that is, in the formula $\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y} + \frac{2e\dot{x}\dot{y}}{m}}$, instead of $\dot{x}$, it's value given by y being substituted, we shall have $\frac{2\dot{y}}{a}\sqrt{yy + \frac{aey}{m} + \frac{1}{4}aa}$; the integral of which will be partly algebräical, and will depend partly on the quadrature of the hyperbola.

EXAMPLE XXVII.

119. Let the equation be $x^t = y$, which is to infinite parabolas, and to infinite hyperbolas between the asymptotes. By differencing, it will be $x^{t-1}\dot{x} = \dot{y}$, and $x^{2t-2}\dot{x}\dot{x} = \dot{y}\dot{y}$; whence $\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}$, or the element of the curve, will be $\dot{x}\sqrt{x^{2t-2} + 1}$. Proceeding to the integration, I shall have recourse to the method of § 61, and shall exhibit the formula in the following manner, $\frac{\dot{x}}{\overline{x^{2t-2} + 1}|^{-\frac{1}{2}}}$. The canonical formula of the said article, or $\frac{x^n\dot{x}}{\overline{x^m + a^m}|^u}$, is algebräically integrable when $\frac{1 - m + n}{m}$ is an integer affirmative number; and if it be an integer negative number, it will be reduced to known simple quadratures. Now, by comparing this formula $\frac{\dot{x}}{\overline{x^{2t-2} + 1}|^{-\frac{1}{2}}}$ with the canonical, we have $n = 0$, $2t - 2 = m$, and $a = 1$. By which it will be necessary that $\frac{1 - 2t + 2}{2t - 2}$ shall be an integer, which I call h. Then $\frac{1 - 2t + 2}{2t - 2}$, that is, $\frac{3 - 2t}{2t - 2} = h$, and consequently $\frac{3 + 2h}{2 + 2h} = t$, the determining exponent of the infinite curves.

Let

Let h be a positive integer, beginning from o. Now, if $h = 0$, it will be $t = \frac{3}{2}$; if $h = 1$, it will be $t = \frac{5}{4}$; if $h = 2$, it will be $t = \frac{7}{6}$, &c. Let h be any one of the series of natural numbers, 0, 1, 2, 3, 4, 5, &c. the innumerable values of the exponent t will be expressed by the following progression, $t = \frac{3}{2}, \frac{5}{4}, \frac{7}{6}, \frac{9}{8}, \frac{11}{10}, \frac{13}{12}$, &c. the law of which series is manifest; and in all these cases the parabolical curves will be algebräically rectifiable; the first of which is the second cubical parabola.

Let h be equal to an integer negative number; and, first, make $h = -0$, in which case the same cubical parabola arises, because -0 and $+0$ are the same thing. Make $h = -1$, and the exponent t becoming $= \frac{1}{0}$, it is consequently infinite. Make $h = -2$, then $t = \frac{1}{2}$. Make $h = -3$, then $t = \frac{3}{4}$. And so on. Therefore the infinite values of the exponent t will be expressed by this progression, $t = \frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}, \frac{9}{10}$, &c. and the parabolical curves thence arising will be rectifiable by means of known quadratures.

The first curve which presents itself is the conic parabola, the rectification of which requires the quadrature of the hyperbola, § 110.

The other case, in which the general formula of § 61 is either rectifiable algebräically, or by means of known quadratures, is when $u - \frac{1}{m} - 1 - \frac{n}{m}$ is an integer. That is, by substituting the particular species of this example, $\frac{-3t + 2}{2t - 2} = h$, and therefore $\frac{2 + 2h}{3 + 2h} = t$, the determining exponent of the infinite curves.

Let h be a positive integer number, beginning at o; we shall have the following progression, $t = \frac{2}{3}, \frac{4}{5}, \frac{6}{7}, \frac{8}{9}, \frac{10}{11}$, &c.

Let h be a negative integer, and, first, let $h = -0$. Then the same exponent $t = \frac{2}{3}$ returns upon us, because -0 is equivalent to $+0$. Let $h = -1$, the exponent t becomes equal to the fraction $\frac{0}{1}$, and consequently is nothing. Let $h = -2$, $h = -3$, &c. and we shall have this following progression, $t = \frac{2}{1}, \frac{4}{3}, \frac{6}{5}, \frac{8}{7}, \frac{10}{9}$, &c.

The fraction which gives the value of the determining exponent t, is the same in both cases, only that in the second it is the reciprocal of the first; so that the progressions ought to come out reciprocal, as in effect they do. Therefore the curves determined by means of each formula are the same, but with reciprocal exponents, that is, they are referred to two different axes. As for example, the two exponents $\frac{1}{2}$ and $\frac{2}{1}$ belong to the *Apollonian* parabola, which offers itself in two manners, $x^{\frac{1}{2}} = y$, that is, $x = yy$, and likewise $x^{\frac{2}{1}} = y$, or $xx = y$; both local equations to the parabolical *trilineum*.

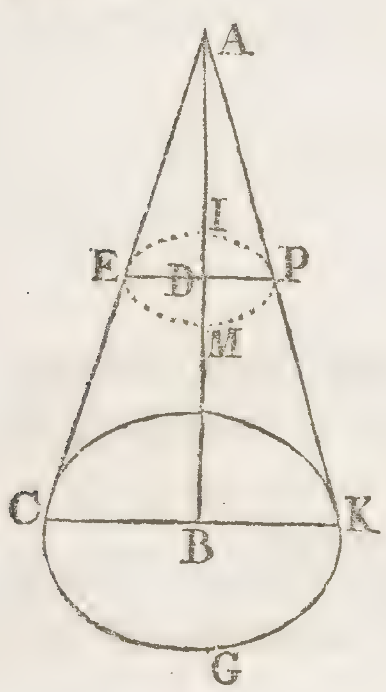
Wherefore

Wherefore these curves, which are included in the foregoing progressions, are either algebräically integrable, or do not require quadratures beyond the circle or hyperbola. But the other curves, infinite in number, require quadratures of a higher order.

It appears from our progressions, that the value of the exponent t is never negative. Hence no hyperbola admits of a rectification, either algebräical, or depending on the forementioned simple quadratures.

EXAMPLE XXVIII.

Fig. 127.



120. Let ACGKA be an erect cone, AB $= a$, BC $= b$; let AD $= x$ be any portion of the axis AB; it will be DE $= y = \frac{bx}{a}$, and therefore, substituting this value instead of y in the general formula, $\frac{cyy\dot{x}}{2r}$, it will be $\frac{cbbxx\dot{x}}{2aar}$, and by integration, $\frac{cbbx^3}{6aar}$, in respect to any portion taken from the vertex; omitting the constant quantity, which here is needless. And making $x = a$, the whole cone ACGKA will be $= \frac{cbba}{6r} = \frac{cbb}{2r} \times \frac{a}{3}$, that is, equal to the product of the base into a third part of the altitude. Of cubatures.

And, because the solid content of a cylinder is the product of the base into it's height, the cylinder will be to the inscribed cone as 3 to 1.

The cone ACGKA is therefore $\frac{cbba}{6r}$, and the cone AIEMP $= \frac{cbbx^3}{6aar}$; therefore the frustum of the cone IMCK will be $\frac{cbb}{6r} \times \overline{a - \frac{x^3}{aa}}$, and therefore will be to the whole cone in the ratio of $a^3 - x^3$ to a^3. Whence, for example, if we should make AD $= \frac{1}{2}$AB $= \frac{1}{2}a$, the frustum will be to the whole cone as $a^3 - \frac{1}{8}a^3$, or $\frac{7}{8}a^3$, to a^3, or as 7 to 8; and to the cone AEMPI, as 7 to 1.

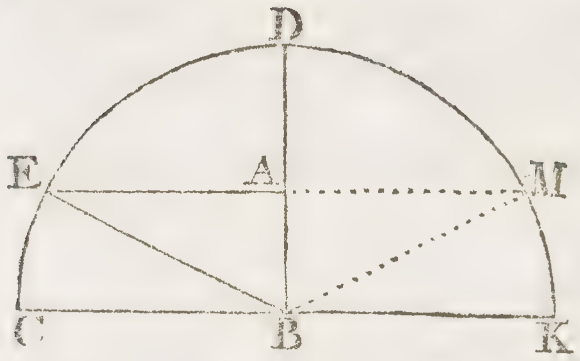
Therefore, as often as we are to measure any solid, it is necessary to consider, of what elements we design to have it composed, according to the different sections that may be adapted to it; varying it sometimes one way, sometimes another, as circumstances and conveniency may require. Then, among the aforesaid elements, to choose those which may be managed with the greatest facility,

facility, and to which the calculation may be moſt naturally adapted. In the erect cone for example, of which we are treating, we have as many circles as we pleaſe parallel to the baſe; and alſo as many triangles, which have their vertex the ſame as the cone, and for a baſe the parallel ordinates of the circle CGK. We may alſo cut the cone according to ſo many parabolas, which are equidiſtant from each other, and with axes parallel to the ſide AK; and many other ſections may be made.

Nevertheleſs it is true, that, to find the ſolidity of the cone, ſuch means as theſe are to be conſidered as not to the purpoſe, as being too compounded for the caſe propoſed. But it may be propoſed to cut the cone, or other ſolid, according to any plane whatever, and then to meaſure the two ſegments into which it is divided; and, in this caſe, it is convenient to make uſe of ſuch elements as ſhall correſpond to that ſection; as may be ſeen in Examples XXXVII. and XXXVIII. following.

EXAMPLE XXIX.

Fig. 128.



121. Let CDK be a ſemicircle, which is converted about a fixed radius DB, by which a hemiſphere will be produced; and make DB $= a$, DA $= x$, and it will be AE $= y = \sqrt{2ax - xx}$. Then, ſubſtituting this value in the general formula, it will be $\frac{cx}{2r} \times \overline{2ax - xx}$; and, by integration, the ſolidity of the indefinite ſegment AEM will be $= \frac{3caxx - cx^3}{6r}$. And making $x = a$, the ſolidity of the hemiſphere will be $= \frac{ca^3}{3r}$, and it's double, $\frac{2ca^3}{3r}$, will be the whole ſphere.

And becauſe the cylinder, the height of which is equal to the diameter of the baſe, or $2a$, is $\frac{ca^3}{r}$; the cylinder circumſcribed will be to the ſphere inſcribed, as $\frac{ca^3}{r}$ is to $\frac{2ca^3}{3r}$, or as 3 to 2. And conſequently the half cylinder will be to the hemiſphere in the ſame ratio. But the cone alſo, whoſe height is equal to the radius of the baſe, (or equal to a, the radius of the ſphere,) is $= \frac{ca^3}{6r}$; therefore the hemiſphere will be to the cone inſcribed as 2 to 1.

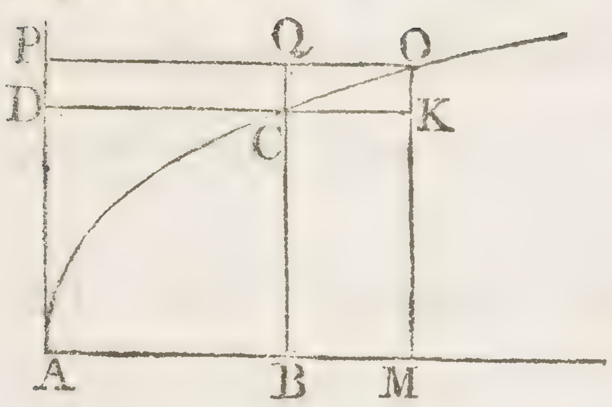
 Furthermore,

Furthermore, as it is known that $\frac{\sqrt{3aa}}{2}$ is the radius of the baſe of an equilateral cone inſcribed in a ſphere, the radius of which is $= a$; and the height of the ſame being $= \frac{3a}{2}$, the cone will be $= \frac{9ca^3}{48r}$, and the ſphere will be $\frac{2ca^3}{3r}$, and therefore the ſphere to the cone as $\frac{2}{3}$ to $\frac{9}{48}$, or as 32 to 9. In like manner may be demonſtrated as many Theorems of *Archimedes* as we pleaſe, which are of a like nature.

Hence the manner is plain, of obtaining any ſector of the ſphere, which is generated (for example) by the ſector of the circle BEDM. For to the ſegment of the ſphere generated by the figure AED, which we know to be $= \frac{3caxx - cx^3}{6r}$, muſt be added the cone generated by the triangle EBA, and which is found to be $= \frac{c}{6r} \times \overline{2ax - xx} \times \overline{a - x}$, and the ſum, $\frac{caax}{3r}$, will be the ſector required.

EXAMPLE XXX.

Fig. 129.



122. Let there be a parabola of any order, whoſe equation is $y^m = a^{m-1}x$; which, being converted about the axis AM, generates a parabolical conoid. Then it will be $y = a^{\frac{m-1}{m}} x^{\frac{1}{m}}$, and $yy = a^{\frac{2m-2}{m}} x^{\frac{2}{m}}$; and therefore, ſubſtituting this value, the general formula will be $\frac{ca^{\frac{2m-2}{m}} x^{\frac{2}{m}} \dot{x}}{2r}$; and, by integrating, $\frac{mca^{\frac{2m-2}{m}} x^{\frac{m+2}{m}}}{2r \times \overline{m+2}}$ will be the ſolid content of the indefinite conoid. Or elſe, becauſe $x^{\frac{2}{m}} = \frac{yy}{a^{\frac{2m-2}{m}}}$, and therefore $x^{\frac{2+m}{m}} = \frac{xyy}{a^{\frac{2m-2}{m}}}$, by ſubſtituting this value in the integral now found, it will be $\frac{mcxyy}{2r \times \overline{m+2}}$.

Make $m = 2$, that is, let it be the *Apollonian* parabola; the conoid will be $= \frac{cxyy}{4r}$, that is, the product of the base into half the height; and, by consequence, the said conoid will be half a cylinder of the same height, and of the same base.

If we would have the solid content of the dish, or of the solid generated by the figure ACD, converted about the axis AB; from the cylinder described by the rectangle ABCD, which we know to be $= \frac{cxyy}{2r}$, we must subtract the parabolical conoid $\frac{mcxyy}{2r \times \overline{m+2}}$, the remainder, $\frac{cxyy}{r \times \overline{m+2}}$, will be the content of the dish. And making $m = 2$, in respect of the *Apollonian* parabola, the dish will be $\frac{cxyy}{4r}$, which is half the cylinder, just as it ought to be, the conoid being also half of the same cylinder.

Let the figure move about the ordinate MO, and make $AM = b$, $MO = f$, $AB = x$, $BC = y$, $CK = b - x$, $KO = f - y$. The circle, with radius CK, will be $= \frac{c}{2r} \times \overline{b - x}|^2$, and therefore the product of this circle into $\dot{y}$ will be the differential of KM; that is, $\frac{c}{2r} \times \overline{bb\dot{y} - 2bx\dot{y} + xx\dot{y}}$ will be the element of the solid generated by the figure MACK. Therefore, by integrating, and, instead of x, putting it's value given by y, it will be $\frac{c}{2r} \times \overline{bby - \frac{2by^{m+1}}{\overline{m+1} \times a^{m-1}} + \frac{y^{2m+1}}{\overline{2m+1} \times a^{2m-2}}}$, equal to the indefinite solid. Or, putting x in the place of $\frac{y^m}{a^{m-1}}$, it will be $\frac{c}{2r} \times \overline{bby - \frac{2bxy}{m+1} + \frac{xxy}{2m+1}}$. Now, putting $x = b$, $y = f$, in respect to the whole solid generated by the figure ACOM, it will be $\frac{c}{2r} \times \overline{bbf - \frac{2bbf}{m+1} + \frac{bbf}{2m+1}}$, that is, $\frac{2mmbbf}{\overline{2m+1} \times \overline{m+1}} \times \frac{c}{2r}$. And if we would have the parabola to be that of *Apollonius*, that is, if $m = 2$, then the solid will be $= \frac{4cbbf}{15r}$.

It is easy to perceive, that, in the *Apollonian* parabola, a cylinder on the same base, and of the height of the said solid, shall be to the solid as 15 to 8; and that the solid generated by the figure OAP shall be $= \frac{7cbbf}{30r}$.

 Let

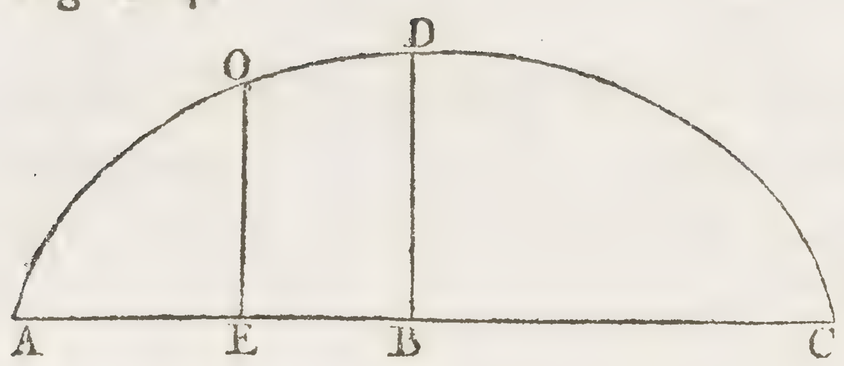
Let the figure move about the right line AP, and let it be, as before, $AB = x$, $BC = y$; then $\frac{cxx}{2r}$ will be a circle with radius DC, and $\frac{cxx\dot{y}}{2r}$ will be the element of the folid generated by the figure ACD. And, inftead of x, putting it's value given by y, and then integrating, it will be $\frac{c}{2r} \times \frac{y^{2m+1}}{\overline{2m+1} \times a^{2m+2}}$, that is, $\frac{c}{2r} \times \frac{xxy}{2m+1}$, equal to the indefinite folid. And making $x = b$, $y = f$, it will be $\frac{cbbf}{2r \times \overline{2m+1}}$, in refpect to the whole folid, generated by the figure AOP.

But the cylinder on the fame bafe and altitude is $= \frac{cbbf}{2r}$; therefore the folid generated by the figure AMO is $= \frac{c}{2r} \times \frac{2mbbf}{2m+1}$.

But ftill, in another manner, we may obtain the folid generated by the figure AOM, revolving about the axis AP. Make $AM = b$, $MO = f$. A circle with radius DC will be $= \frac{cxx}{2r}$, and the circle with radius DK will be equal to $\frac{cbb}{2r}$. Therefore $\frac{c}{2r} \times \overline{bb - xx}$ will be the annulus defcribed by the line CK, and $\frac{c\dot{y}}{2r} \times \overline{bb - xx}$ will be the element of the folid generated by the figure CKMA; and, inftead of x, putting it's value given by y, it will be $\frac{c}{2r} \times \overline{bb\dot{y} - \frac{y^{2m}\dot{y}}{a^{2m-2}}}$, and by integration, $\frac{c}{2r} \times \overline{bby - \frac{y^{2m+1}}{\overline{2m+1} \times a^{2m-2}}}$. Laftly, making $y = f$, in refpect of the whole folid, generated by the figure AMOA, it will be $\frac{c}{2r} \times bbf - \frac{f^{2m+1}}{\overline{2m+1} \times a^{2m-2}}$. But, when $y = f$, becaufe of the parabola, it will be $x = b = \frac{f^m}{a^{m-1}}$, and $bb = \frac{f^{2m}}{a^{2m-2}}$. Therefore, in the integral, fubftituting the value given by b, the folid will be $\frac{c}{2r} \times bbf - \frac{bbf}{2m+1} = \frac{c}{2r} \times \frac{2mbbf}{2m+1}$, as above.

EXAMPLE XXXI.

Fig. 124.

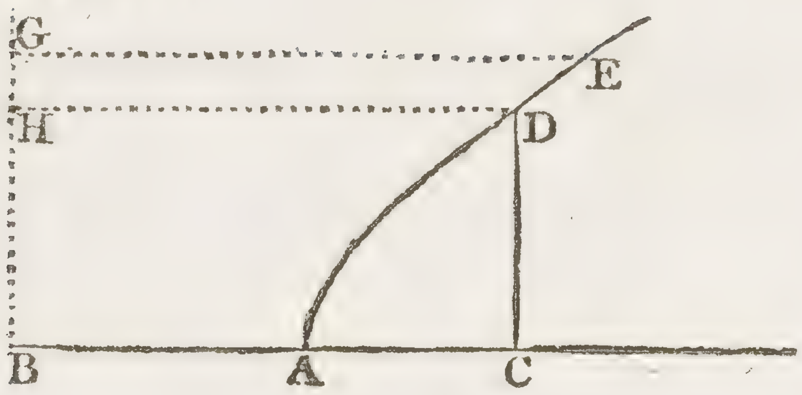


123. Let ADC be an ellipsis, $AB = a$, $BD = b$, $AE = x$, $EO = y$; and therefore the equation is $\frac{bb}{aa} \times \overline{2ax - xx} = yy$. Therefore, in the general formula, substituting the value of y given from the equation, it will be $\frac{cbb}{2aar} \times \overline{2axx - xxx}$; and by integration, it will be $\frac{cbb}{2aar} \times \overline{axx - \frac{1}{3}x^3}$, equal to the indefinite solid generated by the figure AEO, turning about the axis AC. Making $x = a$, it will be $\frac{cbba}{3r}$, half of the spheroid; and putting $x = 2a$, it will be $\frac{2cbba}{3r}$, the whole spheroid.

And, because the cone of the same altitude AC, and of a base the radius of which is the conjugate semiaxis BD, is $= \frac{cbba}{3r}$, and the cylinder is $= \frac{cbba}{r}$, the spheroid will be two third parts of the cylinder, and double to the cone.

EXAMPLE XXXII.

Fig. 121.



124. Let AD be an hyperbola, which is converted about BC, and let it's transverse semiaxis be $BA = \frac{1}{2}a$, the centre B, and it's parameter $= b$, $AC = x$, $CD = y$, and the equation is $\overline{ax + xx} \times \frac{b}{a} = yy$. Substituting the value of y in the general formula, it will be $\frac{cb\dot{x}}{2ar} \times \overline{ax + xx}$; and by integration, it will be $\frac{cb}{2ar} \times \overline{\frac{1}{2}axx + \frac{1}{3}x^3}$, equal to the indefinite hyperbolical conoid, generated by the figure ADC.

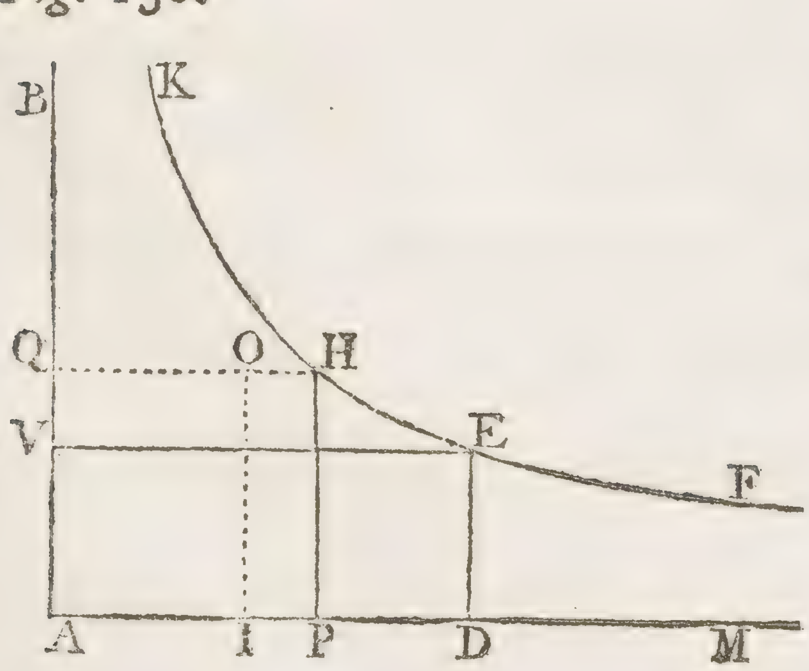
Make

Make BC $= x$, and the reft as above. The equation will be $\frac{b}{a} \times \overline{xx - \frac{1}{4}aa} = yy$, and therefore the formula will be $\frac{cb\dot{x}}{2ar} \times \overline{xx - \frac{1}{4}aa}$, and by integration, $\frac{cb}{2ar} \times \overline{\frac{1}{3}x^3 - \frac{1}{4}aax} + f$. I add the conftant quantity f, which, in this cafe, will be fomething. To determine what, it muft be obferved that in the point A, when $x = \frac{1}{2}a$, the folid ought to be nothing. Wherefore, inftead of x, putting $\frac{1}{2}a$ in the integral, it ought to be $f + \frac{cb}{2ar} \times \overline{\frac{1}{24}a^3 - \frac{1}{8}a^3} = 0$, and therefore $f = \frac{caab}{24r}$. Therefore the complete integral will be $\frac{cb}{2ar} \times \overline{\frac{1}{3}x^3 - \frac{1}{4}a^2x + \frac{1}{12}a^3}$.

Let the hyperbola be converted about the conjugate femiaxis HB, and make the tranfverfe femiaxis AB $= a$, the conjugate femiaxis $= b$, BC $= x$, CD $= y$. The circle with radius HD will be $= \frac{cxx}{2r}$, and therefore $\frac{cxx\dot{y}}{2r}$ will be the element of the folid generated by the plane or figure BHDA. And, inftead of xx, fubftituting it's value given from the equation of the curve, we fhall have $\frac{c\dot{y}}{2r} \times \frac{aayy + aabb}{bb}$; and by integration, $\frac{c}{2r} \times \overline{\frac{aay^3}{3bb} + aay}$; and making $y = b$, the folid will be $= \frac{2caab}{3r}$.

EXAMPLE XXXIII.

Fig. 130.



125. Let KHF be an hyperbola between the afymptotes; AD $= a$, DE $= b$, AP $= x$, PH $= y$, and the equation $xy = ab$. Let the curve revolve about the afymptote AB. Then the circle with radius QH will be $= \frac{cxx}{2r}$, and therefore $\frac{cxx\dot{y}}{2r}$ will be the element of the folid generated by the figure AQHFMA, infinitely produced towards M. And, inftead of x, putting it's value given from the equation, it

it will be $\frac{caabb\dot{y}}{2ryy}$, and by integration, $f - \frac{caabb}{2ry}$. Now, to determine f, it may be obſerved, that, when it is $y = 0$, the ſolid ought to be nothing, and therefore $f = \frac{caabb}{2r \times 0}$, an infinite quantity, and therefore the complete integral will be $- \frac{caabb}{2ry} + \infty$; ſo that the ſolid is of an infinite value.

Inſtead of ſubſtituting in the formula the value given by y from the equation, in the place of xx, if we ſhould ſubſtitute the value of y; it would be $- \frac{abc\dot{x}}{2r}$, and by integration, $- \frac{abcx}{2r} + f$. But the ſolid cannot be nothing except when x is infinite, and then the conſtant quantity f to be added ought to be infinite, and therefore the ſolid will be infinite.

To have the ſolid generated by the plane or figure BAPHK infinitely produced towards B, it will be enough to conſider, that as $\frac{cx}{r}$ is the periphery of the circle whoſe radius is QH $= x$, then $\frac{cxy}{r}$ will be the ſuperficies of the cylinder, generated by the plane AQHP, and conſequently $\frac{cxy\dot{x}}{r}$ will be the ſolid content of the hollow cylinder, generated by the infinitely little rectangle IPHO. Therefore the ſum of all theſe, or $\int \frac{cxy\dot{x}}{r}$, will be the ſolid required. Therefore, inſtead of y, putting it's value $\frac{ab}{x}$, the integral will be $\frac{cabx}{r}$, a finite quantity, although the ſolid be of an infinite altitude.

In the expreſſion $\frac{cabx}{r}$ of the ſolid, inſtead of ab putting it's value xy, given from the equation; it will be $\frac{cxxy}{r}$. But $\frac{cxxy}{2r}$ is the cylinder generated by the rectangle APHQ. Therefore the hyperbolical ſolid will be double to this cylinder. And therefore the ſolid generated by the figure BQHK, infinitely produced, will be equal to the cylinder which ſerves it for a baſe. Therefore, taking $x = a$, and conſequently $y = b$, this cylinder will be $= \frac{caab}{2r}$, which is equal to the ſolid erected upon it.

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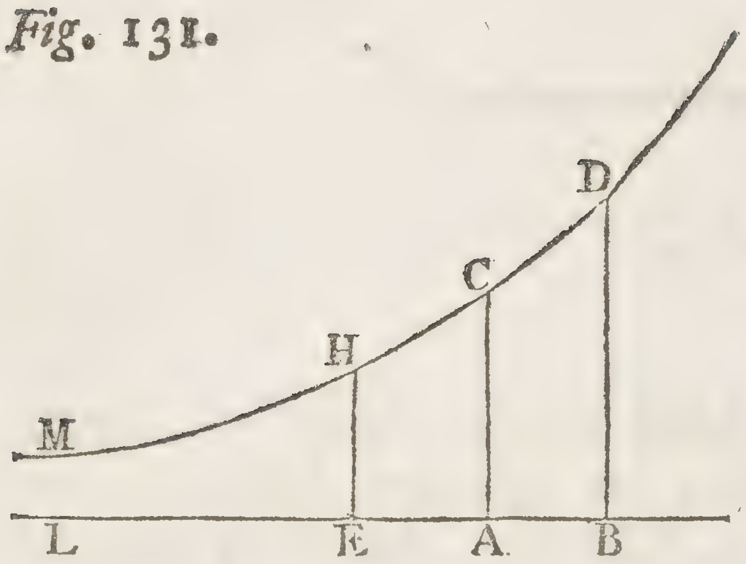
EXAMPLE XXXIV.

Fig. 131.



126. Let HCD be the logarithmic curve, it's subtangent CA $= a$, AB $= x$, BD $= y$, and it's equation $\dot{x} = \frac{a\dot{y}}{y}$. Let it be converted about the asymptote EB. In the general formula, instead of $\dot{x}$, putting it's value given from the equation, it will be $\frac{cay\dot{y}}{2r}$; and by integration, it will be $\frac{cayy}{4r} + f$. But when it is $y =$ AC $= a$, the solid will be $= 0$. Therefore it must be $f = -\frac{ca^3}{4r}$; and the complete integral, that is, the solid generated by the indefinite plane ABDC, will be $= \frac{cayy - ca^3}{4r}$.

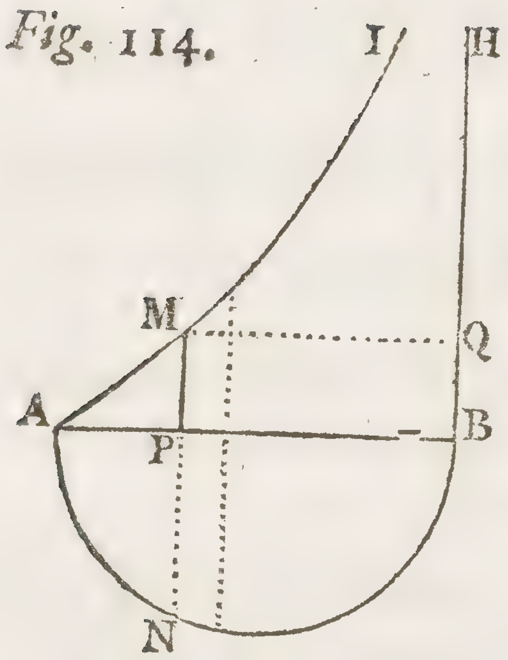
Let the abscis AE be negative, and therefore $= -x$; and it's fluxion also will be negative, or $-\dot{x}$. And because, as the abscis increases, the ordinate will diminish, therefore the fluxion of EH will also be negative, or $-\dot{y}$; so that the equation of the curve will be still the same, $\dot{x} = \frac{a\dot{y}}{y}$. But, because $\dot{x}$ is negative, the general formula will be negative also, or $-\frac{cyy\dot{x}}{2r}$. Substituting therefore, the value of $\dot{x}$, it will be $-\frac{cay\dot{y}}{2r}$, and by integration, $-\frac{cayy}{4r} + f$. But when the solid is nothing, it will be $y = a$; therefore $f = \frac{ca^3}{4r}$, and the complete integral will be $\frac{ca^3 - cayy}{4r}$, equal to the solid generated by the plane ACHE. Putting $y = 0$, that is, supposing the solid to be infinitely produced towards M, the integral will be $= \frac{ca^3}{4r}$, and then the solid itself, infinitely produced, will be $= \frac{ca^3}{4r}$. But the solid generated by the plane ACHE we have seen to be $\frac{ca^3 - cayy}{4r}$; then the solid infinitely produced, generated by the plane LEMH, is $\frac{cayy}{4r}$.

Now,

Now, becauſe the cylinder, the radius of whoſe baſe is AC $= a$, and it's height alſo $= a$, is $\frac{ca^3}{2r}$; the ſolid of the logarithmic curve, infinitely produced towards M, on the baſe with radius AC $= a$, will be to the ſaid cylinder, in the ratio of $\frac{1}{4}$ to $\frac{1}{2}$, or as 1 to 2.

EXAMPLE XXXV.

Fig. 114.

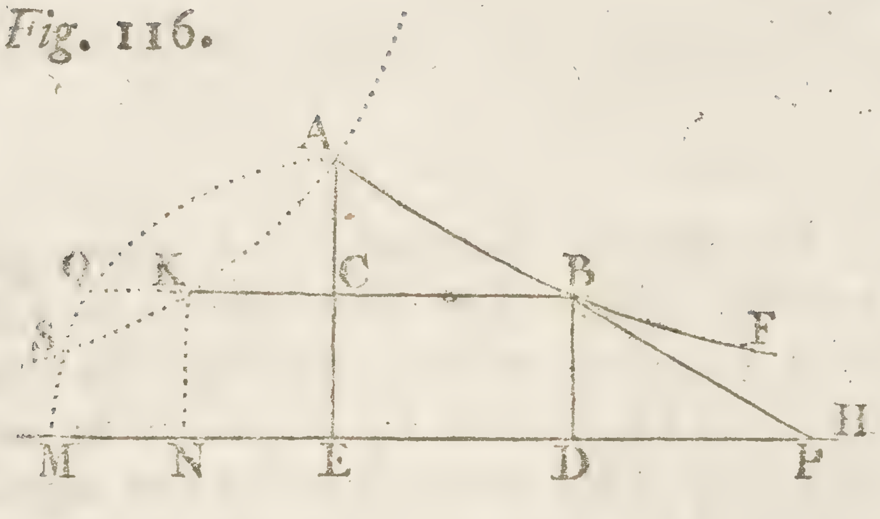


127. Let the curve AMI be the ciſſoid of *Diocles*, which, by revolving about the right line AB, deſcribes a ſolid. Make AP $= x$, PM $= y$, AB $= a$, and the equation will be $yy = \frac{x^3}{a-x}$. Therefore, the value of yy being ſubſtituted, the general formula of ſolids will be $\frac{cx^3\dot{x}}{2r \times \overline{a-x}}$, and by integration, $-\frac{cx^3}{6r} - \frac{cax^2}{4r} - \frac{caax}{2r} - \frac{caaa}{2r} \times l\,\overline{a-x} + f$. But, making $x = 0$, the ſolid ought to be nothing, and therefore $f = \frac{caaa}{2r}la$. And the complete integral $\frac{caaa}{2r}la - \frac{caaa}{2r}l\,\overline{a-x} - \frac{caax}{2r} - \frac{caxx}{4r} - \frac{cx^3}{6r}$ is equal to the ſolid generated by the figure APM. And making $x = a$, the whole ſolid will be $= \frac{caaa}{2r}la - \frac{caaa}{2r}l0 - \frac{11ca^3}{12r}$. But the logarithm of 0 is an infinite quantity and negative, which, multiplied into $-\frac{caaa}{r}$, makes an affirmative quantity; ſo that the intire ſolid will be infinite. It is to be obſerved, that the aforeſaid logarithms are to be taken from the logarithmic curve, the ſubtangent of which $= a$.

By the help of infinite ſeries, it will be $\frac{cx^3\dot{x}}{2r \times \overline{a-x}} = \frac{cx^3\dot{x}}{2ar} + \frac{cx^4\dot{x}}{2raa} + \frac{cx^5\dot{x}}{2ra^3} + \frac{cx^6\dot{x}}{2ra^4}$, &c.; and by integration, the ſolid generated by the plane APM will be $= \frac{cx^4}{8ar} + \frac{cx^5}{10ra^2} + \frac{cx^6}{12ra^3} + \frac{cx^7}{14ra^4}$, &c. And making $x = a$, in reſpect of the intire ſolid, it will be $\frac{ca^3}{2r}$ into $\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}$, &c. the total value of which ſeries is infinite.

EXAMPLE XXXVI.

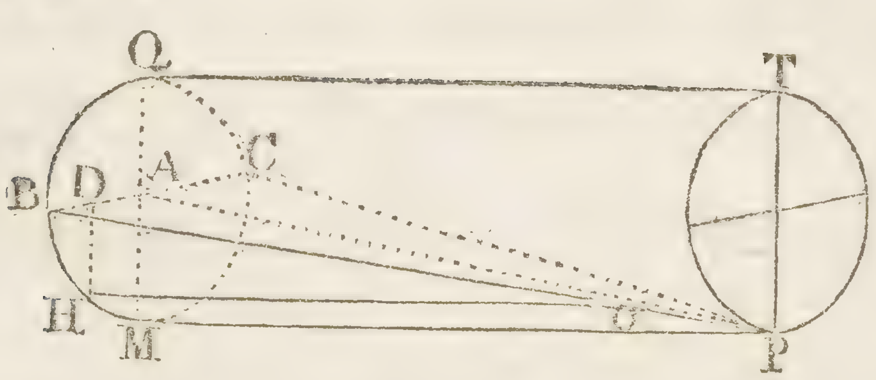
Fig. 116.



128. Let the *tractrix* ABF be converted about the asymptote EH. In the general formula $\frac{cyy\dot{x}}{2r}$, substituting the value of $\dot{x}$ given from the equation $\dot{x} = -\frac{\dot{y}\sqrt{aa - yy}}{y}$, § 103, we shall have $-\frac{cy\dot{y}\sqrt{aa - yy}}{2r}$. And by integration, it will be $\frac{c}{6r} \times \overline{aa - yy}^{\frac{3}{2}}$, equal to the solid generated by the figure AEDB, omitting the addition of a constant, which is here unnecessary. Wherefore, making $y = 0$, the solid infinitely produced will be $= \frac{ca^3}{6r}$. But the solid content of the sphere whose radius is AE $= a$, (§ 121.) will be $= \frac{2ca^3}{3r}$; and therefore the solid infinitely produced will be a fourth part of that sphere.

EXAMPLE XXXVII.

Fig. 132.

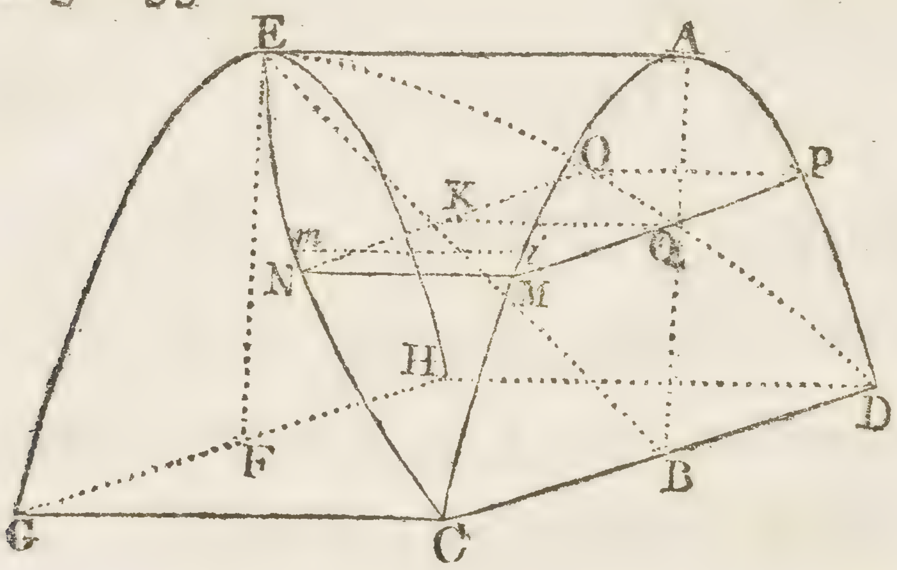


129. Let QBMCPT be a cylinder, from which, by a plane through the diameter BC, and in the direction AP, a portion or *ungula*, BMCPB, is cut off; the solid content of this is required.

Make BC $=$ QM $= 2a$, MP $=$ QT $= b$, AD $= x$, and DH being drawn, shall be an ordinate in the circle $= \sqrt{aa - xx}$. From the point H let the right line HO be drawn parallel to MP or QT, which shall be in the superficies of the cylinder. Then from D to the point O let the right line DO be drawn, which shall be in the plane BOPC. Then we shall have formed in the solidity of the *ungula* the triangle DHO, which

which is similar to the triangle AMP, and therefore it will be HO $= \frac{b\sqrt{aa-xx}}{a}$. But the aggregate of all these triangles, DHO, is just the solidity required of half the *ungula*, and therefore it will be $= \int \frac{b\dot{x}}{2a} \times \overline{aa - xx}$; and by integration, $\frac{abx}{2} - \frac{bx^3}{6a}$. And making $x = a$, the aforesaid half *ungula* will be finally $= \frac{1}{3}aab$, and the whole $= \frac{2}{3}aab$.

Fig. 133.



In another manner, and more generally, thus. Let DACHEG be half of a cylinder, which, through the diameter CD, is cut by a plane in the direction DE, whence arises the *ungula* DBCEAD, the solidity of which is required. Make BA $= a$, AE $= b$, BQ $= x$, QM $= y$; it will be QK $= \frac{bx}{a}$, and therefore the rectangle PONM $= \frac{2bxy}{a}$. And this being drawn into $\dot{x}$, or $\frac{2byx\dot{x}}{a}$, will be the element of the solidity of the *ungula*.

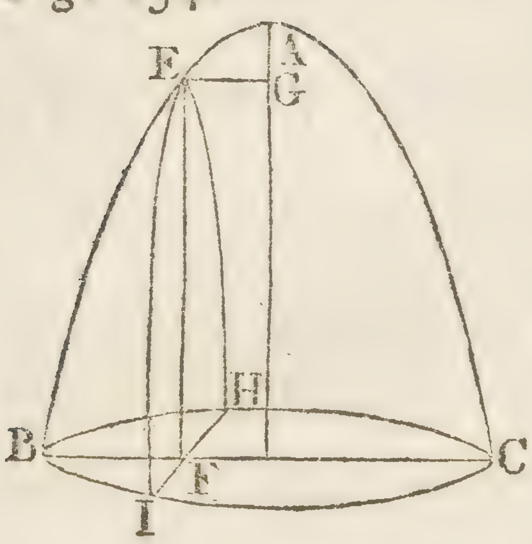
Let the curve DAC be a semicircle; then $y = \sqrt{aa - xx}$, and the formula will be $\frac{2bx\dot{x}}{a}\sqrt{aa - xx}$; and by integration, $- \frac{2b}{3a} \times \overline{aa - xx}|^{\frac{3}{2}} + m$. Now, by putting $x = 0$, the constant, m, will be found to be $= \frac{2}{3}baa$, and therefore the integral of the solid complete will be $\frac{2}{3}baa - \frac{2b}{3a} \times \overline{aa - xx}|^{\frac{3}{2}}$; and making $x = a$, in respect of the whole *ungula*, it will be $\frac{2}{3}baa$, as before.

Let the curve DAC be one of the parabolas *ad infinitum*, and it's equation $y^m = a - x$. Substituting the value of y, the formula will be $\frac{2bx\dot{x}}{a} \times \overline{a - x}|^{\frac{1}{m}}$, which being integrated according to § 29, and a constant being joined, and making $x = a$; it will give $\frac{2bm^2 a^{\frac{m+1}{m}}}{2m + 1 \times m + 1}$ for the solidity of the whole *ungula*. And taking $m = 2$, or the *Apollonian* parabola, it will be $\frac{8ba^{\frac{3}{2}}}{15}$. Now, supposing that, when $x = 0$, it is BC $= y = c$; it will be $a^{\frac{1}{2}} = c$, and therefore

therefore the *ungula* $= \frac{8}{15}abc$. After the same manner we may find the *ungula* of the elliptical cylinder to be $\frac{2}{3}abc$, supposing the transverse semiaxis $= a$, and the conjugate semiaxis $= c$.

EXAMPLE XXXVIII.

Fig. 134.



130. Let the parabolical conoid BAC be cut by any plane IEH, perpendicular to the circular base BICH; it is required to find the measure of the segment, comprehended by the section IEH, and by the plane parallel to it, through the axis AD.

Make the parameter $= a$ of the generating parabola BAC, the given abscis AD $= b$, then the ordinate DB $= \sqrt{ab}$. Let the co-ordinates be DF $= x$, FE $= y$, and therefore the equation of the aforesaid curve BAC will be $ab - xx = ay$. By the nature of the circle BICH, the rectangle CFB $= ab - xx$, equal to the square FH $= zz$. But $ab - xx = ay$; therefore $ay = zz$, and consequently the section IEH will be a parabola, with the same parameter as the principal. Wherefore the rectangle EFH remains fixed, $= yz = y\sqrt{ay}$; and because this is to the area IEH, as 3 to 4, this area will be $= \frac{4}{3}y\sqrt{ay}$, and the product of this area IEH into the infinitely little height $\dot{x}$, the fluxion of DF $= x$, will be the element of the solid in question, that is, $\frac{4}{3}y\dot{x}\sqrt{ay}$. But $y = \frac{ab - xx}{a}$; therefore the element will be $\frac{4}{3}\dot{x} \times \frac{ab - xx}{a}\sqrt{ab - xx}$, or $\frac{4}{3}b\dot{x}\sqrt{ab - xx} - \frac{4}{3a}x^2\dot{x}\sqrt{ab - xx}$.

The fluent of the first term depends on the quadrature of the circle BHC; the second is reduced to known quadratures, by means of the first formula of § 61.

131. I forbear from giving examples of solids generated by curves with the co-ordinates at oblique angles to each other; because, the formula for these cases being different from the usual and ordinary ones, only by constant quantities, no difficulties can be met with of a different nature from these already produced.

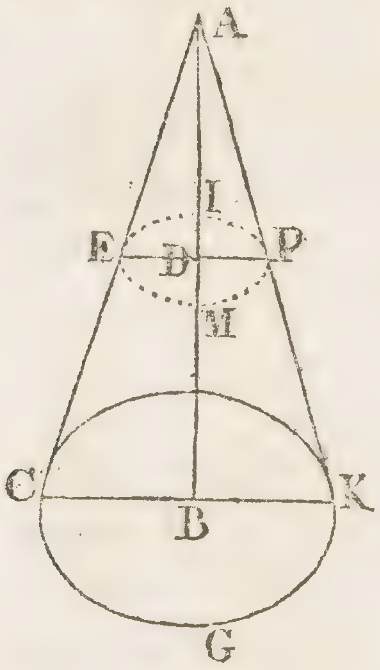
Thus, also, I omit examples of solids generated by curves which are referred to a *focus*, because I am not willing to introduce the Theory of the Centers of Gravity, as I have said before. The given curves may be reduced to others referred to an axis, about which I have already treated.

N. B. The letter D is omitted in the center of the base of Fig. 134.

EXAMPLE XXXIX.

The complanation of curved surfaces.

Fig. 127.

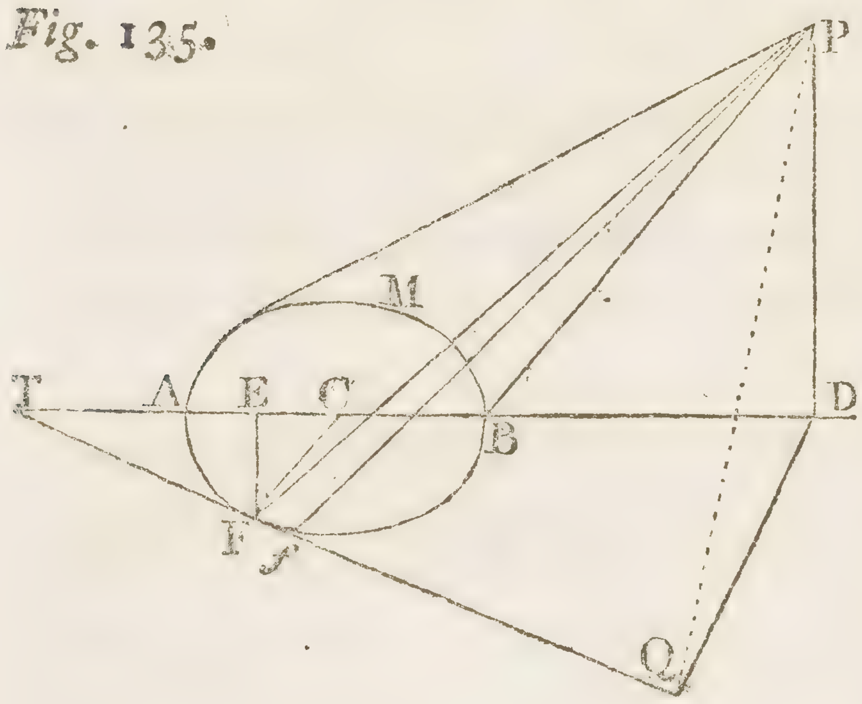


132. Let ACGK be an erect cone, AB $= a$, BC $= b$, any portion of the axis, as AD $= x$; it will be DE $= y$ $= \frac{bx}{a}$, and $\dot{y} = \frac{b\dot{x}}{a}$, $\dot{y}\dot{y} = \frac{bb\dot{x}\dot{x}}{aa}$. Therefore this value of $\dot{y}\dot{y}$, being substituted in the general formula $\frac{cy}{r}\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}$, it will be $\frac{cy}{r}\sqrt{\frac{aa\dot{x}\dot{x} + bb\dot{x}\dot{x}}{aa}} = \frac{cy\dot{x}\sqrt{aa + bb}}{ar}$; and the value of y being substituted, that is, $\frac{bx}{a}$, it will be $\frac{cbx\dot{x}\sqrt{aa + bb}}{aar}$, and by integration, $\frac{cbxx\sqrt{aa + bb}}{2aar}$, in respect of the superficies of the cone AEMPI. And making $x = a$, it will be $\frac{cb\sqrt{aa + bb}}{2r}$, in respect of the superficies of the whole cone, and therefore it is equal to the rectangle of half the circumference of the base into the side AC.

The same conclusion would have been had, if, instead of substituting in the general formula the value of $\dot{y}\dot{y}$, we had substituted the value of $\dot{x}\dot{x}$.

Wherefore the surface of the frustum of the cone IMKCG will be $=$ $\frac{cb}{2r}\sqrt{aa + bb} - \frac{cbxx}{2aar}\sqrt{aa + bb}$, that is, $\frac{cb\sqrt{aa + bb} \times \overline{aa - xx}}{2raa}$; and therefore it will be to the surface of the whole cone, as $aa - xx$ to aa.

Fig. 135.



133. But if the cone be scalene, it is necessary to proceed after another manner. Let PAFBM be a scalene cone, the base of which is the circle AFBM; and on the diameter produced (if need be) let fall PD perpendicular to the plane of the circle, or the base. Let two points F, f, be taken in the periphery of the circle, infinitely near to each other, and let the two sides of the cone FP, fP, be drawn. It is plain that the infinitesimal triangle PFf will be the difference or

or element of the ſuperficies of the cone. Then to the point F let the tangent TFQ be drawn, to which let DQ be perpendicular, and let the points P, Q, be joined by the right line PQ.

Now, becauſe the plane of the triangle PDQ paſſes through the right line PD, which is perpendicular to the plane of the baſe of the cone, the plane PQD will alſo be perpendicular to the ſame plane of the baſe. But the tangent TQ, which is alſo in the plane of the baſe, makes a right angle with QD, the common ſection of the two planes, and therefore will be perpendicular to the plane PQD, and conſequently to the right line QP; and therefore the triangle $PFf = \frac{PQ \times Ff}{2}$.

Make the radius $CA = r$, $CD = b$, $CE = x$; it will be $FE = \sqrt{rr - xx}$; and becauſe the angle CFT is a right one, TF being a tangent to the circle, the triangles CFE, TCF, will be ſimilar. Whence it will be $CT = \frac{rr}{x}$. But CT . CF :: CF . CE :: TD . DQ. Therefore $DQ = \frac{rr + bx}{r}$. Make the given line $PD = p$. Therefore it will be $PQ = \sqrt{pp + \frac{\overline{rr + bx}^2}{rr}}$. But the element of the circle Ff we know to be $-\frac{r\dot{x}}{\sqrt{rr - xx}}$; therefore $\frac{1}{2}Ff \times PQ$, the element of the ſuperficies, will be $-\frac{r\dot{x}}{2}\sqrt{pp + \frac{\overline{rr + bx}^2}{rr}} \div \sqrt{rr - xx}$; a formula which hitherto has not been reduced to the known quadratures of the circle or hyperbola, becauſe it cannot be freed from radical ſigns, as has been ſeen at § 38, and as we have alſo ſeen, in our attempt to rectify the ellipſis.

If we have recourſe to infinite ſeries, the numerator muſt be reduced to a ſeries, and alſo the denominator; then we muſt proceed in the ſame manner as was done in the ſecond method concerning the ellipſis, in Example XX, § 112.

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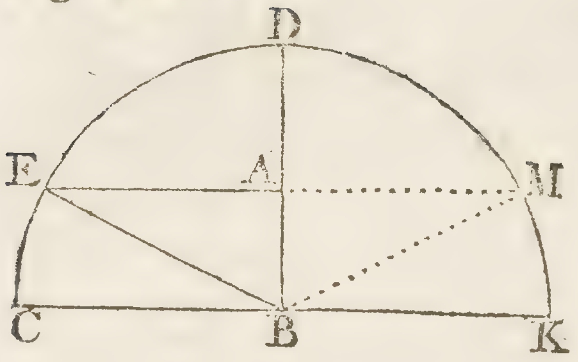
EXAMPLE XL.

Fig. 128.



134. Let there be a hemiſphere, the generating ſemicircle of which is CDK, which is converted about the radius DB $= a$, and make any line DA $= x$; it will be AE $= y = \sqrt{2ax - xx}$, and therefore $\dot{y}\dot{y} = \frac{\overline{a - x}|^2 \times \dot{x}\dot{x}}{2ax - xx}$. And making the ſubſtitutions in the general formula, it will be $= \frac{ca\dot{x}}{r}$, and by integration, $\frac{cax}{r}$ = to the ſuperficies of the ſegment of the ſphere, generated by the arch EDM. And making $x = a$, the ſuperficies of the hemiſphere will be $= \frac{caa}{r}$, and therefore $\frac{2caa}{r}$ will be the ſuperficies of the whole ſphere. Therefore the ſuperficies of any ſegment will be equal to the product of the periphery of the generating circle of the ſphere, into the altitude of that ſegment; of the hemiſphere, equal to the rectangle of the ſame periphery into the radius; and of the ſphere, equal to the rectangle of the periphery into the diameter; and therefore theſe ſuperficies will be to each other in the ratio of their reſpective altitudes, the radius, and the diameter.

And becauſe the area of the generating circle of the ſphere is $= \frac{caa}{2r}$, the ſuperficies of the ſphere will be to the ſame area as 4 to 1, that is, quadruple of the greateſt circle.

And becauſe, alſo, the ſuperficies of the cylinder, (excluding it's baſes,) which is circumſcribed to the hemiſphere, is equal to the product of the periphery of the baſe into the height; it will therefore be $= \frac{caa}{r}$, and conſequently the ſuperficies will be equal to that of the hemiſphere. Now the cone inſcribed in the hemiſphere has alſo it's ſuperficies $= \frac{ca\sqrt{2aa}}{2r}$; therefore the ſuperficies of the cylinder, or of the hemiſphere, to the ſuperficies of the inſcribed cone, will be as $2a$ to $\sqrt{2aa}$, that is, as the diameter to the ſide of the cone.

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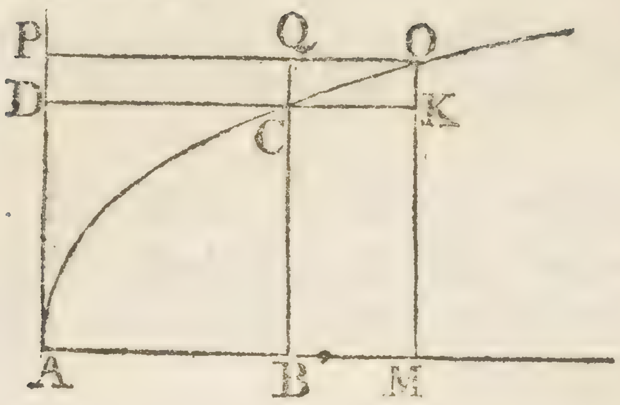
EXAMPLE XLI.

Fig. 129.



135. If the parabola ACO of the equation $ax = yy$, turns about the axis AM; it will be $a\dot{x} = 2y\dot{y}$, and $\dot{x}\dot{x} = \frac{4yy\dot{y}\dot{y}}{aa}$, and therefore, making the substitution, the formula will be $\frac{cy\dot{y}}{ar}\sqrt{4yy+aa}$, and by integration, $\frac{c}{12ra} \times \overline{4yy + aa}|^{\frac{3}{2}}$, equal to the [superficies of the] indefinite parabolical conoid, equal to the fourth proportional of $6a$, $\sqrt{4yy + aa}$, and the area of the circle whose radius is $= \sqrt{4yy + aa}$.

136. More generally, let $\frac{x^t}{t} = y$ be the equation of the parabola ACO, (Fig. 129.) with it's abscis AB $= x$, and with it's ordinate BC $= y$; which equation for the *trilineum* ACD will be $\overline{xt}|^{\frac{1}{t}} = y$, if we take AD $= x$ as abscis, and DC $= y$ as ordinate. At § 119, Example XXVII, it has been seen, that the element of the curve, which I call $\dot{u}$, was $= \frac{\dot{x}}{\overline{x^{2t-2}+1}|^{-\frac{1}{2}}}$; and the differential formula for the superficies is $\frac{cy\dot{u}}{r}$. Then it will be $\frac{cy\dot{u}}{r} = \frac{cy\dot{x}}{r \times \overline{x^{2t-2}+1}|^{-\frac{1}{2}}}$. But, by the local equation, it is $\frac{x^t}{t} = y$. Then it will be

$$\frac{cy\dot{u}}{r} = \frac{cx^t\dot{x}}{rt \times \overline{x^{2t-2}+1}|^{-\frac{1}{2}}}.$$

To proceed to the integrations or quadratures, I shall make use of the method explained at § 61, and applied to the aforesaid Example XXVII. But, first, it is to be observed, that c, being the periphery of the circle whose radius is r, the integral $\int\frac{cy\dot{u}}{r}$ will give us the surface of the conoid. But if c represents any right line whatever, we shall have the measure of the surface of the *ungula*, when a cylindroid is erected upon the base CAB, which is cut by a plane

plane paffing through the axis AB, and with the fubject bafe CAB forms an angle, of which the right fine is to that of the complement, as c is to r. Then the ungular fuperficies is to that of the round folid, as a given right line is to the circumference c.

Operating, therefore, as explained above, at § 61, that our formula may be algebräically integrable, or reducible to known quadratures, we fhall find that it muft be $t = \frac{3 + 2b}{1 + 2b}$, or elfe $t = \frac{b + 1}{b + 2}$, where b denotes any integer number, pofitive or negative.

The firft condition, or $t = \frac{3 + 2b}{1 + 2b}$, making b any integer number, firft pofitive and then negative, will give us thefe two progreffions:

I. $t = \frac{3}{1}, \frac{5}{3}, \frac{7}{5}, \frac{9}{7}, \frac{11}{9}$, &c. II. $t = \frac{-1}{1}, \frac{1}{3}, \frac{3}{5}, \frac{5}{7}, \frac{7}{9}, \frac{9}{11}$, &c.

The fecond condition, or $t = \frac{b + 1}{b + 2}$, making b any integer number, firft pofitive and then negative, will give us thefe other two progreffions:

III. $t = \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}$, &c. IV. $t = \frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}$, &c.

To this I fhall fubjoin a few fhort obfervations.

I. As the two progreffions, the firft and the third, contain the exponents of all thofe parabolas, which, by circulating about the axis, generate conoids, the fuperficies of which are analytically quadrable, fuppofing only the rectification of the circular periphery; and confequently all the *ungulæ* above defcribed, of a given altitude, admit an algebräical quadrature: So, in the cafes of the fecond and fourth progreffions, fomething more is intended, as they require the quadrature of the hyperbola.

II. It is obfervable that, the firft feries being compared with the fecond, and the third with the fourth, the exponents are reciprocal, and belong to the fame curve. This fhows that, as the parabolical area may be converted, either about the axis AB, or about the axis AD, and in each cafe may produce very different fuperficies; if, in the firft cafe, it generates a fuperficies that is abfolutely quadrable, at leaft confidered in the *ungula*; in the fecond cafe, on the contrary, the values being reciprocal, the above-faid fuperficies will arife, which are only hypothetically quadrable. For example, the conoid formed from the firft cubical parabola being turned about AD, furnifhes us with the furface of an *ungula* which is algebräically quadrable, and alfo that of the round folid, provided we have a right line equal to the circumference. But if it be converted about the axis AB, then quadratures are required. The fame thing obtains in the fecond cubical parabola, and quite the contrary in that of *Apollonius*.

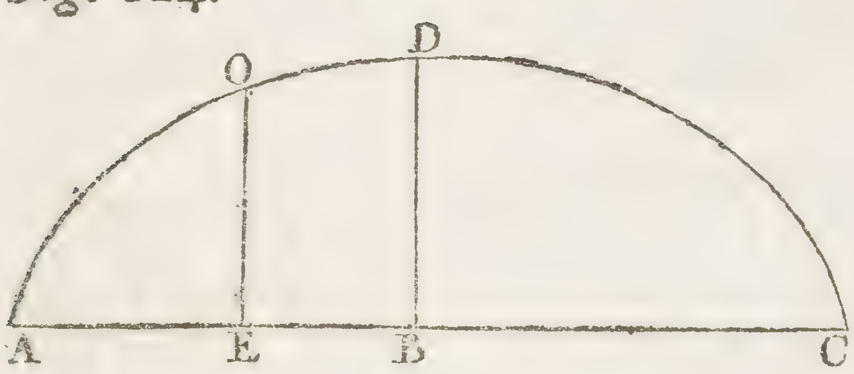
III. Comparing these series with those of § 119, we may discover, that among these there is no parabola of the first or second series, that is rectifiable either analytically, or by the means of known quadratures; on the contrary, those of the third and fourth are all rectifiable, and comprehend all that are contained in the progressions of § 119.

IV. Among the hyperbolas, the common one only between the asymptotes admits a superficies reducible to the quadrature of the said hyperbola; because no other negative exponent appears in the progressions, except — 1.

V. The exponents which are not found in the said series are these, $t = 4$, 5, 6, &c. $\frac{2}{3}$, $\frac{5}{8}$, &c. for which higher quadratures are required, to measure the conoidal surfaces thence arising.

EXAMPLE XLII.

Fig. 124.

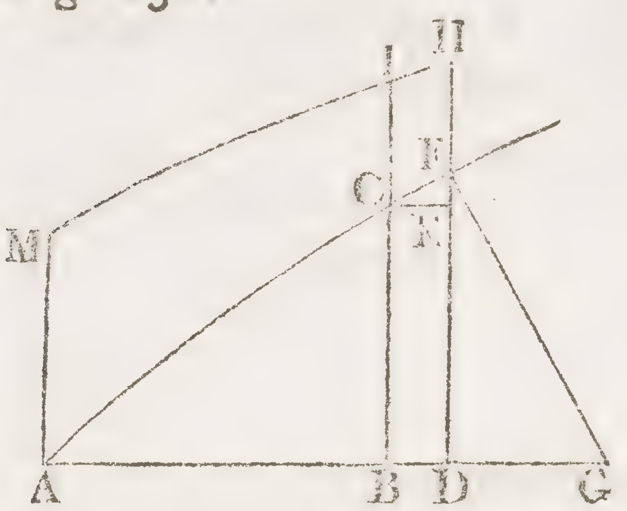


137. Let ADC be an ellipsis, which is converted about the axis AC, and make $AB = a$, $BD = b$, $AE = x$, $EO = y$; and the equation is $\frac{aayy}{bb} = 2ax - xx$. Therefore, by differencing, it will be $\dot{x} = \frac{aay\dot{y}}{bb \times \overline{a-x}}$, and therefore $\dot{x}\dot{x} = \frac{a^4yy\dot{y}\dot{y}}{b^4 \times \overline{a-x}|^2}$; and, instead of $-2ax + xx$, putting it's value $-\frac{aayy}{bb}$ given by the equation, it will be $\dot{x}\dot{x} = \frac{aayy\dot{y}\dot{y}}{bb \times \overline{bb-yy}}$. Then substituting this value in the general formula, we shall have $\frac{cy\dot{y}\sqrt{b^4 + aayy - bbyy}}{rb\sqrt{bb - yy}}$; and, for brevity-sake, making $aa - bb = ff$, supposing a to be greater than b, or that the axis about which the ellipsis circulates to be the greater axis (for, if a were less than b, we ought to make $aa - bb = -ff$), the formula will be $\frac{cy\dot{y}\sqrt{b^4 + ffyy}}{rb\sqrt{bb-yy}}$, which, for reasons already mentioned in their place, may be freed from radicals; and the integral of which, by means of the canon of § 56, we shall find to depend on the quadrature of the circle. But if a shall be less than b, or the axis about which the ellipsis turns be the lesser axis, the superficies of the spheroid will

 depend

depend on both the quadratures, that of the circle and that of the hyperbola. Wherefore the superficies of the *ungula*, in the first case, is equal to a portion of the elliptic space, which is easily determined by means of the perpendicular to the curve. But, in the second case, these perpendiculars will give us an hyperbolical space equal to the same superficies of the *ungula*. That this may be plainly seen, let ACF be the curve on which a cylindroid is supposed to be erected, which is to be cut by a plane which passes through the axis AB, and forms with the subjacent plane CAB half a right angle. It is evident that, making $\dot{u}$ the element of the curve, $\int y\dot{u}$ will be the superficies of the lower *ungula*, and $\int \frac{cy\dot{u}}{r}$ will be the superficies of the conoid, generated by the conversion of the figure CAB about the axis AB; and therefore the superficies of the *ungula* will be to that of the conoid, as radius to the circumference of the circle.

Fig. 136.



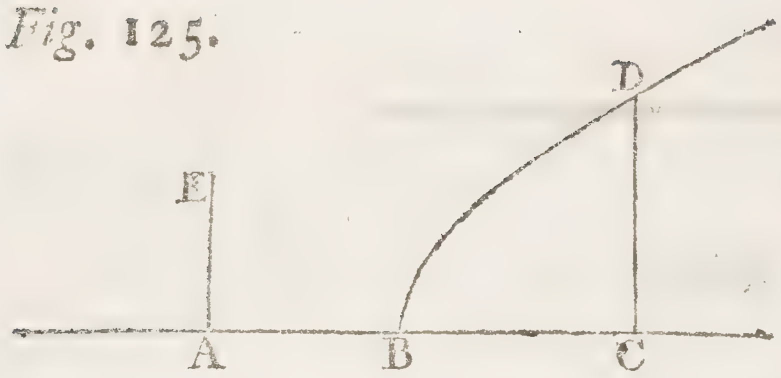
Now let the two ordinates BC, DF, be infinitely near, and drawing the perpendicular FG at the point F, let it be put in DH, and represent the ordinate of a new curve MIH drawn by the method prescribed. I say that the area MABI is equal to the superficies of the *ungula*, which has for it's base the arch AC.

The two triangles FCE, GFD, are similar; then it will be FC . CE :: GF . FD. Therefore FD $\times$ FC $=$ GF $\times$ CE $=$ DH $\times$ DB. But FD $\times$ FC ($y\dot{u}$) is the element of the superficies of the *ungula*, and HD $\times$ DB is the element of the area IMAB. Then, these elements being equal, their integrals will be equal also; that is, the aforesaid areas. This being premised, let the figure ACB be a fourth part of the ellipsis, the equation of which is $\frac{aayy}{bb} = 2ax - xx$. Then the perpendicular will be FG $=$ $\frac{b}{aa}\sqrt{2a^3x - aaxx + bbxx - 2abbx + aabb}$. Then, making the ordinate BI $= z$, it will be $z = \frac{b}{aa}\sqrt{xx - 2ax \times \overline{b^2 - a^2} + a^2b^2}$, an equation to the curve MIH, which will be another ellipsis when a is greater than b, or if AB be the greater axis of the ellipsis ACB; and on the contrary, an hyperbola, when a is less than b, that is, when AB is the lesser axis.

Lastly, in the middle case, or when the ellipsis degenerates into a circle, we know already, that the said surface of the *ungula* is quadrable, as being equal to a rectangle.

EXAMPLE XLIII.

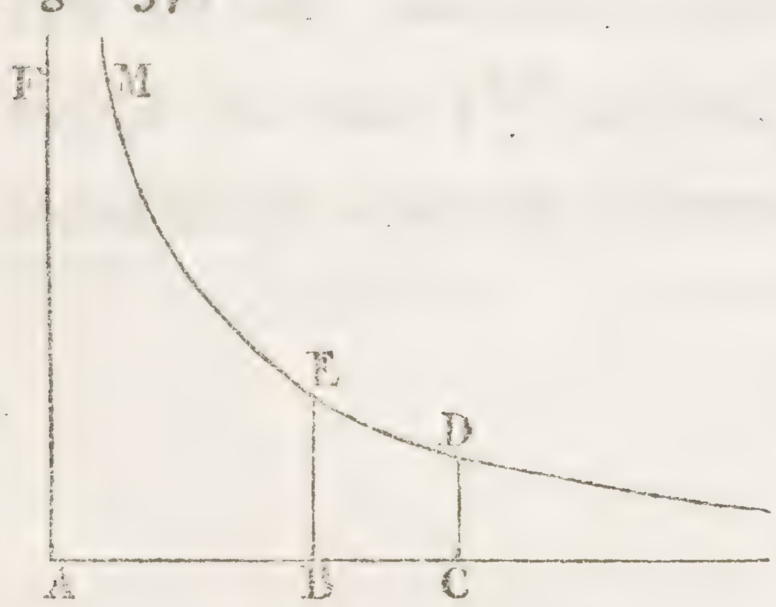
Fig. 125.



138. Let BD be an hyperbola, which circulates about the tranfverfe axis BA. Let A be it's centre, BA $= a$, the conjugate femi-axis AE $= b$, AC $= x$, CD $= y$. The equation will be $xx - aa = \frac{aayy}{bb}$, and therefore $y = \frac{b}{a}\sqrt{xx - aa}$, and $\dot{y} = \frac{bx\dot{x}}{a\sqrt{xx - aa}}$. Therefore the general formula, when the fubftitutions are made, will be $\frac{cb}{ar}\sqrt{xx - aa} \times \sqrt{\frac{a^2x^2\dot{x}^2 + b^2x^2\dot{x}^2 - a^4\dot{x}^2}{a^2 \times xx - aa}}$, that is, $\frac{cb\dot{x}}{aar}\sqrt{aaxx + bbxx - a^4}$; or, making $aa + bb = ff$, it will be $\frac{cbf\dot{x}}{aar}\sqrt{xx - \frac{a^4}{ff}}$, the integral of which, when it is freed from it's radical fign, we fhall find, in like manner, to depend on the quadrature of the hyperbola.

EXAMPLE XLIV.

Fig. 137.



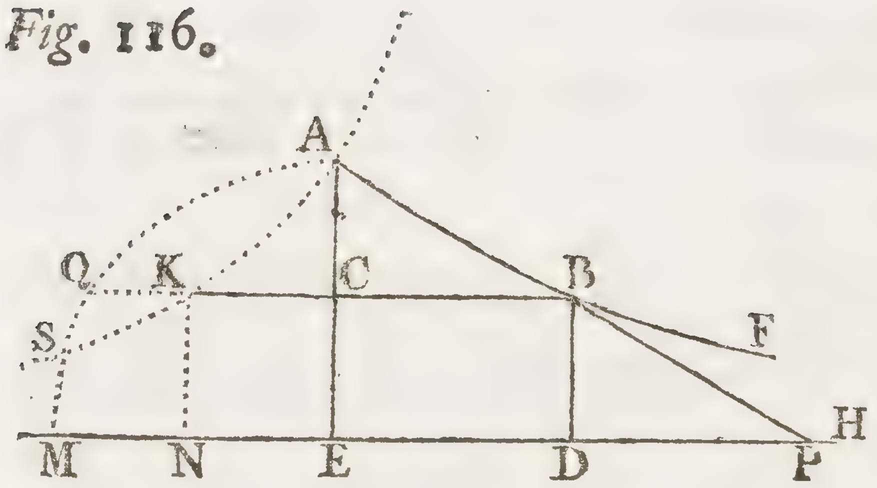
139. Let MD be an equilateral hyperbola, between it's afymptotes, and let it turn about the afymptote AC, of which the equation is $ay + xy = aa$; making AB $= a$, BC $= x$, and CD $= y$. Then it will be $x = \frac{aa}{y} - a$, and $\dot{x} = -\frac{aa\dot{y}}{yy}$, $\dot{x}\dot{x} = \frac{a^4\dot{y}\dot{y}}{y^4}$. Therefore, making the fubftitution, the general formula will be $\frac{c\dot{y}}{ry}\sqrt{y^4 + a^4}$. Put $\sqrt{y^4 + a^4} = z$, and therefore $y^4 = zz - a^4$, $\dot{y} = \frac{z\dot{z}}{2y^3}$. Make thefe fubftitutions, and we fhall have the formula

 transformed

transformed into this other, $\frac{cz z\dot{z}}{2r \times \overline{zz - a^4}}$, which is free from radical figns; the integral of which depends partly on the logarithms, as is eafy to perceive. Therefore the fuperficies required, defcribed by our hyperbola, will alfo depend on the quadrature of the hyperbola.

EXAMPLE XLV.

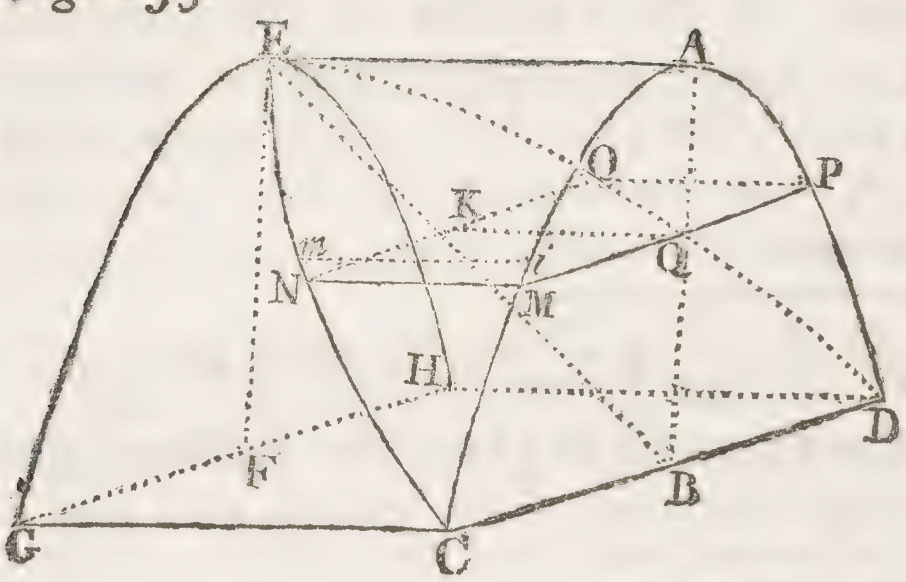
Fig. 116.



140. Let ABF be the folid generated by the *tractrix*, as in Example XXXVI, § 128, of which the fuperficies is required. In the general formula $\frac{cy\dot{u}}{r}$ (where $\dot{u}$ reprefents the element of the curve,) inftead of $\dot{u}$, fubftituting it's value $-\frac{a\dot{y}}{y}$ obtained from the equation of the curve, we fhall have $-\frac{ac\dot{y}}{r}$, and by integration, $-\frac{acy}{r} + n$. But when the fuperficies is nothing, we have $y = a$; therefore the conftant $n = \frac{aac}{r}$, and therefore the complete integral is $\frac{aac}{r} - \frac{acy}{r}$, equal to the furface of the folid generated by the figure AEDB. And making $y = 0$, then $\frac{aac}{r}$ will be equal to the furface of the folid infinitely produced. But the area of the circle, whofe radius is $\sqrt{2aa}$, was found to be $= \frac{caa}{r}$; then the furface of the folid, infinitely produced, is equal to the area of the circle, whofe radius is equal to the diagonal of the fquare of AE.

EXAMPLE XLVI.

Fig. 133.



141. Let CNEODAC be the *ungula* whose superficies is required. Imposing the same names as at § 129, it will be $QK = \frac{bx}{a} = MN$. But Mi, the element of the curve, is $\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}$, and therefore it will be $\frac{bx}{a}\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}$, equal to the infinitesimal *quadrilineum* $MimN$, the element of the superficies of half the *ungula*.

Let the curve DAC be a semicircle; in this case it will be $\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}} = \frac{a\dot{x}}{\sqrt{aa - xx}}$, and therefore the formula is $\frac{bx\dot{x}}{\sqrt{aa - xx}}$. And by integration (according to § 31), it will be $-b\sqrt{aa - xx} + f$. But, making $x = 0$, it will be $f = ab$; therefore the complete integral will be found to be $ab - b\sqrt{aa - xx}$. And making $x = a$, in respect of the whole superficies of the half *ungula*, that superficies will be $= ab$.

Let the curve DAC be the parabola of the equation $yy = a - x$; it will be $\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}} = \frac{1}{2}\dot{x}\sqrt{\frac{4a - 4x + 1}{a - x}}$, and therefore the formula is $\frac{bx\dot{x}}{2a}\sqrt{\frac{4a - 4x + 1}{a - x}}$, the integral of which depends on the quadrature of the hyperbola; so that the superficies of the *ungula* will depend on the same quadrature.

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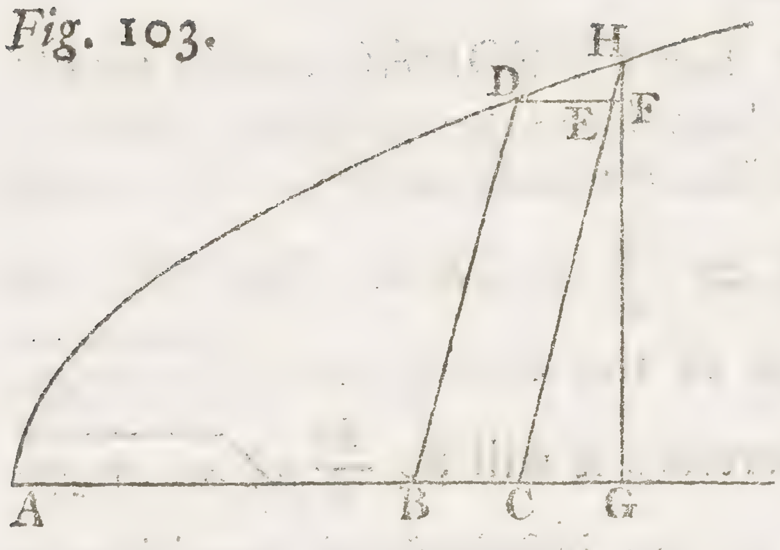
EXAMPLE XLVII.

Fig. 103.



142. Let there be a parabolical conoid, generated by the rotation of the parabola ADH with the co-ordinates at an oblique angle, about the axis AC, whose equation is $ax = yy$. In the formula for the superficies belonging to this case, that is, the formula $\frac{cny}{rm}\sqrt{\dot{y}\dot{y} + \dot{x}\dot{x} + \frac{2e\dot{x}\dot{y}}{m}}$, let there be substituted the value of $\dot{x}$ given by $\dot{y}$, from the differential equation of the curve, and it will be transformed into this other, $\frac{2cny\dot{y}}{arm}\sqrt{yy + \frac{aey}{m} + \frac{1}{4}aa}$; the integral of which will be found to be partly algebräical, and partly logarithmical.

143. In pursuance of the method already explained, for quadratures, rectifications, &c. this would be the proper place to give also formulas for centres of gravity, of oscillation and percussion; but I rather choose to omit them, as they necessarily require some knowledge of the principles of Staticks and Mechanicks, which I shall not suppose my young readers to understand at present.

SECT.

SECT. IV.

The Calculus of Logarithmic and Exponential Quantities.

144. EXPONENTIAL Quantities, (of which, as alſo of logarithmic quantities, we have treated elſewhere,) are thoſe which are raiſed to any indeterminate power. Such would be a^x, y^z, &c. the exponents of which, x, z, are indeterminate or variable quantities. And therefore the method of computation, which is converſant about ſuch quantities, is called *The Exponential Calculus.*

145. But exponential quantities are of ſeveral degrees. Thoſe are ſaid to be of the firſt degree, the exponents of which are the common indeterminates, as are the quantities a^x, y^z. Thoſe are of the ſecond degree, the exponents of which are the ſaid exponential quantities; ſuch would be a^{x^t}, y^{z^p}, where x is raiſed to the power t, and z to the power p. Thoſe are of the third degree, which have an exponential of the ſecond degree for their exponent. And ſo on.

146. Now here we ſhould call to mind what is already ſaid at § 11, that $\int \frac{a\dot{y}}{y} = ly$, in the logarithmic curve, the ſubtangent of which $= a$. Therefore the differential of ly will be $\frac{\dot{y}}{y}$ multiplied into the ſubtangent of the logarithmic, from which the logarithm is taken. Thus, the differential of $l\sqrt{aa - xx}$ will be $-\frac{x\dot{x}}{aa - xx}$ in the logarithmic, in which the ſubtangent $= 1$. And, in general, the differential of any logarithmic quantity whatever will be a formula, compounded of the differential of the quantity itſelf, multiplied into the ſubtangent, and divided by the ſame quantity.

147. This

147. This supposed, let it be required to difference the logarithmic quantity $l^m x$, where m is the exponent of the power of the logarithm. Make $l^m x = y^m$, then it will be $lx = y$, and $\frac{\dot{x}}{x} = \dot{y}$. But the differential of $l^m x$ will be $my^{m-1}\dot{y}$; and it is $y^{m-1} = l^{m-1}x$. So that, instead of y and $\dot{y}$, substituting their values given by x, the differential of $l^m x = ml^{m-1}x \times \frac{a\dot{x}}{x}$, supposing the subtangent of the logarithmic $= a$. Or otherwise, $= ml^{m-1}x \times \frac{\dot{x}}{x}$, supposing that subtangent $= 1$.

148. If we were to difference $l^m x^n$, making $x^n = z$, it will be $l^m z$; and the differential of this will be $ml^{m-1}z \times \frac{\dot{z}}{z}$. But $\dot{z} = nx^{n-1}\dot{x}$, and by substitution, the differential of the proposed formula $l^m x^n$ will be $nml^{m-1}x^n \times \frac{\dot{x}}{x}$.

149. Let it be proposed to difference the formula llx. Make $lx = y$, and therefore $llx = ly$. But it will be $\frac{\dot{x}}{x} = \dot{y}$, in the logarithmic whose subtangent $= 1$ (which is always to be understood, whenever these subtangents are not particularly expressed). But, because $llx = ly$, the differential of llx will be $\frac{\dot{y}}{y}$. Therefore, instead of y and $\dot{y}$, putting their values given by x, it will be $\frac{\dot{x}}{xlx}$ for the differential of the formula proposed.

But, more generally, let it be required to difference $l^m lx$. Put $lx = y$, and therefore $l^m lx = l^m y$, and $\frac{\dot{x}}{x} = \dot{y}$. But the differential of $l^m y$ is $ml^{m-1}y \times \frac{\dot{y}}{y}$; therefore, substituting the values of y and $\dot{y}$ given by x, the differential required will be $ml^{m-1}lx \times \frac{\dot{x}}{xlx}$.

Still more generally. Let it be required to difference $l^n l^m x$. Make $l^m x = y^m$, and therefore $lx = y$, and $\frac{\dot{x}}{x} = \dot{y}$. Then it will be $l^n l^m x = l^n y^m$.

But

But the differential of $l^n y^m$ is $mnl^{n-1}y^m \times \frac{\dot{y}}{y}$. So that, making the fubftitutions, $mnl^{n-1}l^m x \times \frac{\dot{x}}{xlx}$ will be the differential required.

150. Now for the method of differencing exponential quantities. Let the quantity to be differenced be z^x. Make $z^x = t$, and confequently it will be $lz^x = lt$. But, by § 14, it is $lz^x = xlz$, and therefore it will be $xlz = lt$, and therefore, by differencing, $\dot{x}lz + \frac{x\dot{z}}{z} = \frac{\dot{t}}{t}$. But $t = z^x$, whence $\dot{x}lz + \frac{x\dot{z}}{z} = \frac{\dot{t}}{z^x}$, and finally, $\dot{t} = z^x\dot{x}lz + xz^{x-1}\dot{z}$, which is the differential required.

151. Let it be required to difference the exponential quantity of the fecond degree, z^{x^p}. Make $z^{x^p} = t$, and therefore it will be $x^p lz = lt$. And, by differencing, the differential of $x^p \times lz + x^p \times \frac{\dot{z}}{z}$ will be $= \frac{\dot{t}}{t}$. But, by the foregoing article, we know the differential of x^p to be $x^p \dot{p}lx + px^{p-1}\dot{x}$; and therefore it will be $\overline{x^p \dot{p}lx + px^{p-1}\dot{x}} \times lz + \frac{x^p\dot{z}}{z} = \frac{\dot{t}}{t}$. But $t = z^{x^p}$. Therefore it will be $\dot{t} = z^{x^p}x^p \dot{p}lxlz + z^{x^p}px^{p-1}\dot{x}lz + z^{x^p}z^{-1}x^p\dot{z}$ for the differential required.

In the fame manner, we may proceed to exponential quantities of any other degrees.

152. Likewife, in the fame manner, we may have the differentials of quantities, which are the products of exponential quantities; as, for example, of $x^p y^u$. For the differential of this will be the product of x^p into the differential y^u, together with the product of y^u into the differential of x^p. But it has been fhown how to find the differentials of x^p and y^u. Therefore, &c.

153. From the order in which logarithmic differentials proceed, we may derive rules for the integration of logarithmic differential formulas. And, firft, thofe canons which ferve for the integration of common differential quantities, will alfo ferve for logarithmical differentials which are like to them; becaufe

these are divided also by the variable, and the integrals of these will be the same as the integrals of those, putting only in these, instead of the variable or it's power, the logarithm or power of the logarithm of the same variable; dividing the whole by the subtangent of the logarithmic.

Thus, because the integral of $m x^{m-1} \dot{x}$ is x^m, also the integral of $m l^{m-1} x \times \frac{\dot{x}}{x}$ will be $\frac{l^m x}{a}$.

In the same manner, because $\int x^{-1} \dot{x} = lx$; so likewise $\int l^{-1} x \times \frac{\dot{x}}{x}$, or $\int \frac{\dot{x}}{xlx}$ will be llx; supposing the subtangent $= 1$.

And, because $\int y\dot{y}\sqrt{aa+yy} = \frac{1}{3} \times \overline{aa+yy}|^{\frac{3}{2}}$; it will be also $\int ly\sqrt{aa+l^2y} \times \frac{\dot{y}}{y} = \frac{1}{3} \times \overline{aa+l^2y}|^{\frac{3}{2}}$.

Let $m l^{m-1} lx \times \frac{\dot{x}}{xlx}$ be given to be integrated. Make $lx = y$; then $\frac{\dot{x}}{x} = \dot{y}$. And making the substitution, it will be $m l^{m-1} y \times \frac{\dot{y}}{y}$. But we know the integral of $m y^{m-1} \dot{y}$ to be y^m, and therefore the integral of $m l^{m-1} y \times \frac{\dot{y}}{y}$ will be $l^m y$. But $y = lx$, and therefore $ly = llx$, and $l^m y = l^m lx$. Therefore $\int m l^{m-1} lx \times \frac{\dot{x}}{xlx} = l^m lx$.

Let it be $n m l^{n-1} x^m \times \frac{\dot{x}}{x}$. Make $x^m = y$, and therefore $\dot{x} = \frac{\dot{y}}{m x^{m-1}}$. And making the substitutions, it will be $n m l^{n-1} y \times \frac{\dot{y}}{m x^{m-1} \times x}$, that is, $n l^{n-1} y \times \frac{\dot{y}}{y}$, the integral of which is $l^n y$. Then restoring the value of y, it will be $\int n m l^{n-1} x^m \times \frac{\dot{x}}{x} = l^n x^m$.

Let it be $n m l^{n-1} l^m x \times \frac{\dot{x}}{xlx}$. Make $lx = y$; then $\frac{\dot{x}}{x} = \dot{y}$, and $l^m x = y^m$. Making the substitution, it will be $n m l^{n-1} y^m \times \frac{\dot{y}}{y}$. But the integral of this is $l^n y^m$. Therefore, restoring the value, it will be $\int n m l^{n-1} l^m x \times \frac{\dot{x}}{x} = l^n l^m x$.

154. To

154. To this I ſhall add a general rule for the integration of the formula $y^m l^n y \times \dot{y}$, and ſay, in general, it will be $\int y^m l^n y \times \dot{y} = \frac{y^{m+1} l^n y}{m+1} - \frac{ny^{m+1} a l^{n-1} y}{\overline{m+1}|^2}$ $+ \frac{n \times \overline{n-1} \times y^{m+1} a^2 l^{n-2} y}{\overline{m+1}|^3} - \frac{n \times \overline{n-1} \times \overline{n-2} \times y^{m+1} a^3 l^{n-3} y}{\overline{m+1}|^4}$, &c. And thus the ſeries may be continued *in infinitum*, by obſerving the law of it's progreſſion, which is manifeſt of itſelf.

Hence, if the exponent n ſhall be a poſitive integer number, it is eaſy to obſerve, that the ſeries will break off of itſelf, and conſequently the integral of the propoſed formula will be given in a finite number of terms.

For example, make $n = 2$; then it will be $n - 2 = 0$, and therefore the co-efficient of the fourth term will be nothing, and of all that follow, becauſe every one is multiplied by $n - 2$. So, if $n = 3$, the ſeries will break off at the fifth term; and ſo of others.

Make $n = 2$, $m = 1$; then the formula to be integrated will be $yl^2y \times \dot{y}$. Therefore the fourth term, and all the ſubſequent terms, will be nothing. Therefore the integral will be $\frac{yyl^2y}{2} - \frac{2yyaly}{4} + \frac{2yyaa}{8}$.

Now, if it were $m = -1$, the ſeries would be of no uſe, becauſe it would be $m + 1 = 0$, which makes every term infinite. But, in this caſe, there would be no need of a ſeries, becauſe we know already how to integrate ſuch formulas, by what has been ſaid before.

It remains to give the demonſtration of this rule. To do which, make $ly = z$, and therefore $\frac{a\dot{y}}{y} = \dot{z}$. Then making the ſubſtitution, it will be $y^m l^n y\dot{y} = y^m z^n \dot{y}$. But $y^m z^n \dot{y} = y^m z^n \dot{y} + \frac{n}{m+1} y^{m+1} z^{n-1} \dot{z} - \frac{n}{m+1} y^m z^{n-1} a\dot{y}$ $- \frac{n \times \overline{n-1}}{\overline{m+1}|^2} y^{m+1} z^{n-2} a\dot{z} + \frac{n \times \overline{n-1}}{\overline{m+1}|^2} y^m z^{n-2} a^2\dot{y}$, &c. And ſo on *in infinitum*; becauſe, in this manner, every term, except the firſt, will be deſtroyed by that immediately following, becauſe it is $\dot{z} = \frac{a\dot{y}}{y}$. Now, becauſe ſuch an infinite ſeries is integrable, by taking the terms two by two; for the integral of the firſt and ſecond term is $\frac{y^{m+1} z^n}{m+1}$, of the third and fourth is $- \frac{any^{m+1} z^{n-1}}{\overline{m+1}|^2}$, of the fifth and ſixth is $\frac{aan \times \overline{n-1} \times y^{m+1} z^{n-2}}{\overline{m+1}|^3}$; and ſo of the reſt: in this

integral,

integral, instead of z, restoring it's value ly, we shall find it to be at last $\int y^m l^n y\dot{y} = \frac{y^{m+1} l^n y}{m+1} - \frac{any^{m+1} l^{n-1} y}{\overline{m+1}|^2}$, &c. as before.

155. The artifice of finding the aforesaid series is this. We may conceive the integral of $y^m l^n y\dot{y}$ to be $\frac{y^{m+1} l^n y}{m+1}$, as it really would be, if $l^n y$ were not a variable quantity; but, supposing the subtangent $= a$, the differential of this integral is $y^m l^n y\dot{y} + \frac{ny^m al^{n-1} y\dot{y}}{m+1}$. This is found greater than the proposed formula by $\frac{ny^m al^{n-1} y\dot{y}}{m+1}$, so that the integral assumed is greater than it ought to be, by the integral of $\frac{nyal^{n-1} y\dot{y}}{m+1}$, and therefore the integral of this ought to be subtracted from the supposed integral.

And here again I conceive that the integral of $\frac{ny^m al^{n-1} y\dot{y}}{m+1}$ is $\frac{ny^{m+1} al^{n-1} y}{\overline{m+1}|^2}$; whence the integral of the proposed formula will be $\frac{y^{m+1} l^n y}{m+1} - \frac{ny^{m+1} al^{n-1} y}{\overline{m+1}|^2}$. But, by differencing $\frac{ny^{m+1} al^{n-1} y}{\overline{m+1}|^2}$, we shall have $\frac{ny^m al^{n-1} y\dot{y}}{m+1} + \frac{n \times \overline{n-1} \times y^m a^2 l^{n-2} y\dot{y}}{\overline{m+1}|^2}$. Therefore the integral of $\frac{ny^m al^{n-1} y\dot{y}}{m+1}$ is not $\frac{ny^m al^{n-1} y}{\overline{m+1}|^2}$, but is greater than it ought to be by the integral of $\frac{n \times \overline{n-1}}{\overline{m+1}|^2} \times y^m a^2 l^{n-2} y\dot{y}$. Therefore too much is subtracted, and this integral is to be added, which again I imagine to be $\frac{n \times \overline{n-1} \times y^{m+1} a^2 l^{n-2} y}{\overline{m+1}|^3}$. So that the integral of the proposed formula will be $\frac{1}{m+1} y^{m+1} l^n y - \frac{n}{\overline{m+1}|^2} y^{m+1} al^{n-1} y + \frac{n \times \overline{n-1}}{\overline{m+1}|^3} y^{m+1} a^2 l^{n-2} y$, &c. And thus proceeding in the same manner, the series may be continued *in infinitum*.

156. We

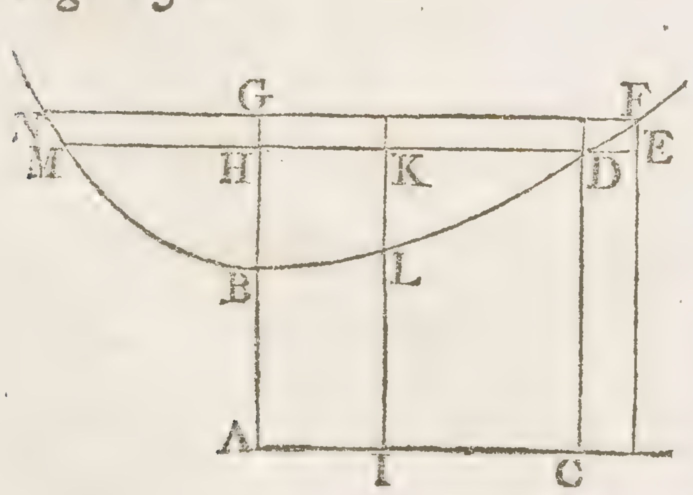
156. We may also have the integrals of logarithmic differential formulæ by the help of series, which shall not contain logarithmic quantities, but only common quantities; which series, therefore, will never break off, but are always infinite.

Let $xlx \times \dot{x}$ be proposed to be integrated. Make $x = z + a$; then, by substitution, it will be $\overline{z + a} \times l\overline{z + a} \times \dot{z}$. But, by § 70, it is $l\overline{z + a} = \frac{z}{a} - \frac{z^2}{2a^2} + \frac{z^3}{3a^3} - \frac{z^4}{4a^4}$, &c. Supposing the subtangent $= 1$. Then, by actually multiplying, we shall have $\overline{z + a} \times l\overline{z + a} \times \dot{z} = z\dot{z} + \frac{z^2\dot{z}}{a} - \frac{z^3\dot{z}}{2a^2} + \frac{z^4\dot{z}}{3a^3} - \frac{z^5\dot{z}}{4a^4}$, &c. $- \frac{z^2\dot{z}}{2a} + \frac{z^3\dot{z}}{3a^2} - \frac{z^4\dot{z}}{4a^3} + \frac{z^5\dot{z}}{5a^4}$, &c.; that is, $z\dot{z} + \frac{z^2\dot{z}}{2a} - \frac{z^3\dot{z}}{6a^2} + \frac{z^4\dot{z}}{12a^3} - \frac{z^5\dot{z}}{20a^4}$, &c.; and, by integration, it will be $\frac{z^2}{2} + \frac{z^3}{6a} - \frac{z^4}{24a^2} + \frac{z^5}{60a^3} - \frac{z^6}{120a^4}$, &c. $= \int \overline{z + a} \times l\overline{z + a} \times \dot{z}$.

So, if the formula were $x^m lx \times \dot{x}$, that is, $\overline{z + a}|^m \times l\overline{z + a} \times \dot{z}$, we must multiply the series expressing the logarithm into the power $\overline{z + a}|^m$. And moreover, if the logarithm also were raised to a power, as $x^m l^n x \times \dot{x}$, that is, $\overline{z + a}|^m \times l^n \overline{z + a} \times \dot{z}$, there would be occasion, besides, to raise the infinite series, expressing the logarithm, to the power n, and to do the rest, as above.

157. Differential formulas, or equations affected by logarithmic quantities, very often admit of integrations which are geometrical, and which depend on quadratures of curvilinear spaces, which may easily be described, supposing the logarithmic curve to be given. Here are some examples selected out of the more simple ones.

Fig. 138.



Let the equation be $\dot{y}ly = \dot{x}$, and in the logarithmic described let CD $= y$; and taking the subtangent for unity, we shall have AC $=$ HD $= ly$. Whence the infinitesimal rectangle DG, of which the base is GH $=$ FE $= \dot{y}$, will be $= \dot{y}ly$. But this rectangle is the element of the increasing area BDH, and therefore the sum or integral $\int \dot{y}ly$ is equal to the said area. In fact, the area itself is equal to the rectangle AD, subtracting the logarithmic space ABDC. But this

this ſpace, as is known, is meaſured by the rectangle AB × CD $= y$. Therefore the area BDH $= \int \dot{y}ly = yly - y$; as may be found by the way of analyſis.

I ſhall conſider another formula, $\dot{y}l^2y = \dot{x}$. The firſt member is no other than the ſolid generated by the fluxion HG, multiplied into the ſquare of the ordinate GF; which ſolid is analogous to the element of the conoid, generated by the area BDH, revolving about the axis BG. Therefore the integral $\int \dot{y}l^2y = yl^2y - 2yly + 2y$ is to the ſaid conoid in a given ratio.

More generally, let us have $\dot{y}l^my$. Raiſing the ordinate HD to the power m, (the index m being either an affirmative or negative number, either whole or broken, it will ſuffice that the ordinate HM may be made equal to the dignity HD^m, and that through the point M, and infinite others to be determined in the ſame manner, the curve BMN may paſs; in order that the area BMH $= \int \mathrm{MH} \times \dot{y}$ may be equal to, or analogous to, the integral $\int \dot{y}l^my$.

The difficulty will not be greater, even though the logarithms of logarithms ſhould alſo be found in our expreſſions. Let there be propoſed $\dot{y}lly = \dot{x}$. Whereas AC is the logarithm of CD; if, in the logiſtic, the new ordinate IL, equal to the abſciſs AC, ſhould be adapted; AI will be the logarithm of IL, and conſequently the logarithm of the logarithm of CD. Let the right line IL be prolonged, ſo as to cut HD, parallel and equal to AC, in the point K; through which and infinite others, determined in the ſame manner, let a new curve paſs, drawn relatively to the logiſtic. I ſay, that the quadrature of the ſpace belonging to this curve will give us the integral of the formula $\dot{y}lly = \dot{x}$.

After another manner. I take the fluxion of the quantity $ylly$, that is, $\dot{y}lly + \frac{\dot{y}}{ly}$, and adding the term $\frac{\dot{y}}{ly}$ to both ſides of our expreſſion, we ſhall have $\dot{y}lly + \frac{\dot{y}}{ly} = \dot{x} + \frac{\dot{y}}{ly}$; and by integration, $ylly = x + \int \frac{\dot{y}}{ly}$. Therefore, to the abſciſs AH annexing the correſponding ordinate in the reciprocal ratio of HD $= ly$, a curve will be produced, the quadrature of which will expreſs the integral $\int \frac{\dot{y}}{ly}$. And this will be enough to ſhow how the method proceeds.

158. I ſhall now go on to the integration of differential formulæ, which contain exponential quantities; and let us integrate $x^x\dot{x}$. Put $x = 1 + y$, (taking unity for any conſtant quantity,) then it will be $x^x\dot{x} = \overline{1+y}|^{1+y}\dot{y}$.

 This

This ſuppoſed, make alſo $\overline{1+y}|^{1+y} = 1 + u$, and then it will be $\overline{1+y} \times l\,\overline{1+y} = l\,\overline{1+u}$. Now let the two logarithms be converted into ſeries, by § 70; and making an actual multiplication of the firſt ſeries by $1 + y$, we ſhall have $y + \frac{1}{2}y^2 - \frac{1}{6}y^3 + \frac{1}{12}y^4 - \frac{1}{20}y^5$, &c. $= u - \frac{1}{2}u^2 + \frac{1}{3}u^3 - \frac{1}{4}u^4 + \frac{1}{5}u^5$, &c. Then make a fictitious equation, ſuppoſing it to be $u = y + Ay^2 + By^3 + Cy^4 + Dy^5$, &c. (where A, B, C, D, &c. are quantities to be determined by the proceſs.)

$$\text{Therefore } uu = y^2 + 2Ay^3 \begin{array}{l} + A^2y^4 + 2ABy^5, \text{ \&c.} \\ + 2By^4 + 2Cy^5 \end{array}$$

$$u^3 = y^3 + 3Ay^4 \begin{array}{l} + 3A^2y^5, \text{ \&c.} \\ + 3By^5 \end{array} \qquad u^4 = y^4 + 4Ay^5, \text{ \&c.} \qquad u^5 = y^5, \text{ \&c.}$$

Whence $u - \frac{1}{2}u^2 + \frac{1}{3}u^3 - \frac{1}{4}u^4 + \frac{1}{5}u^5$, &c. $=$

$$\left.\begin{array}{rrrrl} y & + Ay^2 & + By^3 & + Cy^4 & + Dy^5, \text{ \&c.} \\ & -\frac{1}{2}y^2 & - Ay^3 & -\frac{1}{2}A^2y^4 & - ABy^5 \\ & & & - By^4 & - Cy^5 \\ & & +\frac{1}{3}y^3 & + Ay^4 & + A^2y^5 \\ & & & & + By^5 \\ & & & -\frac{1}{4}y^4 & - Ay^5 \\ & & & & +\frac{1}{5}y^5 \end{array}\right\} = y + \frac{1}{2}y^2 - \frac{1}{6}y^3 + \frac{1}{12}y^4 - \frac{1}{20}y^5, \text{ \&c.}$$

Now, by comparing homologous terms, we ſhall find the values of the aſſumed quantities to be $A = 1$, $B = \frac{1}{2}$, $C = \frac{1}{3}$, $D = \frac{1}{12}$, &c.; ſo that, putting theſe values in the places of the capitals, we ſhall have $1 + u = \overline{1+y}|^{1+y} = 1 + y + y^2 + \frac{1}{2}y^3 + \frac{1}{3}y^4 + \frac{1}{12}y^5$, &c. Whence $\overline{1+y}|^{1+y}\dot{y} = \dot{y} + y\dot{y} + y^2\dot{y} + \frac{1}{2}y^3\dot{y} + \frac{1}{3}y^4\dot{y}$, &c.; and laſtly, by integration, $\int \overline{1+y}|^{1+y} \times \dot{y} = y + \frac{1}{2}y^2 + \frac{1}{3}y^3 + \frac{1}{8}y^4 + \frac{1}{15}y^5 + \frac{1}{72}y^6$, &c.

159. We may find the integral of the formula $x^x\dot{x}$ thus, in another manner. Make $x^x = 1 + y$, then $xlx = l\,\overline{1+y}$. Reduce $l\,\overline{1+y}$ to a ſeries, and it will be $l\,\overline{1+y} = y - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \frac{1}{4}y^4 + \frac{1}{5}y^5$, &c. This ſuppoſed, make $y = l\,\overline{1+y} + Al^2\,\overline{1+y} + Bl^3\,\overline{1+y} + Cl^4\,\overline{1+y} + Dl^5\,\overline{1+y}$, &c. (where A, B, C, D, &c. are quantities to be determined,) and it will be

$y^2 =$

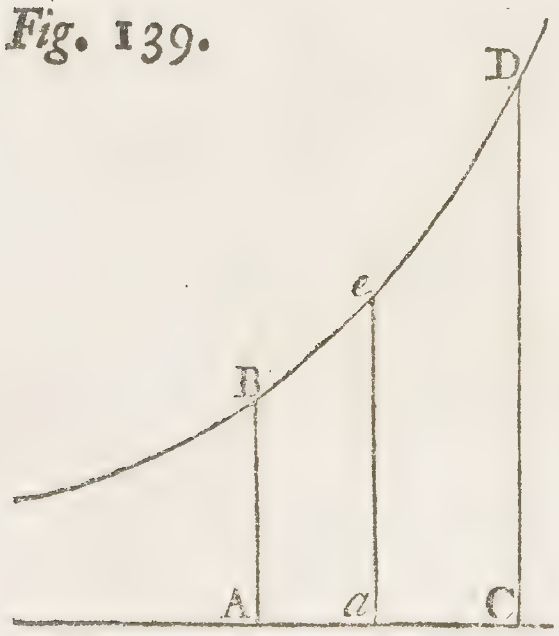
$$\begin{array}{llllll}
y^2 = & l^2\overline{1+y} & + 2Al^3\overline{1+y} & + A^2l^4\overline{1+y} & + 2ABl^5\overline{1+y}, \&c. \\
 & & & + 2Bl^4\overline{1+y} & + 2Cl^5\overline{1+y} \\
y^3 = & & l^3\overline{1+y} & + 3Al^4\overline{1+y} & + 3A^2l^5\overline{1+y} \\
 & & & & + 3Bl^5\overline{1+y} \\
y^4 = & & & l^4\overline{1+y} & + 4Al^5\overline{1+y} \\
y^5 = & & & & l^5\overline{1+y}
\end{array}$$

Therefore $y - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \frac{1}{4}y^4 + \frac{1}{5}y^5$, &c. $= l\overline{1+y} =$

$$\begin{array}{llllll}
l\overline{1+y} & + Al^2\overline{1+y} & + Bl^3\overline{1+y} & + Cl^4\overline{1+y} & + Dl^5\overline{1+y}, \&c. \\
 & -\frac{1}{2}l^2\overline{1+y} & - Al^3\overline{1+y} & -\frac{1}{2}A^2l^4\overline{1+y} & - ABl^5\overline{1+y} \\
 & & & - Bl^4\overline{1+y} & - Cl^5\overline{1+y} \\
 & & +\frac{1}{3}l^3\overline{1+y} & + Al^4\overline{1+y} & + A^2l^5\overline{1+y} \\
 & & & & + Bl^5\overline{1+y} \\
 & & & -\frac{1}{4}l^4\overline{1+y} & - Al^5\overline{1+y} \\
 & & & & +\frac{1}{5}l^5\overline{1+y}
\end{array}$$

Now, by the comparifon of homologous terms, we fhall find $A = \frac{1}{2}$, $B = \frac{1}{6}$, $C = \frac{1}{24}$, $D = \frac{1}{120}$, &c.; whence $1 + y = 1 + l\overline{1+y} + \frac{1}{2}l^2\overline{1+y} + \frac{1}{6}l^3\overline{1+y} + \frac{1}{24}l^4\overline{1+y} + \frac{1}{120}l^5\overline{1+y}$, &c. But $l\overline{1+y} = xlx$, and $1 + y = x^x$; therefore, making the fubftitutions, and multiplying by $\dot{x}$, it will be $x^x\dot{x} = \dot{x} + x\dot{x}lx + \frac{1}{2}x^2\dot{x}l^2x + \frac{1}{6}x^3\dot{x}l^3x + \frac{1}{24}x^4\dot{x}l^4x + \frac{1}{120}x^5\dot{x}l^5x$, &c.; and integrating, by the known rules above delivered, it will be $\int x^x\dot{x} = x + \frac{1}{2}x^2lx - \frac{1}{4}x^2 + \frac{1}{6}x^3l^2x - \frac{1}{9}x^3lx + \frac{1}{27}x^3 + \frac{1}{24}x^4l^3x - \frac{1}{32}x^4l^2x + \frac{1}{64}x^4lx - \frac{1}{256}x^4$, &c.

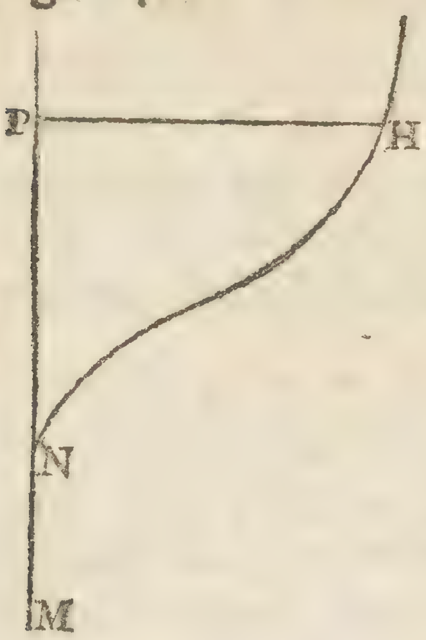
Fig. 139.



160. Now, to add fomething concerning the conftruction of curves expreffed by logarithmic and exponential equations. Firft, let it be required to defcribe the curve of the equation $x = \frac{l^{\frac{3}{2}}y}{a^{\frac{1}{2}}}$. Let BD (Fig. 139.) be the logarithmic, in which we are to take the logarithms of the propofed equation, whofe fubtangent (for example) is $= a =$ AB. This

fuppofed,

Fig. 140.



ſuppoſed, taking $y = a = AB$, the logarithm of y will be $= 0$, and therefore $x = 0$. Making, then, $MN = y = a$ (Fig. 140), N will be a point in the curve. Taking y leſs than AB, ly will be a negative quantity, and therefore $l^{\frac{3}{2}}y$ will be an imaginary quantity, becauſe the even number 2 is the index of the root of a negative quantity; whence x will be imaginary whenever y is leſs than a. Taking y greater than AB, ſuppoſe $=$ CD, it will be $AC = ly$. But, by the given equation, it is $a^{\frac{1}{2}} . l^{\frac{1}{2}}y :: ly . x$, or $a . \sqrt{aly} :: ly . x$; and therefore, making $MP = CD$, we muſt take PH equal to the fourth proportional of AB, a mean proportional between AB and AC, and the ſaid AC; which fourth proportional will be $= x$, and H will be a point in the curve. After this manner we may find as many points as we pleaſe, and ſo deſcribe the curve, which will go on *ad infinitum*, as is eaſy to perceive.

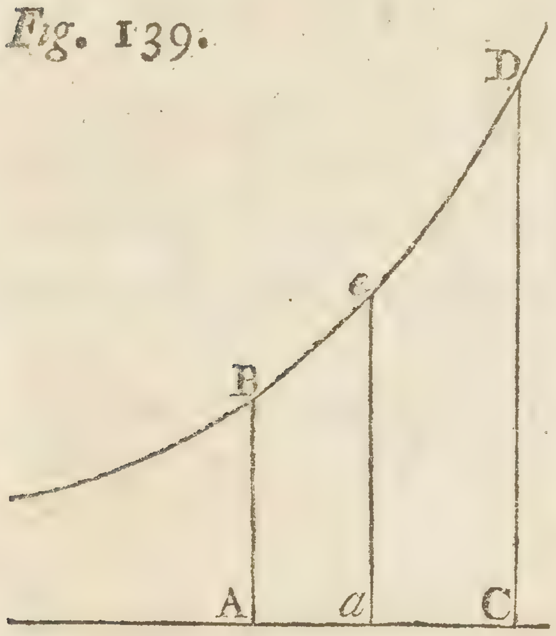
To have the ſubtangent of the given curve, I take the differential formula $\frac{y\dot{x}}{\dot{y}}$ of the ſubtangent, find the difference of the equation of the curve, which is $\dot{x} = \frac{3}{2}l^{\frac{1}{2}}y \times \frac{a^{\frac{1}{2}}\dot{y}}{y}$. Making the ſubſtitution in the place of $\dot{x}$, we ſhall have the ſubtangent $= \frac{3}{2}l^{\frac{1}{2}}y \times a^{\frac{1}{2}} = \frac{3ax}{2ly} = \frac{3}{2}a^{\frac{2}{3}}x^{\frac{1}{3}}$.

Alſo, our curve will have a contrary flexure; to find which I take the ſecond fluxion of the given equation, ſuppoſing $\dot{x}$ conſtant, and I find $\frac{\frac{3}{2}a^{\frac{1}{2}}y\, l^{\frac{1}{2}}y \times \ddot{y} + \frac{3}{4}a^{\frac{3}{2}}\dot{y}\dot{y}l^{-\frac{1}{2}}y - \frac{3}{2}a^{\frac{1}{2}}\dot{y}\dot{y}l^{\frac{1}{2}}y}{yy} = 0$; and therefore $\ddot{y} = \frac{\frac{3}{2}a^{\frac{1}{2}}\dot{y}\dot{y}\,l^{\frac{1}{2}}y - \frac{3}{4}a^{\frac{3}{2}}\dot{y}\dot{y}l^{-\frac{1}{2}}y}{\frac{3}{2}a^{\frac{1}{2}}y\,l^{\frac{1}{2}}y}$. But, by the method of contrary flexures, it ought to be $\ddot{y} = 0$. Therefore it will be $\frac{3}{2}a^{\frac{1}{2}}\dot{y}\dot{y}l^{\frac{1}{2}}y - \frac{3}{4}a^{\frac{3}{2}}\dot{y}\dot{y}l^{-\frac{1}{2}}y = 0$; that is, $l^{\frac{1}{2}}y - \frac{1}{2}al^{-\frac{1}{2}}y = 0$, or $ly = \frac{1}{2}a$. Therefore the point of contrary flexure will be there, where it is $ly = \frac{1}{2}a$.

If the curve propoſed to be deſcribed were $xlx = y$, reſolving the equation into an analogy, it will be $1 . lx :: x . y$, which may be conſtructed in a like manner.

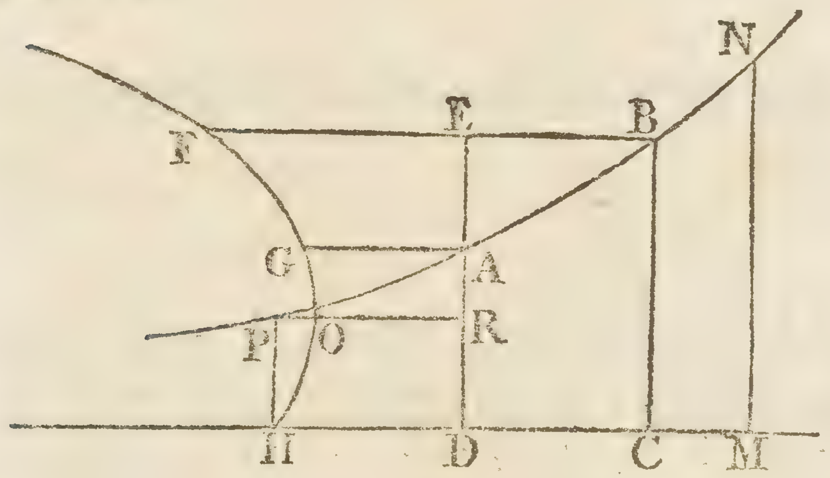
If the curve were $x^2 lx = y$, or $x^3 lx = y$, or $x^{\frac{1}{3}} lx = y$, or, more generally, $x^n lx = y$, ſuppoſing n to denote any power of x, whether integer or fraction; this equation being likewiſe reſolved into an analogy, $1 . lx :: x^n . y$, and taking in the logarithmic any line $CD = x$, whence $AC = lx$; the multiple of AC, according to the number n, if it be an integer, the ſubmultiple, if a fraction, will give the correſponding ordinate in the logarithmic itſelf, which ſhall be x^n, by the property of the logarithmic.

Fig. 139.



If the curve ſhould contain quantities that are logarithms of logarithms, ſuch as $x^n llx = y$, we ſhould eaſily have in the logarithmic the line expreſſed by llx, by taking any line $CD = x$ (Fig. 139.), whence it is $AC = lx$; and then putting AC for an ordinate in (ae). For Aa would be the logarithm of (ae), that is, llx; as has before been taken notice of at § 157.

Fig. 141.



161. Let it be required to conſtruct the exponential curve of the equation $x^x = y$. Now, taking the logarithms, it will be $xlx = ly$; and deſcribing the logarithmic curve PAB, (Fig. 141.) with the ſubtangent $AD = 1$, and taking any line $CB = DE = x$, it will be $DC = lx$. Then, becauſe the equation may be reſolved into the analogy, $1 . x :: lx . ly$; the fourth proportional to AD, BC, and DC, which ſuppoſe is DM, will be ly; ſo that $MN = y$. Therefore, if it be made $EF = MN$, it will be $DE = x$, $EF = y$, and F will be a point in the curve to be deſcribed.

The curve will cut the aſymptote HM in H, making $DH = DA$. For, putting $x = 0$, it will be $ly = 0$, that is, $y = DA$. Making, therefore, $AG = DH$, G will be a point in the curve.

From the point H drawing HP, an ordinate to the logarithmic, and drawing POR parallel to HD, then OR will be the leaſt ordinate, y, to the curve. For, taking the difference of the equation, it will be $\dot{x} + \dot{x}lx = \frac{\dot{y}}{y}$, that is, $y\dot{x} + y\dot{x}lx = \dot{y}$. But, by the method *de maximis et minimis*, it muſt be $\dot{y} = 0$; therefore $y\dot{x} + y\dot{x}lx = 0$, and therefore $-lx = 1 = HD = DA$.

8 Becauſe

Becauſe $\frac{y\dot{x}}{\dot{y}}$ is the general formula for the ſubtangent, and having $\dot{x} = \frac{\dot{y}}{y \times \overline{1 + lx}}$ from the given equation of the curve, by ſubſtituting this value in the formula, the ſubtangent belonging to any point of the curve will be $= \frac{1}{1 + lx}$; and for the point G, in reſpect of which it is $x = AD$, and conſequently $lx = 0$, the ſubtangent will be $= 1 = AD$, which is the ſubtangent of the logarithmic.

As to the area, take the general formula $y\dot{x}$; but $y = x^x$, in the equation of the curve. Therefore, ſubſtituting the value of y in the formula, it will become $x^x\dot{x}$, and therefore $\int x^x\dot{x}$ is the indefinite area HOFEADH; which, being integrated according to § 159, will be $= x + \frac{x^2lx}{2} - \frac{1}{4}x^2 + \frac{x^3l^2x}{6} - \frac{x^3lx}{9} + \frac{x^3}{27} + \frac{x^4l^3x}{24} - \frac{x^4l^2x}{32} + \frac{x^4lx}{64}$, &c. And taking $x = AD = 1$, it will be $lx = 0$, and therefore the area HOGAD $= 1 - \frac{1}{4} + \frac{1}{27} - \frac{1}{256}$, &c.; that is, $= 1 - \frac{1}{2^2} + \frac{1}{3^3} - \frac{1}{4^4} + \frac{1}{5^5}$, &c.

162. Let $x^y = a$ be the equation of the curve. Then $ylx = la$, and therefore it may be conſtructed by means of the logarithmic. By taking the fluxion of the equation, we ſhall have $\frac{y\dot{x}}{x} + \dot{y}lx = 0$, making the ſubtangent of the logarithmic $= 1$. And therefore it will be $\dot{x} = -\frac{x\dot{y}lx}{y}$; and therefore the ſubtangent $= -xlx$.

163. Let it be $x^x = a^y$; therefore $xlx = yla$, which may be conſtructed as uſual. Taking the fluxion, it will be $\dot{x} + \dot{x}lx = \dot{y}la$; and the ſubtangent $= \frac{xlx}{1 + lx}$.

Here, becauſe $y = \frac{xlx}{la}$, it will be $y\dot{x}$, or the element of the area, $= \frac{x\dot{x}lx}{la}$; and integrating, by § 154, it is $\frac{2xxlx - xx}{4la} =$ area.

164. Other queſtions may be ſtill propoſed, relating to exponential equations; as, for example, in exponential equations compoſed of only known quantities, but with variable exponents, to find thoſe exponents. So, let it be $c^x = ab^{x-1}$; the value of the unknown exponent, x, is required, a, b, c, being given.

Because $\frac{c^x}{a} = b^{x-1}$, it will be $xlc - la = \overline{x-1}\, lb$, and therefore $xlc - xlb = la - lb$. Whence $x = \frac{la - lb}{lc - lb}$.

165. Another question shall be this. To find such a number x, as that it may be $x^x = a$, and also $x^{x+p} = b$. Now, by the first condition, we shall have $xlx = la$, and therefore $x = \frac{la}{lx}$, or $lx = \frac{la}{x}$. By the second condition, we shall have $\overline{x+p}\, lx = lb$. Therefore it will be $x = \frac{lb - plx}{lx}$, or $lx = \frac{lb}{x+p}$. Then it will be $\frac{la}{x} = \frac{lb}{x+p}$, that is, $xla + pla = xlb$, or $x = \frac{pla}{lb - la}$; or else $\frac{la}{lx} = \frac{lb - plx}{lx}$, that is, $lx = \frac{lb - la}{p}$. This supposed, I shall propose to myself to resolve the following Problem.

166. A vessel being given of a known capacity, full of any liquor, suppose wine, out of which is drawn a draught of a given quantity, and then the vessel is filled up with water. Of this mixture of wine and water another draught is drawn equal to the former, and the vessel is again filled up with water. Again, of this mixed liquor another such draught is drawn out; and the same operation is continually repeated in the same manner. It is demanded how many such draughts may be drawn out, or how many times the operation must be repeated, that a given quantity of wine may be left in the vessel.

Let the capacity of the vessel be $= a$, and the quantity of each draught $= b$. Therefore, at the first draught, will be drawn such a quantity of wine as will be expressed by b; and as much water will be poured in again; whence, after the first draught, will be left in the vessel the quantity of wine $= a - b$.

At the second draught will be drawn out the quantity b of the mixture; so that, to have the quantity of pure wine contained in it, we must make this analogy; as the capacity of the vessel (a) is to the quantity of the draught (b), so is the wine which is in the vessel ($a - b$) to a fourth proportional $\frac{ab - bb}{a}$, which will be the quantity of pure wine which is drawn out at the second draught. Then there remains in the vessel the quantity of pure wine, $\frac{aa - 2ab + bb}{a}$, that is, $\frac{\overline{a-b}|^2}{a}$.

Therefore, for the third draught, making also this analogy; as the capacity of the vessel (a) is to the quantity of a draught (b), so is the wine in the vessel,

veffel, $\frac{\overline{a-b}|^2}{a}$, to a fourth, $\frac{\overline{a-b}|^2}{a} \times \frac{b}{a}$. This will be the quantity of pure wine, which was drawn out at the third draught; fo that there will remain in the veffel the quantity of pure wine, $\frac{\overline{a-b}|^2}{a} - \frac{b}{a} \times \frac{\overline{a-b}|^2}{a}$, or $\frac{\overline{a-b}|^3}{aa}$. And thus, after the fourth draught, there will be left in the veffel the quantity of pure wine, $\frac{\overline{a-b}|^4}{a^3}$; and, in general, after a number of draughts denoted by n, there will be left in the veffel the quantity of pure wine $= \frac{\overline{a-b}|^n}{a^{n-1}}$. Therefore, if we would know how many draughts muft be taken, fo that there fhould remain in the veffel a given quantity of pure wine, fuppofe, for example, $\frac{a}{m}$ part of the whole; we muft make the equation $\frac{\overline{a-b}|^n}{a^{n-1}} = \frac{a}{m}$; which, becaufe n is an unknown number, will be an exponential quantity. Wherefore, the equation being reduced to the logarithms, it will be $l\frac{\overline{a-b}|^n}{a^{n-1}} = l\frac{a}{m}$, that is, $nl\overline{a-b} = la - lm + \overline{n-1}\,la$, or $nl\overline{a-b} = -lm + nla$, and therefore $n = \frac{lm}{la - l\overline{a-b}}$; fo that it will be eafy from hence to find the number n, by the help of a Table of Logarithms.

END OF THE THIRD BOOK.

ANALYTICAL INSTITUTIONS.

BOOK IV.

THE INVERSE METHOD OF TANGENTS.

1. AS, when any curve is given, the manner of finding it's tangent, subtangent, perpendicular, or any line of that kind, is called the Direct Method of Tangents; so, when the tangent, subtangent, perpendicular, or any such line is given,—or when the rectification or area is given, to find the curve to which such properties belong, is called the Inverse Method of Tangents.

In the second and third Books are found the general differential expressions of the tangent, or other lines analogous to it; as also, of rectifications and areas. Therefore, by comparing the given property of the tangent, rectification, &c. with the respective expression or general differential formula, there will arise a differential equation of the first degree, or of a superior degree, which, being integrated, either algebraically, or reduced to known quadratures, will give the curve required, to which belongs the given property. For example, let the curve be required of which the subtangent is double to the abscis. Calling the abscis x, and the ordinate y, the formula of the subtangent is $\frac{y\dot{x}}{\dot{y}}$, and therefore the equation will be $\frac{y\dot{x}}{\dot{y}} = 2x$. Again, let us seek the curve, the area of which

which muſt be equal to two third parts of the rectangle of the co-ordinates; the element of the area is $y\dot{x}$, and therefore it ought to be $\int y\dot{x} = \frac{2}{3}xy$, or $y\dot{x} = \frac{2x\dot{y} + 2y\dot{x}}{3}$. If we would find the curve whoſe property it is, that any arch taken from the vertex ſhall be equal to the reſpective ſubnormal; the expreſſion of the arch is $\int\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}$, and that of the ſubnormal is $\frac{y\dot{y}}{\dot{x}}$; ſo that we ſhall have $\int\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}} = \frac{y\dot{y}}{\dot{x}}$, and therefore $\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}} = \frac{y\dot{x}\ddot{y} + \dot{x}\dot{y}\dot{y}}{\dot{x}\dot{x}}$, (taking $\dot{x}$ for conſtant,) which is a differential equation of the ſecond degree.

2. The equations which ariſe by proceeding after this manner, will always have (as is eaſy to perceive,) the indeterminates and differentials intermixed and blended with each other, ſo that at preſent they cannot be managed, in order to proceed to their integration, ſo as to obtain the curves required; and much more if they contain differentials of the ſecond, third, and higher degrees. For, in the third Section aforegoing, the differential formulæ have always been ſuppoſed to be compounded of one indeterminate only, with it's difference or fluxion. Therefore other expedients are neceſſary, to try to reduce ſuch equations to integration, or quadratures, which is called the Conſtruction of Differential Equations, of the firſt, ſecond, &c. Degrees. And, as to the conſtruction of thoſe of the firſt degree, we may proceed two ways; one is, to paſs immediately to integrations or quadratures, without any previous ſeparation of the indeterminates and their differentials; the other is, firſt to ſeparate the indeterminates, and ſo to make the equations fit for integration or quadrature.

I ſhall proceed to ſhow ſeveral particular methods for both the ways, by which we may attain our purpoſe in moſt equations. But very often we ſhall meet with others, which will be found ſo ſtubborn, as not to ſubmit to any methods hitherto diſcovered, or which have not the univerſality that is neceſſary.

SECT.

SECT. I.

Of the Construction of Differential Equations of the First Degree, without any previous Separation of the Indeterminates.

3. The most simple formulæ which have the two variables mixed together, are these two, $x\dot{y} + y\dot{x}$, and $\frac{y\dot{x} - x\dot{y}}{yy}$. The integral of the first is xy, and of the second $\frac{x}{y}$, as is manifest. To these, therefore, we should endeavour to reduce the more compounded, and that by the usual helps of the common Analyticks, by adding, subtracting, multiplying, dividing, &c. by any quantities that will make for the purpose, which will be different according to different cases. We shall here see something of the practice.

Let it be $y\dot{x} = x\dot{x} - x\dot{y}$. By transposing the last term, it will be $y\dot{x} + x\dot{y} = x\dot{x}$, and therefore, by integration, $xy = \frac{1}{2}xx \pm bb$. Let the equation be $x^4\dot{y}\dot{y} + 2x^3y\dot{x}\dot{y} = a^4\dot{x}\dot{x} - xxyy\dot{x}\dot{x}$; then transposing the last term, and dividing by xx, it is $x^2\dot{y}^2 + 2xy\dot{x}\dot{y} + y^2\dot{x}^2 = \frac{a^4\dot{x}^2}{x^2}$, and extracting the square-root, $x\dot{y} + y\dot{x} = \frac{a^2\dot{x}}{x}$; and by integration, $xy = alx \pm b$, in the logarithmic with subtangent $= a$. Let the equation be $y\dot{x} = y^3\dot{y} + y^2\dot{y} + x\dot{y}$, that is, $y\dot{x} - x\dot{y} = y^3\dot{y} + y^2\dot{y}$. The first member would be integrable if it were divided by yy; therefore I divide the equation, and it will be $\frac{y\dot{x} - x\dot{y}}{yy} = y\dot{y} + \dot{y}$, and, by integration, it is $\frac{x}{y} = \frac{1}{2}yy + y \pm b$.

4. Let the equation be $y^r\dot{y} = my\dot{x} + x\dot{y}$. If there was not here the coefficient m, the matter would be easy, because the integral of the second member would

would be xy. The operation would not ſucceed any better, by tranſpoſing the member $x\dot{y}$ to the other ſide, or by writing $y^r\dot{y} - x\dot{y} = my\dot{x}$; yet I obſerve, that the differential of $mxy^{\frac{1}{m}}$ is $m\dot{x}y^{\frac{1}{m}} + xy^{\frac{1}{m}-1}\dot{y}$, different from that propoſed, $my\dot{x} + x\dot{y}$, only in this, that it is multiplied by $y^{\frac{1}{m}-1}$. Therefore, to make the quantity $my\dot{x} + x\dot{y}$ become integrable, it will be ſufficient to multiply it by $y^{\frac{1}{m}-1}$, and, to preſerve the equality, to multiply alſo the correſponding member of the equation $y^r\dot{y}$; therefore it will be $y^{r+\frac{1}{m}-1}\dot{y} = my^{\frac{1}{m}}\dot{x} + xy^{\frac{1}{m}-1}\dot{y}$, and, by integration, $\int y^{r+\frac{1}{m}-1}\dot{y} = mxy^{\frac{1}{m}} \pm b$.

Let the equation be the ſame, but with a different co-efficient in each of the two laſt terms; that is, let it be $y^r\dot{y} = my\dot{x} + nx\dot{y}$. The ſecond member is not integrable; yet I obſerve, that the differential of $mxy^{\frac{n}{m}}$ is $my^{\frac{n}{m}}\dot{x} + nxy^{\frac{n}{m}-1}\dot{y}$. Therefore the *homogeneum comparationis* would be integrable, if it were multiplied by $y^{\frac{n}{m}-1}$. Therefore I multiply the whole equation, and it will become $y^{r+\frac{n}{m}-1}\dot{y} = my^{\frac{n}{m}}\dot{x} + nxy^{\frac{n}{m}-1}\dot{y}$, and the integral will be $\int y^{r+\frac{n}{m}-1}\dot{y} = mxy^{\frac{n}{m}} \pm b$.

5. The differential of x^ny is $x^n\dot{y} + nyx^{n-1}\dot{x}$. This ſuppoſed, let the equation be $y^r\dot{y} = x^n\dot{y} + yx^{n-1}\dot{x}$. If the laſt term had n for it's co-efficient, the integral of the ſecond member of the equation would be x^ny. I obſerve, therefore, that the differential of x^ny^n is $nx^ny^{n-1}\dot{y} + ny^nx^{n-1}\dot{x}$; therefore, multiplying the equation by ny^{n-1}, there will ariſe $ny^{r+n-1}\dot{y} = nx^ny^{n-1}\dot{y} + ny^nx^{n-1}\dot{x}$, which is found to be integrable, it's integral being $\int ny^{r+n-1}\dot{y} = x^ny^n \pm b$.

But if the laſt term, inſtead of the co-efficient n, had any other, or, in general, if both the laſt terms were affected by different co-efficients; or if the equation

equation were $y^r\dot{y} = cx^n\dot{y} + eyx^{n-1}\dot{x}$; I obferve, that the differential of $\frac{e}{n}x^n y^{\frac{cn}{e}}$ is $cx^n y^{\frac{cn}{e}-1}\dot{y} + ey^{\frac{cn}{e}}x^{n-1}\dot{x}$. Therefore multiply the equation by $y^{\frac{cn}{e}-1}$, that it may be $y^{r+\frac{cn}{e}-1}\dot{y} = cx^n y^{\frac{cn}{e}-1}\dot{y} + ey^{\frac{cn}{e}}x^{n-1}\dot{x}$, which is integrable, and it's integral is $\int y^{r+\frac{cn}{e}-1}\dot{y} = \frac{e}{n}x^n y^{\frac{cn}{e}} \pm b$.

Here make $r = 1$, $c = 3$, $n = 1$, $e = 1$, that is, the equation $y\dot{y} = 3x\dot{y} + y\dot{x}$; the integral will be $\frac{1}{4}y^4 = xy^3$. Make $c = 2$, $e = 3$, $n = 1$, $r = 1$, that is, the equation will be $y\dot{y} = 2x\dot{y} + 3y\dot{x}$, and the integral will be $\frac{y^{1+\frac{2}{3}}}{1+\frac{2}{3}} = 3xy^{\frac{2}{3}}$, or $\frac{3}{5}y^{\frac{5}{3}} = 3xy^{\frac{2}{3}}$. Make $c = 2$, $e = 2$, $n = 3$, $r = 3$, or the equation $y^3\dot{y} = 2x^3\dot{y} + 2yx^2\dot{x}$; and the integral will be $\frac{1}{6}y^6 = \frac{2}{3}x^3y^3$.

If the equation were expreffed thus, $y^{1-\frac{cn}{e}}x^r\dot{x} = cx^n\dot{y} + eyx^{n-1}\dot{x}$, it is eafy to fee, that it would be integrable. For, multiplying by $y^{\frac{cn}{e}-1}$, it would be $x^r\dot{x} = cx^n y^{\frac{cn}{e}-1}\dot{y} + ey^{\frac{cn}{e}}x^{n-1}\dot{y}$. But the integral of the fecond member is known to be $\frac{e}{n}x^n y^{\frac{cn}{e}}$; &c.

6. Now let the equation be $y^r\dot{y} = \frac{2x\dot{y} - y\dot{x}}{xx}$. If it were not for the co-efficient 2, the integral of the fecond member would be $\frac{y}{x}$. But it will be to no purpofe to tranfpofe to the other fide the term $\dot{y}x$, and to write it $y^r\dot{y} - \frac{x\dot{y}}{xx} = \frac{x\dot{y} - y\dot{x}}{xx}$. But I obferve that the differential of $\frac{yy}{x}$ is $\frac{2xy\dot{y} - yy\dot{x}}{xx}$; fo that if the propofed equation be multiplied by y, that it may be $y^{r+1}\dot{y} = \frac{2xy\dot{y} - yy\dot{x}}{xx}$, it will be integrable, and it's integral will be $\int y^{r+1}\dot{y} = \frac{yy}{x} \pm b$. But, more generally, let there be any co-efficient n, and therefore the equation is $y^r\dot{y} = \frac{nx\dot{y} - y\dot{x}}{xx}$. I obferve that the differential of $\frac{y^n}{x}$ is $\frac{nxy^{n-1}\dot{y} - y^n\dot{x}}{xx}$; therefore, if it be

be multiplied by y^{n-1}, so that the equation may be $y^{r+n-1}\dot{y} = \frac{nxy^{n-1}\dot{y} - y^n\dot{x}}{xx}$, it will be integrable, and it's integral will be $\int y^{r+n-1}\dot{y} = \frac{y^n}{x} \pm b$.

Thus, let both the last terms have different co-efficients, and let the equation be $y^r\dot{y} = \frac{nx\dot{y} - my\dot{x}}{xx}$. I observe, that the differential of $\frac{my^{\frac{n}{m}}}{x}$ is $\frac{nxy^{\frac{n}{m}-1}\dot{y} - my^{\frac{n}{m}}\dot{x}}{x}$; therefore, if the equation be multiplied by $y^{\frac{n}{m}-1}$, so that it may be $y^{r+\frac{n}{m}-1}\dot{y} = \frac{nxy^{\frac{n}{m}-1}\dot{y} - my^{\frac{n}{m}}\dot{x}}{xx}$, it will be integrable, and it's integral will be $\int y^{r+\frac{n}{m}-1}\dot{y} = \frac{my^{\frac{n}{m}}}{x} \pm b$.

If the equation were $y^{1-\frac{n}{m}}x^r\dot{x} = \frac{nx\dot{y} - my\dot{x}}{xx}$, it would also be integrable. For, multiplying it by $y^{\frac{n}{m}-1}$, it will be $x^r\dot{x} = \frac{nxy^{\frac{n}{m}-1}\dot{y} - my^{\frac{n}{m}}\dot{x}}{xx}$. But the integral of the second member is known to be $\frac{my^{\frac{n}{m}}}{x}$; therefore, &c.

Let the denominator xx be wanting in the aforesaid equations, and let the equation be $y^r\dot{y} = nx\dot{y} - y\dot{x}$. To integrate the second part of the equation, there would be occasion to multiply it by y^{n-1}, and to divide it by xx. But as this must be done also in respect to the first part, it would be $\frac{y^{r+n-1}\dot{y}}{xx}$, which cannot by any means be integrated. Therefore let the signs of the equation be changed, and it will be $-y^r\dot{y} = y\dot{x} - nx\dot{y}$. I observe that the differential of $\frac{x}{y^n}$ is $\frac{y^n\dot{x} - nxy^{n-1}\dot{y}}{y^{2n}}$. Therefore, if the equation be multiplied by y^{n-1}, and then divided by y^{2n}, so that it may be $\frac{-y^{r+n-1}\dot{y}}{y^{2n}} =$

$\frac{y^n\dot{x} - nxy^{n-1}\dot{y}}{y^{2n}}$, it will be integrable, and the integral is $\int \frac{-y^{r+n-1}\dot{y}}{y^{2n}} = \frac{x}{y^n} \pm b$.

Let the equation have both the laſt terms with a co-efficient, and let it be $y^r\dot{y} = nx\dot{y} - my\dot{x}$. Let the ſigns be changed, and it will be $-y^r\dot{y} = my\dot{x} - nx\dot{y}$. I obſerve that the differential of $\frac{x}{my^{\frac{n}{m}}}$ is $\frac{my^{\frac{n}{m}}\dot{x} - nxy^{\frac{n}{m}-1}\dot{y}}{mmy^{\frac{2n}{m}}}$.

Therefore, if the equation be multiplied by $y^{\frac{n}{m}-1}$, and divided by $mmy^{\frac{2n}{m}}$, ſo that it may be $\frac{-y^{r+\frac{n}{m}-1}\dot{y}}{mmy^{\frac{2n}{m}}} = \frac{my^{\frac{n}{m}}\dot{x} - nxy^{\frac{n}{m}-1}\dot{y}}{mmy^{\frac{2n}{m}}}$, it will be integrable, and the integral is $\int \frac{-y^{r+\frac{n}{m}-1}\dot{y}}{mmy^{\frac{2n}{m}}} = \frac{x}{my^{\frac{n}{m}}} \pm b$.

7. Let the equation be $y^r\dot{y} = x^n\dot{y} - nyx^{n-1}\dot{x}$. Change the ſigns, and it will be $-y^r\dot{y} = nyx^{n-1}\dot{x} - x^n\dot{y}$. I obſerve that the differential of $\frac{x^n}{y}$ is $\frac{nyx^{n-1}\dot{x} - x^n\dot{y}}{yy}$. Therefore, dividing the equation by yy, it will become $-y^{r-2}\dot{y} = \frac{nyx^{n-1}\dot{x} - x^n\dot{y}}{yy}$, which will be integrable, and it's integral is $\int -y^{r-2}\dot{y} = \frac{x^n}{y} \pm b$.

But if the co-efficient n had been wanting, and the equation were $y^r\dot{y} = x^n\dot{y} - yx^{n-1}\dot{x}$; change the ſigns, and it will be $-y^r\dot{y} = yx^{n-1}\dot{x} - x^n\dot{y}$. It may be obſerved, that the differential of $\frac{x^n}{y^n}$ is $\frac{ny^nx^{n-1}\dot{x} - nx^ny^{n-1}\dot{y}}{y^{2n}}$. Therefore, multiplying the equation by ny^{n-1}, and dividing it by y^{2n}, it will become

$\dfrac{-ny^{r+n-1}\dot{y}}{y^{2n}} = \dfrac{ny^n x^{n-1}\dot{x} - nx^n y^{n-1}\dot{y}}{y^{2n}}$, which will be integrable, and it's integral is $\int -\dfrac{ny^{r+n-1}\dot{y}}{y^{2n}} = \dfrac{x^n}{y^n} \pm b$.

But if, inftead of the co-efficient n, there fhould be another of a different nature; or if both the laft terms were affected by a different co-efficient, as if the equation were $y^r\dot{y} = cx^n\dot{y} - eyx^{n-1}\dot{x}$; change the figns, and it will be $-y^r\dot{y} = eyx^{n-1}\dot{x} - cx^n\dot{y}$. I obferve that the differential of $\dfrac{x^n}{ey^{\frac{nc}{e}}}$ is $\dfrac{ney^{\frac{nc}{e}}x^{n-1}\dot{x} - ncx^n y^{\frac{nc}{e}-1}\dot{y}}{eey^{\frac{2nc}{e}}}$. Therefore, multiplying the equation by $ny^{\frac{nc}{e}-1}$, and dividing it by $eey^{\frac{2nc}{e}}$, it will be $-\dfrac{ny^{r+\frac{nc}{e}-1}\dot{y}}{eey^{\frac{2nc}{e}}} = \dfrac{ney^{\frac{nc}{e}}x^{n-1}\dot{x} - ncx^n y^{\frac{nc}{e}-1}\dot{y}}{eey^{\frac{2nc}{e}}}$, which will be integrable, and it's integral will be

$$\int -\frac{ny^{r+\frac{nc}{e}-1}\dot{y}}{eey^{\frac{2nc}{e}}} = \frac{x^n}{ey^{\frac{nc}{e}}} \pm b.$$

But if the equation were thus expreffed, $y^{1-\frac{ne}{e}}x^r\dot{x} = cx^n\dot{y} - eyx^{n-1}\dot{x}$; without changing the figns, I obferve, that the differential of $\dfrac{ey^{\frac{nc}{e}}}{x^n}$ is $\dfrac{ncx^n y^{\frac{nc}{e}-1}\dot{y} - ney^{\frac{nc}{e}}x^{n-1}\dot{x}}{x^{2n}}$; therefore, multiplying the equation by $ny^{\frac{nc}{e}-1}$, and dividing it by x^{2n}, we fhall have $\dfrac{nx^r\dot{x}}{x^{2n}} = \dfrac{ncx^n y^{\frac{nc}{e}-1}\dot{y} - ney^{\frac{nc}{e}}x^{n-1}\dot{x}}{x^{2n}}$, which will be integrable; for it's integral is $\int \dfrac{nx^r\dot{x}}{x^{2n}} = \dfrac{ey^{\frac{nc}{e}}}{x^n} \pm b$.

8. I have

8. I have already ſaid, in the foregoing Book, § 17, that as often as the numerator of a fraction, compoſed of only one variable and conſtants, is the exact differential of the denominator, or proportional to that differential; the integral of ſuch a formula is the logarithm of the denominator, or in a given proportion to that logarithm. This alſo obtains when the formula contains two variables, intermixed with each other and with their differentials. Therefore the integral of $\frac{\dot{x} + \dot{y}}{x + y} = \dot{z}$, ($\dot{z}$, after any manner, being given by x or by y,) will be $l\,\overline{x + y} = z \pm b$. The integral of $\frac{\dot{x} + \dot{y}}{2x + 2y} = \dot{z}$ will be $l\sqrt{x + y} = z + b$. The integral of $\frac{4x\dot{x} - 4y\dot{y}}{xx - yy} = \dot{z}$ will be $2l\,\overline{xx - yy} = z \pm b$. The integral of $\frac{y\dot{x} + x\dot{y} - 2y\dot{y}}{2xy - 2yy} = \dot{z}$ will be $l\sqrt{xy - yy} = z \pm b$. And, in general, the integral of $\frac{my^n x^{m-1}\dot{x} + nx^m y^{n-1}\dot{y} - \overline{m+n}\,y^{m+n-1}\dot{y}}{r \times \overline{x^m y^n - y^{m+n}}} = \dot{z}$ will be $l\sqrt[r]{x^m y^n - y^{m+n}} = z \pm b$. And ſo of any other equation whatever, which ſhall have the condition aſſigned.

9. Wherefore many equations, though they have not the neceſſary condition, yet may eaſily be made to acquire it, with the aſſiſtance of ſome calculation. Thus, the equation $\frac{x\dot{y} + y\dot{x}}{x} = -\dot{y}$, has not the required condition in the firſt member; but it will have it if it be divided by y. Then it will be $\frac{x\dot{y} + y\dot{x}}{xy} = -\frac{\dot{y}}{y}$; and therefore, by integration, $l\,xy = ly^{-1} \pm lb$.

Let the equation be $ax\dot{y} + 2ay\dot{x} = xy\dot{y}$. I divide it by axy, and it will be $\frac{x\dot{y} + 2y\dot{x}}{xy} = \frac{\dot{y}}{a}$. This would be integrable if it were not for the co-efficient 2 in the ſecond term of the firſt member; therefore I ſubtract the quantity $\frac{y\dot{x}}{xy}$ from each member, and it will be $\frac{x\dot{y} + y\dot{x}}{xy} = \frac{\dot{y}}{a} - \frac{y\dot{x}}{xy}$, that is, $\frac{x\dot{y} + y\dot{x}}{xy} = \frac{\dot{y}}{a} - \frac{\dot{x}}{x}$; and therefore, by integration, $lxy = \frac{y}{a} - lx \pm lb$.

Let the equation be $yx\dot{x} = \overline{x^2y\dot{y} + y^3\dot{y}} \times \sqrt{y} - y^2\dot{y}$. I divide it by y, and it will be $x\dot{x} = \overline{x^2\dot{y} + y^2\dot{y}} \times \sqrt{y} - y\dot{y}$, that is, $x\dot{x} + y\dot{y} = \overline{x^2\dot{y} + y^2\dot{y}} \times \sqrt{y}$. And dividing again by $xx + yy$, it will be $\frac{x\dot{x} + y\dot{y}}{xx + yy} = \dot{y}\sqrt{y}$. And therefore, by integration, $l\sqrt{xx + yy} = \frac{2}{3}y^{\frac{3}{2}} \pm b$.

 10. From

10. From § 31, 32, of the ſaid Book III, we may gather, that any formula compoſed of one variable only, if it be the product of any complicate quantities raiſed to a poſitive or negative power, integer or fraction, into the exact differential, or into a proportional of the differential of the terms of the quantity; it will always be integrable. And the integral will be the ſame quantity, the exponent of which will be that as at firſt, but increaſed by unity, and multiplied into the ſame exponent ſo increaſed, but taken inverſely: Or, which is the ſame thing, divided by it; or elſe this integral ſhall be proportional to it. Nevertheleſs the rule obtains when the differential formulæ are likewiſe compoſed of two variables and their differentials promiſcuouſly, provided they have the condition required.

Thus, the integral of $\overline{\dot{x} + \dot{y}} \times \sqrt{x + y} = \dot{z}$, (where $\dot{z}$ is any how given by x or by y,) will be $\frac{2}{3} \times \overline{x + y}|^{\frac{3}{2}} = z \pm b$. The integral of $\overline{\frac{1}{2}\dot{x} + \frac{1}{2}\dot{y}} \times \sqrt{x + y} = \dot{z}$ will be $\frac{1}{2} \times \frac{2}{3} \times \overline{x + y}|^{\frac{3}{2}} = z \pm b$, that is, $\frac{1}{3} \times \overline{x + y}|^{\frac{3}{2}} = z \pm b$. The integral of $\frac{p^3\dot{q} + 3qp^2\dot{p} + 3pq^2\dot{q} + q^3\dot{p}}{2\sqrt{p^3q + q^3p}} = \dot{z}$, will be $\sqrt{p^3q + q^3p} = z \pm b$.

The integral of $\overline{x\dot{y} + y\dot{x} + 2y\dot{y}} \times b \times \overline{xy + yy}|^{\frac{n}{m}} = \dot{z}$ will be $\frac{mb}{m + n} \times \overline{xy + yy}|^{\frac{m+n}{m}} = z \pm b$. The integral of $\frac{x\dot{y} + y\dot{x} + 2y\dot{y}}{b \times \overline{xx + yy}|^{\frac{n}{m}}} = \dot{z}$, will be $\frac{m \times \overline{xy + yy}|^{\frac{m-n}{m}}}{\overline{m - n} \times b} = z \pm b$: And ſo of infinite others of the like kind.

But ſome equations of this kind will firſt have need of ſome preparation. Let the equation be $xx\dot{x} + xy\dot{y} + yy\dot{x} = \dot{z}$, (where $\dot{z}$ is any how given by x,) I multiply it by x, and it will be $x^3\dot{x} + x^2y\dot{y} + xy^2\dot{x} = x\dot{z}$, or $x\dot{x} \times \overline{xx + yy} + xx \times y\dot{y} = x\dot{z}$, which has not yet the neceſſary condition. But it would have it if $y\dot{y}$ were alſo multiplied into yy; therefore I add to each member the term $y^3\dot{y}$, and it will be $x\dot{x} \times \overline{xx + yy} + y\dot{y} \times xx + y^3\dot{y} = x\dot{z} + y^3\dot{y}$, that is, $\overline{x\dot{x} + y\dot{y}} \times \overline{xx + yy} = x\dot{z} + y^3\dot{y}$, which is capable of integration, and it's integral is $\frac{1}{4} \times \overline{x^2 + y^2}|^2 = \frac{1}{4}y^4 \pm b + \int x\dot{z}$.

But it is not always eaſy to perceive, what quantities are to be added or ſubtracted, or what other alterations muſt be made in the equations, that they may be brought under the foregoing method; eſpecially when the equations are ſomething compounded. In this way, to arrive at a ſolution is rather the work of

of chance than of art. In ſuch caſes, therefore, we muſt have recourſe to the Methods of Separation of the Indeterminates, which ſhall now follow.

SECT. II.

Of the Conſtruction of Differential Equations, by a Separation of the Indeterminates.

11. The Separation of the Indeterminates in ſome equations, although but few, may be performed by the firſt operations only of the common Algebra. Such would be the equation $x^2\dot{x}^2 + xy\dot{x}\dot{y} = a^2\dot{y}^2$, in which I obſerve, that the firſt member is a formula of an affected quadratick, which would be made a complete ſquare if the term $\frac{y^2\dot{y}^2}{4}$ were added to it. Therefore I add this quantity on each ſide, and the equation will be $xx\dot{x}\dot{x} + xy\dot{x}\dot{y} + \frac{1}{4}yy\dot{y}\dot{y} = aa\dot{y}\dot{y} + \frac{1}{4}yy\dot{y}\dot{y}$. And extracting the root, it will be $x\dot{x} + \frac{1}{2}y\dot{y} = \dot{y}\sqrt{\frac{1}{4}yy + aa}$, in which the variables are ſeparated, and therefore, by integration, $\frac{1}{2}xx + \frac{1}{4}yy = \int\dot{y}\sqrt{aa + \frac{1}{4}yy} \pm b$. The integral of the ſecond member depends on the quadrature of the hyperbola.

12. But moſt frequently it will be convenient to make uſe of ſubſtitutions. Let the equation be $aa\dot{x} = xx\dot{y} + 2xy\dot{y} + yy\dot{y}$. Make $x + y = z$, aſſuming z as a new indeterminate; and therefore $\dot{x} + \dot{y} = \dot{z}$, and $xx + 2xy + yy = zz$. Then making the ſubſtitutions, it will be $aa\dot{z} - aa\dot{y} = zz\dot{y}$, that is, $\frac{aa\dot{z}}{aa + zz} = \dot{y}$, an equation in which the variables are ſeparate. The integration of the firſt member depends on the rectification of the circle.

Let the equation be $\overline{x\dot{y} + y\dot{x}} \times \sqrt{a^4 - xxyy} = \frac{x\dot{x} + y\dot{y}}{\sqrt[4]{xx + yy} \times \sqrt{xx + yy}}$. Here I obſerve in the firſt member, that the integral of $x\dot{y} + y\dot{x}$ is xy, and that the ſquare of this integral is found exactly in the quantity $\sqrt{a^4 - xxyy}$; therefore, if I put $xy = z$, in the firſt member the variables will be ſeparated, and it will be $\dot{z}\sqrt{a^4 - zz}$. I obſerve further, that, in the ſecond member, the integral of $x\dot{x} + y\dot{y}$ is $\frac{xx + yy}{2}$, and that the quantities in the denominator are like to this integral. Therefore, by the ſubſtitution $xx + yy = 2p$, the indeterminates of the ſecond member will alſo be ſeparated, and the equation will be $\dot{z}\sqrt{a^4 - zz} = \frac{\dot{p}}{\sqrt[4]{2p} \times \sqrt{2p}}$.

 Let

Let the equation be $\frac{2x\dot{y} - 2y\dot{x}}{\overline{x-y}|^2} = \dot{z}$, (where $\dot{z}$ is any how given by x or y;) the integral of $x\dot{y} - y\dot{x}$ will be had, if we divide by xx, and it will be $\frac{y}{x}$. Let us ſuppoſe, then, $\frac{y}{x} = \frac{p}{a}$, and therefore $\frac{x\dot{y} - y\dot{x}}{xx} = \frac{\dot{p}}{a}$, and $\frac{2x\dot{y} - 2y\dot{x}}{xx} = \frac{2\dot{p}}{a}$, and $2x\dot{y} - 2y\dot{x} = \frac{2xx\dot{p}}{a}$. Making, therefore, the ſubſtitutions, it will be $\frac{2xx\dot{p}}{a \times \overline{xx - 2xy + yy}} = \dot{z}$, and dividing the numerator and denominator of the firſt member by xx, it will be $\frac{2\dot{p}}{a \times \overline{1 - \frac{2y}{x} + \frac{yy}{xx}}} = \dot{z}$. But it was put $\frac{y}{x} = \frac{p}{a}$, and $\frac{yy}{xx} = \frac{pp}{aa}$; therefore it will be $\frac{2a\dot{p}}{aa - 2ap + pp} = \dot{z}$. And, becauſe the integral of this equation is algebräical, I will go on to the integration. Make, therefore, $a - p = q$, and it will be $-\frac{2a\dot{q}}{qq} = \dot{z}$, and by integration, $\frac{2a}{q} \pm b = z$. But $q = a - p$, and $p = \frac{ay}{x}$; therefore it is $q = \frac{ax - ay}{x}$. Now, reſtoring this value, it will be $\frac{2x}{x-y} \pm b = z$, which is the curve belonging to the differential equation propoſed. If, inſtead of making $a - p = q$, I had made $p - a = q$, another integral would have been found, but differing from this only in the ſigns.

13. The above equation gives me an occaſion of making an uſeful obſervation; which is, that ſometimes curves do not only change their nature by taking their integrals, either ſimply or with the addition of conſtants, which has been already obſerved from the firſt original of infiniteſimal quantities; but ſometimes alſo preſent us with ſuch formulæ, as admit of integrations which are really different, and ſupply us with curves of various kinds, even without the addition of any conſtant quantity; which is a matter deſerving conſideration.

By means of the ſuppoſition $\frac{y}{x} = \frac{p}{a}$, the equation $\frac{2x\dot{y} - 2y\dot{x}}{\overline{x-y}|^2} = \dot{z}$ is preſently integrated, and the integration is found to be $\frac{2x}{x-y} = z$, omitting the conſtant. Now I make the ſuppoſition of $\frac{x}{y} = \frac{p}{a}$, and attempt the integration. It will be, therefore, $\frac{y\dot{x} - x\dot{y}}{yy} = \frac{\dot{p}}{a}$, and thence $2x\dot{y} - 2y\dot{x} = -\frac{2yy\dot{p}}{a}$. And, by ſubſtitution, the equation will be $\frac{-2\dot{p}}{a \times \overline{\frac{xx}{yy} - \frac{2x}{y} + 1}} = \dot{z}$. But $\frac{x}{y} = \frac{p}{a}$; therefore $\frac{-2a\dot{p}}{pp - 2ap + aa} = \dot{z}$. And making $p - a = q$, it will

will be $-\frac{2a\dot{q}}{qq} = \dot{x}$; and, by integration, $\frac{2a}{q} = z$. Now, reſtoring the values, it is $\frac{2y}{x-y} = z$, the integral of the propoſed differential equation, which is different from the firſt.

Another integral of the propoſed formula, different from the two firſt, is $\frac{x+y}{x-y} = z$. For, by differencing, it is $\frac{x\dot{x} - y\dot{x} + x\dot{y} - y\dot{y} - x\dot{x} - y\dot{x} + x\dot{y} + y\dot{y}}{\overline{x-y}|^2}$ $= \dot{z}$, and ſtriking out the terms that deſtroy one another, it is $\frac{2x\dot{y} - 2y\dot{x}}{\overline{x-y}|^2} = \dot{z}$, which is the equation at firſt propoſed.

Make $\dot{z} = \dot{y}$, and the propoſed equation is $\frac{2x\dot{y} - 2y\dot{x}}{\overline{x-y}|^2} = \dot{y}$. If I make uſe of the ſecond integral found above, there ariſes the equation $\frac{2y}{x-y} = y$, and therefore $2 + y = x$, which is a *locus* to a triangle. Then, if I make uſe of the firſt, and of the third integral, by putting $\frac{2x}{x-y} = y$, or $\frac{x+y}{x-y} = y$, the curve will be of the ſecond degree.

In general, let it be $\frac{2x\dot{y} - 2y\dot{x}}{\overline{x-y}|^2} = y^m\dot{y}$. The firſt and the third integration being performed, the curve thence ariſing will aſcend to a degree denoted by $m + 2$, if m be a poſitive number. But, making uſe of the ſecond, the curve will ſtop one degree ſhort.

14. But, however, the method of ſubſtitutions is nevertheleſs univerſal, the greateſt difficulty of which is, that it is often very hard to know what ſubſtitutions ought to be made, that we may not work by chance, and beſtow much labour unſucceſsfully. However, we ſhall proceed with the greateſt ſecurity in all ſuch equations, in which the ſum of the exponents of the variable quantities is the ſame in every term, and the ſeparation of the indeterminates will always ſucceed. It matters not that theſe equations are affected by radicals, or by fractions, or by ſeries, and that the co-efficients and ſigns are of any kind. The ſubſtitution to be made in all theſe equations will be, by putting one of the variables equal to the product of the other into a new variable, ſo that, if the equation be given by x and y, we muſt make $x = \frac{yz}{a}$, or elſe $y = \frac{xz}{a}$, (where by the denominator a is underſtood any conſtant quantity at pleaſure,) and therefore $\dot{y} = \frac{x\dot{z} + z\dot{x}}{a}$; and, making the ſubſtitutions, we ſhall arrive at another equation, which will always be diviſible by as high a power of the indeterminate x, as was the ſum of the exponents of x and y in every term of the propoſed equa-

tion,

tion. Wherefore, making the divifion, the letter x will not exceed the firft power, and will always be multiplied by $\dot{z}$; whence the equation will be fo reduced, that on one fide there will be $\frac{\dot{x}}{x}$, and on the other fide $\dot{z}$, with only the functions of z; and thus the variables will be feparated. For, calling A all thofe terms which are multiplied into $\dot{y}$, and B thofe which are multiplied into $\dot{x}$, the equation will be $A\dot{y} = B\dot{x}$, and A and B will be given promifcuoufly by x and y. Now, becaufe the dimenfions of the letter y, together with the dimenfions of the letter x, in every term make the fame number; if, inftead of y, we put $\frac{xz}{a}$, it will follow from thence, that in every term of the quantities A, B, the letter x will have the fame dimenfion which, at firft, x and y had together. Whence, if this dimenfion be called n, the equation will be divifible by x^n, there only remaining z, a, $\dot{y}$, $\dot{x}$. Let it be fuppofed, that after the fubftitution of $\frac{xz}{a}$, and after the divifion by x^n, that which remains in the quantity A may be called C, and that which remains in the quantity B may be called D; the equation will be $C\dot{y} = D\dot{x}$, and C and D will be given by z and by conftants. But $\dot{y} = \frac{x\dot{z} + z\dot{x}}{a}$; therefore the equation will be $\frac{Cx\dot{z} + Cz\dot{x}}{a} = D\dot{x}$, that is, $Da\dot{x} - Cz\dot{x} = Cx\dot{z}$, and therefore $\frac{\dot{x}}{x} = \frac{C\dot{z}}{Da - Cz}$. And thus the indeterminates, with their differentials, will be feparated, and the equation will be conftructible, at leaft by quadratures.

It is indifferent whether you put $y = \frac{zx}{a}$, or $x = \frac{yz}{a}$; for, in either of the two ways, the indeterminates will always be feparated. But fometimes one fubftitution will give a more fimple equation, and of fewer terms, than the other, and the conftruction will be more eafy and elegant. Wherefore it will not be amifs to try them both, and, at laft, to make choice of that which fucceeds beft.

EXAMPLE I.

Let the equation be $xx\dot{y} = yy\dot{x} + xy\dot{x}$. Make $y = \frac{xz}{a}$, and therefore $\dot{y} = \frac{x\dot{z} + z\dot{x}}{a}$. Making the fubftitutions, it will be $\frac{x^3\dot{z} + zx^2\dot{x}}{a} = \frac{xxzz\dot{x}}{aa} + \frac{zxx\dot{x}}{a}$. And reducing to a common denominator, and dividing by xx, it will be $ax\dot{z} + az\dot{x} = zz\dot{x} + az\dot{x}$; that is, $ax\dot{z} = zz\dot{x}$, or $\frac{\dot{x}}{ax} = \frac{\dot{z}}{zz}$.

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EXAMPLE II.

Let the equation be $xx\dot{y} = yy\dot{x} + xx\dot{x}$. Putting $y = \frac{xz}{a}$, it will be $\dot{y} = \frac{x\dot{z} + z\dot{x}}{a}$. And, making the fubftitutions, it will be $\frac{x^3\dot{z} + zx^2\dot{x}}{a} = \frac{z^2x^2\dot{x}}{aa} + x^2\dot{x}$. And, reducing to a common denominator, and dividing by xx, it will be $ax\dot{z} + az\dot{x} = zz\dot{x} + aa\dot{x}$, that is, $zz\dot{x} - az\dot{x} + aa\dot{x} = ax\dot{z}$, and therefore $\frac{\dot{x}}{x} = \frac{a\dot{z}}{zz - az + aa}$. Now, making another fubftitution, $x = \frac{yp}{a}$, it will be $\dot{x} = \frac{y\dot{p} + p\dot{y}}{a}$, and therefore $\frac{ppyy\dot{y}}{aa} = \frac{y^3\dot{p} + pyy\dot{y}}{a} + \frac{y^3pp\dot{p} + p^3yy\dot{y}}{a^3}$; and, dividing by yy, it is $app\dot{y} = aay\dot{p} + aap\dot{y} + ypp\dot{p} + p^3\dot{y}$, that is, $app\dot{y} - aap\dot{y} - p^3\dot{y} = aay\dot{p} + ypp\dot{p}$; and therefore $\frac{\dot{y}}{y} = \frac{aa\dot{p} + pp\dot{p}}{app - aap - p^3}$.

EXAMPLE III.

Let the equation be $\dot{y}\sqrt{xx + yy} = y\dot{x}$. Make $y = \frac{xz}{a}$, and $\dot{y} = \frac{x\dot{z} + z\dot{x}}{a}$; and, making the fubftitutions, it will be $\frac{x\dot{z} + z\dot{x}}{a} \times \frac{\sqrt{xxzz + aaxx}}{a} = \frac{zx\dot{x}}{a}$, that is, $\overline{xx\dot{z} + zx\dot{x}} \times \sqrt{aa + zz} = azx\dot{x}$, and, dividing by x, it will be $x\dot{z}\sqrt{aa + zz} + z\dot{x}\sqrt{aa + zz} = az\dot{x}$, or $x\dot{z}\sqrt{aa + zz} = az\dot{x} - z\dot{x}\sqrt{aa + zz}$. Therefore $\frac{\dot{z}\sqrt{aa + zz}}{az - z\sqrt{aa + zz}} = \frac{\dot{x}}{x}$. If I had made $x = \frac{yp}{a}$, I fhould have had this equation, $\frac{\dot{y}}{y} = \frac{\dot{p}}{\sqrt{aa + pp} - p}$.

15. But fometimes the differentials themfelves, $\dot{x}$ and $\dot{y}$, afcend to higher dimenfions, the condition mentioned before being, however, in the equations. In which cafes, the fubftitution of $\frac{xz}{a}$, inftead of y, being made as before, not meddling with $\dot{y}$ at prefent, will make every term of the equation divifible by the

the ſame power of x, and there will remain in the equation only z, $\dot{x}$, and $\dot{y}$, with the conſtants given or aſſumed, but not x. Now, becauſe, inſtead of $\dot{y}$, we muſt put $\frac{z\dot{x} + x\dot{z}}{a}$, by which the letter x will again be introduced; make $\frac{x\dot{z}}{a} = \dot{t}$, and, inſtead of $\dot{y}$, write $\frac{z\dot{x} + a\dot{t}}{a}$, and the equation will have only z, $\dot{t}$, $\dot{x}$, with conſtant quantities given or aſſumed, but no longer x. Now, if we make $a \,.\, u :: \dot{x} \,.\, \dot{t}$, and if, inſtead of $\dot{t}$, we put every where $\frac{u\dot{x}}{a}$, we ſhall have an equation free from differential quantities, in which will be only u, z, and conſtants, for an algebräical curve. By means of this curve, we may find the real values of u. Let there be, therefore, A, B, C, &c. ſo that it may be $u = A$, $u = B$, $u = C$, &c. and A, B, &c. will be given only by z, and by conſtants, and it will be $\dot{x} = \frac{a\dot{t}}{A}$, $\dot{x} = \frac{a\dot{t}}{B}$, &c.; and therefore $\dot{t} = \frac{x\dot{z}}{a}$ will be $\dot{x} = \frac{x\dot{z}}{A}$, $\dot{x} = \frac{x\dot{z}}{B}$, &c.; whence, laſtly, $\frac{\dot{x}}{x} = \frac{\dot{z}}{A}$, $\frac{\dot{x}}{x} = \frac{\dot{z}}{B}$, &c.; and the logarithms of x will be directly proportional to the ſpaces comprehended by the curves, of which, the abſciſſes being z, the ordinates will be reciprocally proportional to the values of the quantity u before found. And the curves ſatisfying the purpoſe will be ſo many, as are the real values (different from each other) of the letter u; ſtill obſerving, that the adding of a conſtant quantity in the integration of the equations $\frac{\dot{x}}{x} = \frac{\dot{z}}{A}$, $\frac{\dot{x}}{x} = \frac{\dot{z}}{B}$, &c. may again diverſify the curves that ſatisfy the demand, and will often double their number. Then lx will be equal to the area of that curve, which has z for it's abſciſs, and $\frac{1}{A}$, $\frac{1}{B}$, &c. for it's ordinate; that is, it will be equal to the integral of $\frac{\dot{z}}{A}$, $\frac{\dot{z}}{B}$, &c. Wherefore, taking z at pleaſure, the logarithm of x will be given, and conſequently the correſponding ordinate x in the logarithmic will be given alſo. Then, x being given, by means of the equation $y = \frac{xz}{a}$ will y be given alſo, that is, both the co-ordinates of the differential equation propoſed, or of the curve required. Then, in reference to the different values which will be given to z, ſo will be the different points alſo, which will be found in the ſame curve required.

I ſhall apply the rule to an example. Let the equation be $xx\dot{y}\dot{y} + xy\dot{x}\dot{y} = xx\dot{x}\dot{x}$. Make, therefore, $y = \frac{xz}{a}$, and, putting this value in the equation, inſtead of y, we ſhall have $ax^2\dot{y}^2 + x^2z\dot{x}\dot{y} = ax^2\dot{x}^2$, and dividing by xx, it will be $a\dot{y}^2 + z\dot{x}\dot{y} = a\dot{x}^2$. Here we ſee, that x and it's functions entirely diſappear, there

there remaining only z, $\dot{x}$, $\dot{y}$, with their functions. But, because, by substituting, instead of $\dot{y}$, it's value $\frac{z\dot{x} + x\dot{z}}{a}$, we shall again introduce x into the equation; make $\frac{x\dot{z}}{a} = \dot{t}$, and therefore $\dot{y} = \frac{z\dot{x} + a\dot{t}}{a}$, and the equation will be $\frac{zz\dot{x}\dot{x} + 2az\dot{x}\dot{t} + aa\dot{t}\dot{t}}{a} + \frac{zz\dot{x}\dot{x} + az\dot{x}\dot{t}}{a} = a\dot{x}\dot{x}$, that is, $2zz\dot{x}\dot{x} + 3az\dot{x}\dot{t} + aa\dot{t}\dot{t} = aa\dot{x}\dot{x}$; in which only enter z, $\dot{x}$, $\dot{t}$, with their functions. Again, supposing $\dot{t} = \frac{u\dot{x}}{a}$, and making the substitution, we shall arrive at an expression which is purely algebräical, $2zz + 3zu + uu = aa$, so that we shall have the value of u given algebräically by z and constant quantities. But $\dot{t} = \frac{u\dot{x}}{a} = \frac{x\dot{z}}{a}$, whence $\frac{\dot{x}}{x} = \frac{\dot{z}}{u}$, in which equation, u being given by z, the variables will be separated. Therefore the curve being described, of which the abscisses are z, and the ordinates reciprocally proportional to the values of u; we shall have x, and thence y, by making the substitution of $\frac{xz}{a} = y$.

16. Now, from this and other examples, it will succeed also, without making use of this method, that they may easily be reduced by the method of § 14. And, indeed, if to each of the members of the aforesaid equation, $xx\dot{y}\dot{y} + xy\dot{x}\dot{y} = xx\dot{x}\dot{x}$, there be added the square $\frac{1}{4}yy\dot{x}\dot{x}$, it will be $xx\dot{y}\dot{y} + xy\dot{x}\dot{y} + \frac{1}{4}yy\dot{x}\dot{x} = xx\dot{x}\dot{x} + \frac{1}{4}yy\dot{x}\dot{x}$, and extracting the root, $x\dot{y} + \frac{1}{2}y\dot{x} = \dot{x}\sqrt{xx + \frac{1}{4}yy}$; where now it is reduced to the aforesaid general method of § 14. Or else, transposing the term $xy\dot{x}\dot{y}$, and adding the square $\frac{1}{4}yy\dot{y}\dot{y}$, it will be $xx\dot{y}\dot{y} + \frac{1}{4}yy\dot{y}\dot{y} = xx\dot{x}\dot{x} - xy\dot{x}\dot{y} + \frac{1}{4}yy\dot{y}\dot{y}$; and, extracting the root, it is $\dot{y}\sqrt{xx + \frac{1}{4}yy} = x\dot{x} - \frac{1}{2}y\dot{y}$; now reduced to the same method.

17. Equations which contain differentials mixed together, and raised to any power, may not only be constructed in the case considered at § 15, which supposes the sum of the exponents of the variables to be equal in every term; but, in general, in what manner soever those equations are, provided one of the two indeterminates, x or y, be absent. This is done by making $\dot{x} = \frac{z\dot{y}}{a}$, if x be wanting, or $\dot{y} = \frac{z\dot{x}}{a}$, if y be wanting; z being a new indeterminate, and a any constant quantity. For, by such a substitution in the proposed equation, of $\frac{z\dot{y}}{a}$ instead of $\dot{x}$, it is plain that another will arise, which will be divisible by the power of $\dot{y}$; so that it will be composed of finite quantities only, and

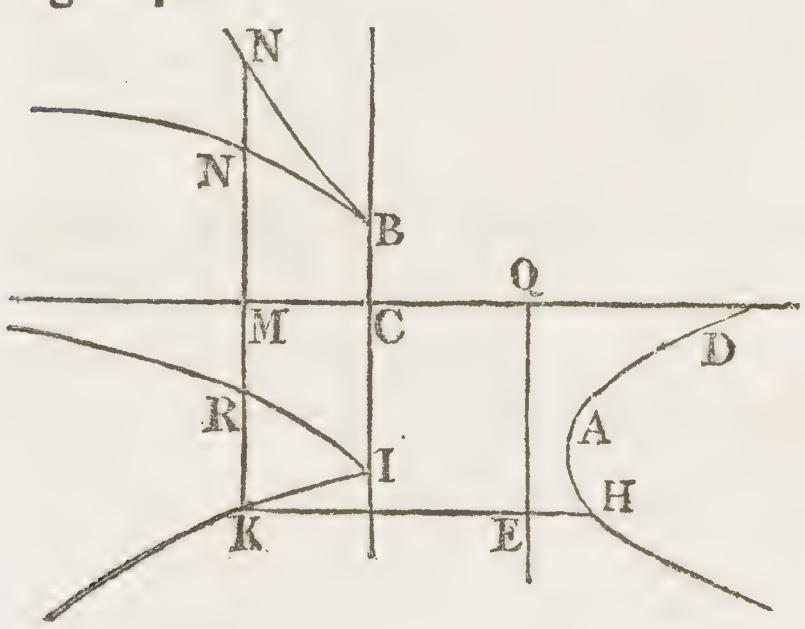
 therefore

therefore will have z given by y and conſtants only, and the relation of y to z will be expreſſed by an equation, or an algebräical curve. Therefore, in the equation $\dot{x} = \frac{z\dot{y}}{a}$, inſtead of $\dot{y}$, putting the value that will be derived from ſuch algebräical equation, we ſhall have the variables ſeparated.

EXAMPLE I.

Let the equation be $y\dot{y}^3\dot{x} = a\dot{x}^4 + 2a\dot{x}^2\dot{y}^2 + a\dot{y}^4$. Make $\dot{x} = \frac{z\dot{y}}{a}$; and, making the ſubſtitutions, inſtead of $\dot{x}$ and it's powers, we ſhall have the equation $\frac{zy\dot{y}^4}{a} = \frac{z^4\dot{y}^4}{a^3} + \frac{2zz\dot{y}^4}{a} + a\dot{y}^4$; and, dividing by $\dot{y}^4$, it will be $\frac{zy}{a} = \frac{z^4}{a^3} + \frac{2z^2}{a} + a$. Or $y = \frac{z^3}{aa} + 2z + \frac{aa}{z}$, and $\dot{y} = \frac{3zz\dot{z}}{aa} + 2\dot{z} - \frac{aa\dot{z}}{zz}$. Therefore $\frac{z\dot{y}}{a} = \dot{x} = \frac{3z^3\dot{z}}{a^3} + \frac{2z\dot{z}}{a} - \frac{a\dot{z}}{z}$. If we go on to the integration, it will be $x = \frac{3z^4}{4a^3} + \frac{zz}{a} - lz$, taking the logarithm from the logarithmic with the ſubtangent $= a$. Whence we have the values of the two co-ordinates x and y of the propoſed differential equation, by means of two curves, which have z for a common indeterminate. Now, as to the conſtruction, we may proceed thus.

Fig. 142.



Taking the abſciſſes in the axis QE, deſcribe the curve DAH of the equation $y = \frac{z^3}{aa} + 2z + \frac{aa}{z}$, and the curve RIK of the equation $x = \frac{3z^4}{4a^3} + \frac{zz}{a} - lz$. Then EH $= y$, and EK $= x$, will be the co-ordinates of the propoſed differential curve; by the conſtruction of which, making CM parallel to EK, then KM is produced to N, whence it will always be MN $=$ EH; and the curve NBN will be that required.

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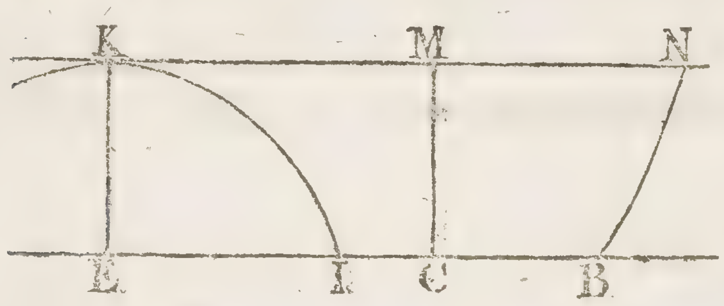
EXAMPLE II.

Let the equation be $y^3\dot{x}^5 + aay\dot{y}\dot{x}^4 = a^3\dot{y}^5$. Make $\dot{x} = \frac{z\dot{y}}{a}$; and, making the ſubſtitutions, we ſhall have $\frac{z^5y^3\dot{y}^5}{a^5} + \frac{aaz^4y\dot{y}^5}{a^4} = a^3\dot{y}^5$. And, dividing by $\dot{y}^5$, it will be $z^5y^3 + a^3z^4y = a^8$. Therefore z will be given only by y and conſtants, and therefore, in the equation $\dot{x} = \frac{z\dot{y}}{a}$, the variables are ſeparated.

Fig. 143.



Now, to have the curve of the propoſed differential equation; to the axis CE let there be deſcribed the curve IK of the equation $z^5y^3 + a^3z^4y = a^8$, it being $CM = y$, and $MK = z$. In KM, produced, take MN equal to the area CMKI, divided by a. Then will it be $MN = \int\frac{z\dot{y}}{a} = x$, and the point N will be in the curve.

18. The method of § 14 may be rendered ſtill more general, by tranſforming the equations which have not the condition required, of the ſum of the exponents being equal, into others which ſhall have thoſe ſums equal, and conſequently ſhall come under the rule of that article. This may be done two ways. One will be, to make uſe of convenient ſubſtitutions, for which there can be no rule, and it muſt be by examples alone that this artifice can be acquired. The other is, by changing the exponents of the propoſed formula or equation, that it may be determined, at leaſt, in what caſes, and with what ſubſtitutions it may ſucceed, to transform the equation into one equivalent to it, in which the condition required may be found. Thus, though the ſeparation of the variables cannot be univerſally performed, yet infinite caſes may be aſſigned, in which that ſeparation will be effected.

EXAMPLE I.

Now, as to the firſt manner. Let the equation be $\dot{x}\sqrt{aaxx + az^3} = zz\dot{z}$, which has not the neceſſary condition. Make $z^3 = ayy$, and, taking the fluxions, $zz\dot{z} = \frac{2}{3}ay\dot{y}$. Therefore, making the ſubſtitutions, $\dot{x}\sqrt{aaxx + aayy} = \frac{2}{3}ay\dot{y}$; an expreſſion that may be managed by the method of § 14. We may alſo have our deſire, by putting $\sqrt{aaxx + az^3} = au$, and therefore $aaxx + az^3 = aauu$, and, by differencing, $2aax\dot{x} + 3azz\dot{z} = 2aau\dot{u}$, that is, $zz\dot{z} = \frac{2}{3}au\dot{u} - \frac{2}{3}ax\dot{x}$; and, making the ſubſtitutions, it is $u\dot{x} = \frac{2u\dot{u} - 2x\dot{x}}{3}$.

EXAMPLE II.

Let the equation be $x^3\dot{x} + \frac{xx\dot{y}}{\sqrt{a + y}} = \dot{y}$. Make $\sqrt{a + y} = z$, and therefore $a + y = zz$, and $\dot{y} = 2z\dot{z}$. And, by ſubſtitution, $x^3\dot{x} + 2xx\dot{z} = 2z\dot{z}$. But this ſtill requires a little further reduction. Therefore make $xx = u$, or $x^4 = uu$, and $4x^3\dot{x} = 2u\dot{u}$; whence, theſe values being ſubſtituted, it will be finally $\frac{1}{2}u\dot{u} + 2u\dot{z} = 2z\dot{z}$, &c.

19. I ſhall go on to the ſecond manner of altering the exponents, and therefore I ſhall take a general equation of three terms, $ay^n x^m \dot{x} + by^q x^p \dot{x} + cx^r y^s \dot{y} = 0$; in which the ſigns may be as we pleaſe, either poſitive or negative. If it were $n + m = q + p = r + s$, it would be the caſe of § 14. But, ſuppoſing ſuch an equality ſhould not be found between the ſums of the exponents; make $y = z^t$, whence $\dot{y} = tz^{t-1}\dot{z}$, $y^s = z^{st}$, $y^q = z^{tq}$, $y^n = z^{nt}$, and making the neceſſary ſubſtitutions in the propoſed equation, it will be $az^{nt}x^m\dot{x} + bz^{qt}x^p\dot{x} + tcx^r z^{st+t-1}\dot{z} = 0$. But, by the condition of the aforeſaid § 14, it is neceſſary that it ſhould be $nt + m = qt + p = r + st + t - 1$. From the firſt equation, therefore, $nt + m = qt + p$, we muſt derive the value of the aſſumed exponent $t = \frac{p - m}{n - q}$, which, being ſubſtituted in

in the second, $qt + p = r + st + t - 1$, or $\overline{s - q + 1} \times t = p - r + 1$, will give $\overline{s - q + 1} \times \overline{p - m} = \overline{p - r + 1} \times \overline{n - q}$; which is the condition that the exponents of the proposed equation ought to have. To verify which, it will always be reducible by the rule of § 14; and the substitution to be made will be $y = z^{\frac{p-m}{n-q}}$.

Instead of making $y = z^t$, if I had made $x = z^t$, I should have found the same condition to be verified in the exponents, but it would have been $t = \frac{n-q}{p-m}$, and therefore the substitution to be made is $x = z^{\frac{n-q}{p-m}}$.

It may happen, that the substitution of $y = z^{\frac{p-m}{n-q}}$ may become impossible, that is, when $p = m$, or $n = q$. But it may be observed, that, in these cases, the indeterminates are separable without need of reduction.

In the canonical equation $ay^n x^m \dot{x} + by^q x^p \dot{x} + cx^r y^s \dot{y} = 0$, if, besides the supposition of $y = z^t$, we shall also make $x = u^w$; making all the substitutions, we shall find $awz^{nt} u^{wm+w-1} \dot{u} + bwz^{qt} u^{wp+w-1} \dot{u} + ctu^{wr} z^{st+t-1} \dot{z} = 0$. By the comparison of the exponents of the first and second terms, we should have $nt + wm + w - 1 = qt + wp + w - 1$, that is, $t = w \times \frac{p-m}{n-q}$. From the comparison of those of the second and third, we shall have $wr + st + t - 1 = qt + wp + w - 1$, or $t \times \overline{s - q + 1} = w \times \overline{p - r + 1}$. And, instead of t, putting it's value, $w \times \overline{p - m} \times \overline{s - q + 1} = w \times \overline{n - q} \times \overline{p - r + 1}$, which is the condition the exponents of the proposed equation ought to have. But the letter w vanishes out of the condition; therefore the second substitution of $x = u^w$ is altogether superfluous; whence it may be inferred, that all the formulæ, in general, cannot be reduced to the rule of § 14, but only such, in which the condition $\overline{p - m} \times \overline{s - q + 1} = \overline{n - q} \times \overline{p - r + 1}$ may be verified. The same thing is to be concluded of others, when compounded of a greater number of terms, which I shall now proceed to treat of.

20. As the number of terms increases beyond three, so, in like manner, the number of conditions increases, which the exponents of the equation must have,

in order to be reducible by the method of § 14. I will take this canonical equation of four terms, $ax^m y^n \dot{x} + bx^p y^q \dot{x} + cx^r y^s \dot{y} + dx^e y^u \dot{y} = 0$. Putting $y = z^t$, $\dot{y} = tz^{t-1}\dot{z}$, and making the substitutions, it is $az^{nt} x^m \dot{y} + bz^{qt} x^p \dot{x} + tcx^r z^{st+t-1}\dot{z} + dtx^e z^{tu+t-1}\dot{z} = 0$. Therefore it ought to be $nt + m = qt + p$. Whence we may derive the value of the assumed exponent $t = \frac{p-m}{n-q}$. Also, it ought to be $r + st + t - 1 = qt + p$, or $st - qt + t = p - r + 1$; and, substituting the value of t, it will be $\overline{s - q + 1} \times \overline{p - m} = \overline{p - r + 1} \times \overline{n - q}$, the first condition. And, besides, it ought to be $e + tu + t - 1 = qt + p$, or $tu - qt + t = p - e + 1$, and, substituting the value of t, $\overline{u - q + 1} \times \overline{p - m} = \overline{p - e + 1} \times \overline{n - q}$, the second condition. If, therefore, the exponents of a proposed equation shall be such, as that both these conditions shall be found therein, it will be reducible to the case of § 14, and the substitution to be made will be $y = z^{\frac{p-m}{n-q}}$.

If the equations shall have five terms, the conditions to be verified will be three; and so on to more terms.

EXAMPLE.

Let the equation be $ay^3x\dot{x} + byyx^{\frac{1}{2}}\dot{x} = cx\dot{y}$. This, being compared with the canonical equation, will give $n = 3$, $m = 1$, $q = 2$, $p = \frac{1}{2}$, $r = 1$, $s = 0$. And, because, in the present case, the condition is verified of $\overline{s - q + 1} \times \overline{p - m} = \overline{p - r + 1} \times \overline{n - q}$, giving $-1 \times -\frac{1}{2} = \frac{1}{2} \times 1$, which is true; the equation will be reducible to the method of § 14, and the substitution to be made will be $y = z^{\frac{p-m}{n-q}} = z^{-\frac{1}{2}}$. Therefore I make $y = z^{-\frac{1}{2}}$, $\dot{y} = -\frac{1}{2}z^{-\frac{3}{2}}\dot{z}$, $y^3 = z^{-\frac{3}{2}}$, $y^2 = z^{-1}$; and, making the substitutions, I find $az^{-\frac{3}{2}}x\dot{x} + bz^{-1}x^{\frac{1}{2}}\dot{x} = -\frac{1}{2}cxz^{-\frac{3}{2}}\dot{z}$; which is now reduced to the case of the said article.

21. But, without applying particular equations to canonical ones, perhaps it may be more commodious to manage them by this method only.

EXAMPLE I.

Let the equation be $ay^{\frac{1}{3}}x^{\frac{11}{6}}\dot{x} - bx^3y^{-1}\dot{y} = cx^2y\dot{y}$. Make $x = z^t$, $\dot{x} = tz^{t-1}\dot{z}$; making the fubftitutions, it will be $tay^{\frac{1}{3}}z^{\frac{11}{6}t+t-1}\dot{z} - bz^{3t}y^{-1}\dot{y} = cz^{2t}y\dot{y}$. But it ought to be $\frac{1}{3} + \frac{11}{6}t + t - 1 = 3t - 1$, whence I obtain $t = 2$, which, being put inftead of t, gives me this equation $2ay^{\frac{1}{3}}z^{\frac{14}{3}}\dot{z} - bz^6y^{-1}\dot{y} = cz^4y\dot{y}$, which is juft the cafe of § 14. Therefore the fubftitution to be made, $x = z^2$.

EXAMPLE II.

Let the equation be $x^{\frac{1}{2}}\dot{x} + y^{\frac{4}{3}}\dot{x} + x^{\frac{3}{4}}y\dot{y} = y^3\dot{y}$. Put $y = z^t$, $\dot{y} = tz^{t-1}\dot{z}$, and, making the fubftitutions, it will be $x^{\frac{1}{2}}\dot{x} + z^{\frac{4t}{3}}\dot{x} + tx^{\frac{3}{4}}z^{2t-1}\dot{z} = tz^{4t-1}\dot{z}$. But it ought to be $\frac{1}{2} = \frac{4}{3}t$, whence I have $t = \frac{3}{8}$; which value, being put inftead of t, gives me the equation $x^{\frac{1}{2}}\dot{x} + z^{\frac{1}{2}}\dot{x} + \frac{3}{8}x^{\frac{3}{4}}z^{-\frac{1}{4}}\dot{z} = \frac{3}{8}z^{\frac{1}{2}}\dot{z}$, which is juft the cafe of § 14. Therefore the fubftitution to be made is $y = z^{\frac{3}{8}}$.

EXAMPLE III.

Let the equation be $ay^2x^2\dot{x} + b\dot{x} + cyx\dot{x} + dx^4y^2\dot{y} = 0$. Put $y = z^t$, $\dot{y} = tz^{t-1}\dot{z}$; making the fubftitutions, it will be $az^{2t}x^2\dot{x} + b\dot{x} + cz^tx\dot{x} + tdx^4z^{3t-1}\dot{z} = 0$. Now it ought to be $2t + 2 = t + 1$, whence $t = -1$; and,

and, putting this inftead of t, gives me the equation $\frac{ax^2\dot{x}}{zz} + b\dot{x} + \frac{cx\dot{x}}{z} - \frac{dx^4\dot{z}}{z^4} = 0$, which is the cafe of § 14. The fubftitution to be made is $y = \frac{1}{z}$.

22. The method of § 14 being thus made more general, I fhall proceed to another, which is alfo general in it's kind. This comprehends all thofe equations, in which neither the indeterminates, nor their differentials, exceed the firft dimenfion.

Wherefore let the general differential equation, which includes all poffible cafes wherein the variables and their fluxions do not afcend beyond one dimenfion, be $ax\dot{x} + by\dot{y} + cy\dot{x} + gx\dot{y} + f\dot{x} + h\dot{y} = 0$. The co-efficients a, b, c, &c. may be pofitive, or negative, or nothing, as the circumftances of the particular equation may require, which is propofed to be conftructed. As to this equation, I obferve, in the firft place, that, if it fhall be $c = g$, both of them being pofitive, or both negative, the equation may be integrated. For then it will be $\pm c \times \overline{y\dot{x} + x\dot{y}} = -ax\dot{x} - by\dot{y} - f\dot{x} - h\dot{y}$, and, by integration, $\pm cxy = -\frac{1}{2}axx - \frac{1}{2}byy - fx - hy$. But, it not being $c = g$, I make $x = p + A$, $y = q + B$, where p and q are two new indeterminates, and A and B are arbitrary conftants, to be determined as the fequel may require. It will be then $\dot{x} = \dot{p}$, $\dot{y} = \dot{q}$, $x\dot{x} = p\dot{p} + A\dot{p}$, $y\dot{y} = q\dot{q} + B\dot{q}$. Thefe values being fubftituted in the principal equation propofed, there will arife this following.

$$\begin{array}{llllll} ap\dot{p} & + aA\dot{p} & + bq\dot{q} & + bB\dot{q} & + cq\dot{p} & + gp\dot{q} = 0. \\ & + cB\dot{p} & & + gA\dot{q} & & \\ & + f\dot{p} & & + h\dot{q} & & \end{array}$$

In this equation, if the fecond and fourth terms be made to vanifh, this will be the cafe of § 14; and we fhall know how to feparate the indeterminates. But the fecond term will vanifh, if it be made $aA + cB + f = 0$, and the fourth, if it be $bB + gA + h = 0$. Whence, from thefe two equations, the values of the affumed quantities A and B will be determined, fo as that the new equation will be a cafe of the aforefaid § 14. Then it will be $A = \frac{-cB - f}{a}$, $B = \frac{-gA - h}{b}$, that is, $A = \frac{bf - ch}{cg - ab}$, $B = \frac{ah - fg}{cg - ab}$. If, therefore, we make the fubftitutions of $x = p + \frac{bf - ch}{cg - ab}$, and of $y = q + \frac{ah - fg}{cg - ab}$, an equation will arife, which may be managed by the method of § 14.

If it fhould happen, in a particular equation, that it fhould be $bf = ch$, or $ah = fg$, fo that either of the affumed conftants fhould be nothing; it would be

be a sure token, that we might obtain our desire by one substitution only. For example-sake, let $\frac{bf - ch}{cg - ab} = A = 0$. In this case, omitting the quantity x with it's fluxion, it will be enough to substitute $q + B$ instead of y, and to proceed in the manner above explained.

Now, if both the quantities A and B should be nothing, in this hypothesis we should have $bf = ch$, and $ah = fg$; and consequently $\frac{ch}{b} = \frac{ah}{g} = f$. Then $cg = ab$, by which we should no longer have any need of these substitutions. Therefore, as often as it is $cg = ab$, make the substitution $ax + cy = z$, and take y and $\dot{y}$ out of the equation. It will be then $y = \frac{z - ax}{c}$, $\dot{y} = \frac{\dot{z} - a\dot{x}}{c}$. Make these substitutions in the principal equation, and we shall have $ax\dot{x} + \frac{bz\dot{z} - abx\dot{z} - abz\dot{x} + aabx\dot{x}}{cc} + z\dot{x} - ax\dot{x} + \frac{gx\dot{z} - agx\dot{x}}{c} + f\dot{x} + \frac{h\dot{z} - ah\dot{x}}{c} = 0$. That is, striking out the first and seventh terms, and, reducing all to a common denominator, $bz\dot{z} - abx\dot{z} - abz\dot{x} + aabx\dot{x} + ccz\dot{x} + cgx\dot{z} - acgx\dot{x} + ccf\dot{x} + ch\dot{z} - ach\dot{x} = 0$. But, because $gc = ab$, the second term will destroy the sixth, and the fourth the seventh, so that there will remain only $bz\dot{z} - abz\dot{x} + ccz\dot{x} + ccf\dot{x} + ch\dot{z} = ach\dot{x}$, or $\dot{x} = \frac{bz\dot{z} + ch\dot{z}}{abz - ccz - ccf + ach}$.

EXAMPLE I.

Let the equation be $ax\dot{x} + 2ay\dot{x} + bx\dot{y} - ab\dot{y} = 0$. Make $x = p + A$, $y = q + B$, $\dot{x} = \dot{p}$, $\dot{y} = \dot{q}$; and, making the substitutions, the equation will be

$$\begin{aligned} &ap\dot{p} + aA\dot{p} + 2aq\dot{p} + bp\dot{q} + bA\dot{q} = 0. \\ &\qquad + 2aB\dot{p} \qquad\qquad\qquad - ab\dot{q} \end{aligned}$$

The last term will vanish if it be $bA - aB = 0$, or $A = a$. The second will vanish if it be $2aB + aA = 0$, or $B = -\frac{1}{2}a$. Therefore the substitutions are $x = p + a$, and $y = q - \frac{1}{2}a$; and the equation will be reduced to the case of § 14.

The aforesaid terms vanishing out of the equation, it may be integrated by means of § 4, without having recourse to § 14.

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EXAMPLE II.

Let the equation be $2ax\dot{x} - 2by\dot{y} - 4ay\dot{x} + bx\dot{y} - aa\dot{x} = 0$. In this the co efficient $2a$ corresponds with a in the canonical equation, $-2b$ with b, $-4a$ with c, b with g; and gives us the case, that it is $cg = ab$, in respect to the constants of the canonical equation. Therefore I make the substitution $2ax - 4ay = z$, and therefore $y = \frac{2ax - z}{4a}$, $\dot{y} = \frac{2a\dot{x} - \dot{z}}{4a}$; wherefore, eliminating y and $\dot{y}$, we shall have $2ax\dot{x} - \frac{8aabx\dot{x} + 4abz\dot{x} + 4abx\dot{z} - 2bz\dot{z}}{16aa} - 2ax\dot{x} + z\dot{x} + \frac{2abx\dot{x} - bx\dot{z}}{4a} - aa\dot{x} = 0$. That is, $4abz\dot{x} - 2bz\dot{z} + 16aaz\dot{x} - 16a^4\dot{x} = 0$, or $\dot{x} = \frac{2bz\dot{z}}{4abz + 16a^2z - 16a^4}$.

23. Equations of this kind, as also those of a higher degree, may be thus managed by the help of one, but a more compounded substitution. I resume the canonical equation above, $ax\dot{x} + by\dot{y} + cy\dot{x} + gx\dot{y} + f\dot{x} + h\dot{y} = 0$, because those of higher degrees would involve us in too long calculations; and what I shall say concerning this, will be sufficient to show us how those others are to be treated. Therefore I make $x = Ay + p + B$, in which subsidiary equation p is a new indeterminate, which has no constant prefixed to it, because that would be unnecessary, as the operation will show. A and B are two constants, to be determined as occasion may require. Making, then, $x = Ay + p + B$, it will be $\dot{x} = A\dot{y} + \dot{p}$, $x\dot{x} = AAy\dot{y} + Ap\dot{y} + AB\dot{y} + Ay\dot{p} + p\dot{p} + B\dot{p}$; so that, these values being substituted in the canonical equation, it will be transformed into this following.

$$\left.\begin{array}{llllll} aAAy\dot{y} & + aAp\dot{y} & + aAy\dot{p} & + ap\dot{p} & + aAB\dot{y} & + aB\dot{p} \\ + by\dot{y} & + gp\dot{y} & + cy\dot{p} & & + gB\dot{y} & + f\dot{p} \\ + cAy\dot{y} & & & & + fA\dot{y} & \\ + gAy\dot{y} & & & & + h\dot{y} & \end{array}\right\} = 0.$$

Now we must contrive to make some of the terms of this equation to vanish, by conveniently determining the assumed arbitrary quantities A and B, and to make it capable of the end proposed; when some of the conditions are to be verified, which arise from the values of A and B. If, therefore, the second and third terms could be destroyed, the variables would be separated, and the equation would become integrable. But, that these two terms may become nothing, it is necessary that it be $aA + g = 0$ in respect of the second,

and

and $aA + c = 0$, in reſpect of the third; and conſequently $g = c$. But, ſuppoſing this, the principal equation will be already integrable, without the help of any operation.

If the two laſt terms were nothing, the equation would be reduced to the canon of § 14. But, that they may vaniſh, it will be neceſſary that $aB + f = 0$, or $B = -\frac{f}{a}$, in reſpect of the laſt, and $aAB + gB + fA + h = 0$, in reſpect of the fifth. But, ſubſtituting the value of B, it will be $-Af - \frac{gf}{a} + Af + h = 0$, that is, $ah = gf$. Therefore the laſt two terms cannot be made to vaniſh, ſo that by them the equation may be reduced, except in the particular caſe, in which is verified the condition of $ah = gf$.

If we endeavour, then, to take away the firſt and fifth terms, by which the equation will be reduced to the caſe of § 4 and § 6; then, in reſpect of the firſt term, it will be $aAA + b + cA + gA = 0$, or $A^2 + \frac{c+g}{a}A = -\frac{b}{a}$, from whence we may deduce the value of A. This being found, the value of B will be diſcovered from the fifth term, and will be $B = \frac{-fA - h}{aA + g}$. And the new equation will become $\overline{aA + g} \times p\dot{y} + \overline{aA + c} \times y\dot{p} = -ap\dot{p} - aB\dot{p} - f\dot{p}$, which may be conſtructed by means of § 4, if the co-efficients of the two firſt terms are both poſitive or negative; but, by means of § 6, if one be poſitive, and the other negative.

But, to obtain the ſeparation required, it will be ſufficient to make the firſt term of the ſubſidiary equation to vaniſh, by making it $aAA + cA + gA + b = 0$. Now, putting the aſſumed conſtant $B = 0$, which, in this caſe, will be unneceſſary, there will remain the equation $-ap\dot{p} - f\dot{p} = \overline{aA + g}. \times p\dot{y} + \overline{fA + h} \times \dot{y} + \overline{aA + c} \times y\dot{p}$, in which the variables may be ſeparated by the method, which ſhall be explained in the following article. Or elſe, by the foregoing, with the help of an eaſy preparation, that is, making $\overline{Aa + g} \times p + fA + h = q$, and taking the fluxion $\overline{Aa + g} \times \dot{p} = \dot{q}$. Then, by ſubſtitution, $-ap\dot{p} - f\dot{p} = q\dot{y} + \frac{\overline{Aa + c} \times y\dot{q}}{Aa + g}$. But we ought to conſider, that, in making uſe of ſuch formulæ, very often imaginary quantities will inſinuate themſelves, ariſing from the extraction of the root A out of the affected quadratick equation $aAA + \overline{c + g} \times A + b = 0$. And theſe will not only obtrude themſelves into the co-efficients, but will often paſs from thence into the exponents. And, becauſe as yet we have not found out the

ways of managing them, it is neceſſary to avoid them as much as poſſible; and, among various methods, to adhere to that which ſhall be found moſt convenient.

For an example; let the equation be $abx x\dot{x} + bbyx\dot{x} + a^3y\dot{x} + aaby\dot{y} + a^3x\dot{y} = 0$. Make $y = Ax + p + B$, whence $\dot{y} = A\dot{x} + \dot{p}$. Here I chooſe to ſubſtitute inſtead of y rather than x, becauſe I foreſee the calculation will be ſhorter. Subſtituting, therefore, we ſhall have this equation following.

$$\begin{array}{llllll} abx^2\dot{x} & + bbpx\dot{x} & + bbBx\dot{x} & + a^3p\dot{x} & + a^3B\dot{x} & + a^2bAx\dot{p} \\ bbAx^2\dot{x} & & + 2a^3Ax\dot{x} & + a^2bAp\dot{x} & + a^2bAB\dot{x} & + a^3x\dot{p} \\ & & + a^2bA^2x\dot{x} & & & \\ + a^2bp\dot{p} & + a^2bB\dot{p} = 0. & & & & \end{array}$$

Here I obſerve, that, in this equation, if I make the firſt, third, fifth, and ſixth terms to vaniſh, we ſhould have the indeterminates ſeparable; for it would be $bbpx\dot{x} + a^3p\dot{x} + a^2bAp\dot{x} + a^2bp\dot{p} + a^2bB\dot{p} = 0$. And, dividing by p, $bbx\dot{x} + a^3\dot{x} + a^2bA\dot{x} = -a^2b\dot{p} - \frac{a^2bB\dot{p}}{p}$. Now, that the firſt may vaniſh, it is neceſſary that $a + bA = 0$, or $A = -\frac{a}{b}$. And, together with this will alſo vaniſh the fifth and ſixth, without any condition ariſing from thence. That the third ſhould vaniſh, it is neceſſary that $bbB + 2a^3A + aabAA = 0$. And ſubſtituting the value of A, it is $bbB - \frac{2a^4}{b} + \frac{a^4b}{bb} = 0$, that is, $B = \frac{a^4}{b^3}$. Therefore the ſubſtitution will be $y = -\frac{ax}{b} + p + \frac{a^4}{b^3}$, and the equation thence ariſing will be $bbx\dot{x} = -aab\dot{p} - \frac{a^6\dot{p}}{b^6p}$.

24. The method of this article conſiſts, firſt, in diſpoſing the propoſed equation in ſuch manner, as that the fluxions may continue accompanied with their indeterminates reſpectively, and that a half-ſeparation (as I may ſo ſay) may be made, by throwing into the common multipliers, or diviſors, ſuch quantities as hinder the operation. Then taking the integrals of the differential thus prepared, compounded of two variables, it muſt be made equal to one aſſumed variable, and, by means of an auxiliary equation, it muſt give a new form to the principal equation. Laſtly, taking obſervation by that which ſucceeds, the operation muſt be repeated till the deſired ſeparation is completed, or till we ſee the formula eludes all our endeavours.

This method has this advantage above the others, that in trying theſe ſubſtitutions, at the ſame time it informs us, which will be ſucceſsful and which uſeleſs. But it muſt be obſerved, that there are ſome equations which will not admit of

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the artifice of the prefent method, unlefs they are firft prepared according to art. The whole will be better underftood by the following Examples.

EXAMPLE I.

Let this equation be propofed, $\frac{x^3\dot{y} + y^3\dot{x}}{\overline{xx+yy} \times \sqrt{xx+yy-xxyy}} = \dot{z}$, in which $\dot{z}$ ftands for any function of x or y whatever. I fet afide the denominator, which is an affection common to the two terms which compofe the firft part of the equation, and the bare differential $x^3\dot{y} + y^3\dot{x}$ will remain. I divide $\dot{x}$ by x^3, and $\dot{y}$ by y^3, and then it will be $x^3\dot{y} + y^3\dot{x} = x^3y^3 \times \overline{\frac{\dot{y}}{y^3} + \frac{\dot{x}}{x^3}}$. Hence the propofed equation will take this new form, $\frac{x^3y^3}{\overline{xx+yy} \times \sqrt{xx+yy-xxyy}} \times \overline{\frac{\dot{x}}{x^3} + \frac{\dot{y}}{y^3}} = \dot{z}$. Having obtained this half-feparation, in which the fluxions $\dot{x}$, $\dot{y}$, appear combined fimply with the functions of their variables x^3, y^3, and the other terms conftitute, as it were, a foreign quantity, which has the appearance of a multiplier; I make $\frac{\dot{x}}{x^3} + \frac{\dot{y}}{y^3} = -\frac{\dot{p}}{a^3}$, and then, by integration, $\frac{a^3}{2xx} + \frac{a^3}{2yy} = p$. Now, finding the value, fuppofe of x, which will be $x = \frac{ya\sqrt{a}}{\sqrt{2yyp-a^3}}$, and fubftituting this inftead of x, and $-\frac{\dot{p}}{a^3}$ inftead of $\frac{\dot{x}}{x^3} + \frac{\dot{y}}{y^3}$ in the equation, it will be $-\frac{\dot{p} \times a\sqrt{a}}{2p\sqrt{2p-a^3}} = \dot{z}$. Wherefore, &c.

It may be recollected, that, taking a quantity at pleafure any how given by p, as $p = \frac{a^3}{2qq}$, it will be $\frac{a^3}{2qq} = \frac{a^3}{2xx} + \frac{a^3}{2yy}$, that is, $q = \frac{xy}{\sqrt{xx+yy}}$; by which, in an inftant, we may perceive infinite fubftitutions, which will promote the defired feparation of the variables. All the other poffible ones will be ufelefs, and will leave the variables as much blended and intermixed as before.

Moreover, let it be obferved, that it often happens with the fubftitutions here explained, that in one member of the equation there may remain fome function of one of the variables x or y; in which cafe, if $\dot{z}$ were given by the variable whofe function remains, one fimple divifion would anfwer the purpofe.

EXAMPLE II.

Let the equation be $\frac{2y\dot{y} + x\dot{y} + y\dot{x}}{a + x + y} = \dot{z}$, in which $\dot{z}$ is any how given by y. To reduce this equation to the method, I take the integral of the numerator of the fraction, that is, $yy + xy$, and make it equal to p. Now, making x and $\dot{x}$ to vanish out of the equation, by substituting their values, I shall have a new equation $\frac{\dot{p}}{a + \frac{p}{y}} = \dot{z}$, which is reduced to the following, $y\dot{p} - p\dot{z} = ay\dot{z}$. And this, being prepared according to the method, will be found to be $p \times \frac{\dot{p}}{p} - \frac{\dot{z}}{y} = a\dot{z}$. I make $\frac{\dot{p}}{p} - \frac{\dot{z}}{y} = \frac{\dot{q}}{q}$, and therefore $lp - \int\frac{\dot{z}}{y} = lq$. I make also $\int\frac{\dot{z}}{y} = ulm$, where lm is some constant logarithm. Then it will be $lp - lq = ulm$. And going on from logarithmic quantities to exponentials, it will be $\frac{p}{q} = m^u$. Therefore, in the reduced equation, making the substitutions of $\frac{\dot{q}}{q}$ instead of $\frac{\dot{p}}{p} - \frac{\dot{z}}{y}$, and of $m^u q$ instead of p, it will be $m^u \dot{q} = a\dot{z}$, that is, $\dot{q} = \frac{a\dot{z}}{m^u}$; in which the variables are separated, because both $\dot{z}$ and m^u are given by y.

EXAMPLE III.

Let the equation be $\frac{2xx\dot{x} + xy\dot{y} + yy\dot{x}}{x^4 + xxyy + a^4} = \frac{x\dot{x} + y\dot{y}}{\sqrt{xx + yy}}$. Before we attempt this formula, it will be best to reduce it. I observe that the second member is integrable, and it's integral is $\sqrt{xx + yy}$ (§ 10). Wherefore I make $\sqrt{xx + yy} = z$, and making y to vanish, finding that it's powers ascend to the square, and putting $zz - xx$ instead of yy, and $z\dot{z} - x\dot{x}$ instead of $y\dot{y}$, we shall have the equation $\frac{2x^2\dot{x} + xz\dot{z} - x^2\dot{x} + z^2\dot{x} - x^2\dot{x}}{xxzz + a^4} = \dot{z}$, that is, $\frac{xz\dot{z} + zz\dot{x}}{xxzz + a^4} = \dot{z}$; which, being

being prepared as usual, will be $\frac{z}{x^2z^2 + a^4} \times \overline{x\dot{z} + z\dot{x}} = \dot{z}$. I make $x\dot{z} + z\dot{x} = \dot{p}$, and, by integration, $xz = p$; and, making x to vanish, we shall have $\frac{z\dot{p}}{pp + a^4} = \dot{z}$, and, finally, $\frac{\dot{p}}{pp + a^4} = \frac{\dot{z}}{z}$.

EXAMPLE IV.

Let it be the last equation of the foregoing article, $-ap\dot{p} - f\dot{p} = \overline{aA + g} \times p\dot{y} + \overline{fA + b} \times \dot{y} + \overline{aA + c} \times y\dot{p}$, which I undertook to construct. This equation being prepared according to the method, and, for brevity, making $aA + g = e$, $fA + b = m$, $aA + c = n$, it will be reduced to this, $-\frac{ap\dot{p} + f\dot{p}}{ep + m} = y \times \overline{\frac{\dot{y}}{y} + \frac{n\dot{p}}{ep + m}}$. Therefore I put $\frac{\dot{y}}{y} + \frac{n\dot{p}}{ep + m} = \frac{\dot{q}}{q}$; and, by integration, $ly + \frac{n}{e} l\overline{p + \frac{m}{e}} = lq$. And therefore $y = \frac{q}{\overline{p + \frac{m}{e}}|^{\frac{n}{e}}}$.

And eliminating y, we shall have $-\frac{ap\dot{p} + f\dot{p}}{ep + m} = \frac{\dot{q}}{\overline{p + \frac{m}{e}}|^{\frac{n}{e}}}$, that is, $-\frac{ap\dot{p} + f\dot{p}}{ep + m} \times \overline{p + \frac{m}{e}}|^{\frac{n}{e}} = \dot{q}$.

EXAMPLE V.

Let the equation be this already prepared, $y^m \times \overline{x\dot{x} + y\dot{y}} = x^n \times \overline{y\dot{x} - x\dot{y}}$, which I write in this manner, $\frac{y^{m-2}}{x^n} \times \overline{x\dot{x} + y\dot{y}} = \frac{y\dot{x} - x\dot{y}}{yy}$, in order to make the second member integrable. In this I make use of a double substitution, and therefore I put $x\dot{x} + y\dot{y} = p\dot{p}$, and, by integration, $xx + yy = pp$. I put also $\frac{y\dot{x} - x\dot{y}}{yy} = \dot{q}$, and by integration, $\frac{x}{y} = q$. Making the substitutions, we shall have $\frac{y^{m-2}}{x^n} \times p\dot{p} = \dot{q}$. But $yy = pp - xx$, and $xx = qqyy$, so that it will

will be $yy = pp - qqyy$, that is, $yy = \frac{pp}{1 + qq}$, and $y^{m-2} = \frac{p^{m-2}}{\overline{1+qq}|^{\frac{m-2}{2}}}$, and

$x^n = \frac{q^n p^n}{\overline{1+qq}|^{\frac{n}{2}}}$. Wherefore, substituting these values of y^{m-2} and x^n, we

shall have $p^{m-2-1}\dot{p} = q^n\dot{q} \times \overline{1+qq}|^{\frac{m-n-2}{2}}$.

EXAMPLE VI.

Let the equation be $\frac{2x\dot{y} - 2y\dot{x}}{\overline{x-y}|^2} = \dot{z}$; in which $\dot{z}$ is any how given by x or y. I observe that the numerator of the first member is integrable, if it were divided by xx, and that it's integral would be $\frac{2y}{x}$, and therefore I thus dispose the equation, $\frac{1}{\overline{x-y}|^2} \times \frac{2x\dot{y} - 2y\dot{x}}{xx} = \frac{\dot{z}}{xx}$. Put $\frac{2y}{x} = p$, whence it will be $\frac{2x\dot{y} - 2y\dot{x}}{xx} = \dot{p}$, and the equation will be changed into this following, $\frac{\dot{p}}{\overline{x-y}|^2} = \frac{\dot{z}}{xx}$. But $2y = px$, and $yy = \frac{1}{4}ppxx$; so that, making the substitutions, it will be $\frac{\dot{p}}{xx - pxx + \frac{1}{4}ppxx} = \frac{\dot{z}}{xx}$; and, multiplying by xx, it is $\frac{\dot{p}}{1 - p + \frac{1}{4}pp} = \dot{z}$, in which the variables are separated. I go on to the integration; and therefore it will be $\frac{2}{1 - \frac{1}{2}p} + c = \int\dot{z}$; and, restoring the value of p, it is $\frac{2}{1 - \frac{y}{x}} + c = \int\dot{z}$, and reducing to a common denominator, it is $\frac{2x + cx - cy}{x - y} = \int\dot{z}$. If we make the constant $c = 0$, we shall have $\frac{2x}{x - y} = \int\dot{z}$; and, making $c = -2$, it will be $\frac{2y}{x - y} = \int\dot{z}$, which is another integral of the proposed formula different from the first. Lastly, putting $c = -1$, a third integral will arise, $\frac{x + y}{x - y} = \int\dot{z}$.

25. The

25. The method I now undertake to explain, although much limited and confined, is yet of great use in some particular cases. By this the variables may be separated in the canonical equation $a\dot{y} = yp\dot{x} + by^n q\dot{x}$, in which the quantities p, q, are to be understood as any how given by x. The quantities a, b, are constant; the signs may be positive or negative at pleasure, and the exponent n may be integer, fraction, positive, negative, or even nothing. In this equation, then, make $y = zu$, where z and u are two new variables; and, by taking the fluxions, it will be $\dot{y} = z\dot{u} + u\dot{z}$; and, by substituting, instead of $\dot{y}$, y, and y^n, their values $z\dot{u} + u\dot{z}$, zu, and $u^n z^n$, we shall have the equation $az\dot{u} + au\dot{z} = uzp\dot{x} + bz^n u^n q\dot{x}$, in which, if two terms shall vanish, the indeterminates will be separated. To do which, let us feign an equation between the two terms $au\dot{z} = uzp\dot{x}$, then $\frac{a\dot{z}}{z} = p\dot{x}$, and, by integration, $alz = \int p\dot{x}$; and, proceeding from logarithms to exponential quantities, it is $z^a = m^{\int p\dot{x}}$, or $z = m^{\frac{\int p\dot{x}}{a}}$, supposing $lm = 1$. This last equation shows us the value of z, and informs us, that, to reduce the equation proposed to two terms only, and to cause the other two to destroy each other, instead of $y = zu$, we ought to put $y = um^{\frac{\int p\dot{x}}{a}}$, that is, $\frac{y}{u} = m^{\frac{\int p\dot{x}}{a}}$, or $ly - lu = \int \frac{p\dot{x}}{a}$. And, by differencing, $\frac{a\dot{y}}{y} - \frac{a\dot{u}}{u} = p\dot{x}$, and therefore $a\dot{y} = yp\dot{x} + \frac{ay\dot{u}}{u}$. Therefore, in the canonical equation $a\dot{y} = yp\dot{x} + by^n q\dot{x}$, instead of $\dot{y}$, I substitute it's value now found, and it will be $yp\dot{x} + \frac{ay\dot{u}}{u} = yp\dot{x} + by^n q\dot{x}$, that is, $\frac{ay\dot{u}}{u} = by^n q\dot{x}$, and therefore $\frac{a\dot{u}}{u} = by^{n-1} q\dot{x}$. But $y = zu$, and $y^{n-1} = z^{n-1} u^{n-1}$; whence, finally, it will be $\frac{a\dot{u}}{u^n} = bz^{n-1} q\dot{x}$; in which equation the variables will be separated, because z is supposed given by x. When we came to the equation $alz = \int p\dot{x}$, it is plain, that if p given by x is such, that the integral $\int p\dot{x}$ depends on the quadrature of the hyperbola, that is, on the logarithms, and the quantity a is any number whatsoever, the relation of z to x will be algebräical, and in all other cases transcendental.

And here it may be observed, that, in order to have a given equation come under the case of the canonical formula, it is necessary that the following conditions should take place. First, that the fluxion $\dot{y}$ may be alone, or, at least, multiplied by a constant, on one side of the equation. Then, that, on the other side, the first term may contain the fluxion $\dot{x}$, multiplied by any function of x expressed by p, and by the indeterminate y. Then, that, in the other term, the quantity $q\dot{x}$ given by x may be multiplied by a power of y. In a word, making

making the divifion by y, it is required, that, on one fide of the equation, there may remain the logarithmical fluxion $\frac{a\dot{y}}{y}$, and, on the other fide, the firft term may be free from the indeterminate y, and the fecond multiplied by the dignity y^{n-1}. If any one of thefe requifites be wanting, this method cannot take place; as we fhould not have them in the following equations, $a\dot{y} = yyp\dot{x} + by^nq\dot{x}$, and $a\dot{y} = yp\dot{x} + \overline{ayy + y^3} \times q\dot{x}$.

But fome formulæ are very eafily reduced to the canon, by a little preparation only. For example, take this equation $a\dot{y} = yp\dot{x} + byq\dot{x} + yyq\dot{x}$. Confider that the quantity $p\dot{x} + bq\dot{x}$, multiplied by y, and that the binomial $p + bq$ is given by x, fo that in it's place may be fubftituted the quantity r, alike given by x; the expreffion then will be changed into the following, $a\dot{y} = yr\dot{x} + yyq\dot{x}$, in which the method here explained will take place. And this will be fufficient to fhow the way of operation in all like cafes.

EXAMPLE I.

Let the equation be $a\dot{y} = \frac{fy\dot{x}}{x} + yy\dot{x}$. Make $y = zu$, and therefore $a\dot{y} = az\dot{u} + au\dot{z}$. And, making the neceffary fubftitutions, we fhall have $az\dot{u} + au\dot{z} = \frac{fuz\dot{x}}{x} + zzuu\dot{x}$. Let $au\dot{z} = \frac{fuz\dot{x}}{x}$, that is, $\frac{a\dot{z}}{z} = \frac{f\dot{x}}{x}$; and integrating, it will be $alz = flx$, and therefore $z^a = x^f$.

If the conftants a, f, fhall be rational numbers, whole or fracted, affirmative or negative, z will be given algebräically by x. For example, make $a = 1$, $f = 2$, fo that it may be $z = xx$. Then eliminating the terms $au\dot{z}$, $\frac{fuz\dot{x}}{x}$, there will remain the two, $az\dot{u} = zzuu\dot{x}$. But $z = xx$, therefore it will be $\frac{a\dot{u}}{uu} = xx\dot{x}$, an equation in which the variables are feparated.

In proceeding to the integration, it will be $-\frac{a}{u} + c = \frac{1}{3}x^3$. But $u = \frac{y}{z} = \frac{y}{xx}$, and therefore $-\frac{axx}{y} + c = \frac{1}{3}x^3$; that is, $3cy - 3axx = x^3y$; which is the algebräical equation concealed under the propofed differential.

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EXAMPLE II.

Let the equation be $\dot{y} = \frac{ay\dot{x}}{xx - aa} + \frac{y^3\dot{x}}{x^3}$. Make, as above, $y = zu$, and $\dot{y} = z\dot{u} + u\dot{z}$; then, making the fubftitutions, we fhall have $z\dot{u} + u\dot{z} = \frac{azu\dot{x}}{xx - aa} + \frac{z^3u^3\dot{x}}{x^3}$. And, fuppofing $u\dot{z} = \frac{azu\dot{x}}{xx - aa}$, that is, $\frac{\dot{z}}{z} = \frac{a\dot{x}}{xx - aa}$, or $z = m^{\int \frac{a\dot{x}}{xx - aa}}$, we fhall have the equation $z\dot{u} = \frac{z^3u^3\dot{x}}{x^3}$, or $\frac{\dot{u}}{u^3} = \frac{zz\dot{x}}{x^3}$, in which the variables are feparated, z being given by x. But it may be obferved, that the quantity $\frac{a\dot{x}}{xx - aa}$ may be reduced to a logarithmic fluxion, by making $x = \frac{\overline{a + n} \times a}{a - n}$; wherefore, making the due fubftitutions, it will be $\frac{a\dot{x}}{xx - aa} = \frac{\dot{n}}{2n}$. Whence $\frac{\dot{z}}{z} = \frac{\dot{n}}{2n}$, and therefore $zz = n = \frac{a \times \overline{x - a}}{x + a}$. And, putting this value, inftead of zz, in the final equation, we fhall have $\frac{\dot{u}}{u^3} = \frac{ax\dot{x} - aa\dot{x}}{x^4 + ax^3}$.

Without making the fubftitution of $x = \frac{\overline{a + n} \times a}{a - n}$, the quantity $\frac{a\dot{x}}{xx - aa}$ may be reduced to a logarithmical fluxion, by means of § 21, Book III; and we fhould have $\frac{a\dot{x}}{xx - aa} = -\frac{\dot{x}}{2 \times \overline{x + a}} + \frac{\dot{x}}{2 \times \overline{x - a}} = \frac{\dot{z}}{z}$, and confequently $zz = \frac{x - a}{x + a}$.

EXAMPLE III.

Let the equation be $\dot{y} = -\frac{y\dot{x}}{x} + y^m\dot{x}$. Make $y = zu$, $\dot{y} = z\dot{u} + u\dot{z}$; therefore, fubftituting, it will be $z\dot{u} + u\dot{z} = -\frac{uz\dot{x}}{x} + u^m z^m \dot{x}$. Suppofing $u\dot{z} = -\frac{uz\dot{x}}{x}$, or $\frac{\dot{z}}{z} = -\frac{\dot{x}}{x}$, and, by integration, $z = \frac{a}{x}$; we fhall have the equation $z\dot{u} = z^m u^m \dot{x}$, that is, $\frac{\dot{u}}{u^m} = z^{m-1}\dot{x}$, or $\frac{\dot{u}}{u^m} = \frac{a^{m-1}\dot{x}}{x^{m-1}}$.

EXAMPLE IV.

Sometimes a two-fold operation is neceſſary; as in certain equations which have more than three terms. Wherefore, let the equation be $x\dot{y} + y\dot{x} = a\dot{u} + x\dot{u}$, and let u be any how given in the terms of y. I diſpoſe the equation in the following manner, $a\dot{u} + x\dot{u} - x\dot{y} = y\dot{x}$, or $\frac{a\dot{u}}{y} + \frac{x\dot{u}}{y} - \frac{x\dot{y}}{y} = \dot{x}$. Make $x = pq$, and $\dot{x} = p\dot{q} + q\dot{p}$; then, making the ſubſtitutions, it will be $\frac{a\dot{u}}{y} + \frac{pq\dot{u}}{y} - \frac{pq\dot{y}}{y} = p\dot{q} + q\dot{p}$. If any one would reduce the formula by one operation only, he muſt put $\frac{pq\dot{u}}{y} - \frac{pq\dot{y}}{y} = p\dot{q}$, that is, $\frac{\dot{u}}{y} - \frac{\dot{y}}{y} = \frac{\dot{q}}{q}$; by which we find q given by y. But the operation will be performed more neatly in the following manner. Make $-\frac{pq\dot{y}}{y} = p\dot{q}$, then $-\frac{\dot{y}}{y} = \frac{\dot{q}}{q}$, and, by integration, $\frac{a}{y} = q$. Taking, therefore, the other terms of the equation $\frac{a\dot{u}}{y} + \frac{pq\dot{u}}{y} = q\dot{p}$, and, inſtead of q, ſubſtituting it's value $\frac{a}{y}$, it will be $\frac{a\dot{u}}{y} + \frac{ap\dot{u}}{yy} = \frac{a\dot{p}}{y}$, that is, $\dot{u} + \frac{p\dot{u}}{y} = \dot{p}$. Make $p = mn$, then $\dot{p} = m\dot{n} + n\dot{m}$, and making the ſubſtitution, it will be $\dot{u} + \frac{mn\dot{u}}{y} = m\dot{n} + n\dot{m}$. Suppoſe $\frac{mn\dot{u}}{y} = m\dot{n}$, that is, $\frac{\dot{u}}{y} = \frac{\dot{n}}{n}$. Therefore n will be given by y, and in the remaining equation, after the terms $\frac{mn\dot{u}}{y}$, $m\dot{n}$, have been eliminated, that is, in the equation $\dot{u} = n\dot{m}$, the variables will be ſeparated, and it will be $\frac{\dot{u}}{n} = \dot{m}$.

26. Still, after another manner, the variables may be ſeparated in the canonical equation $\dot{y} = py\dot{x} + qy^n\dot{x}$. Make $p\dot{x} = \frac{\dot{z}}{\overline{1-n} \times z}$, $\dot{x} = \frac{\dot{z}}{\overline{1-n} \times pz}$; Making the ſubſtitutions, it will be $\dot{y} = \frac{y\dot{z}}{\overline{1-n} \times z} + \frac{qy^n\dot{z}}{\overline{1-n} \times pz}$; that is, $\dot{y} = \frac{py\dot{z} + qy^n\dot{z}}{\overline{1-n} \times pz}$, or $\overline{1-n} \times pz\dot{y} = py\dot{z} + qy^n\dot{z}$; and therefore, dividing by py^n, it is $\frac{\overline{1-n} \times z\dot{y} - y\dot{z}}{y^n} = \frac{q\dot{z}}{p}$. Laſtly, dividing by zz, it will be

$\frac{\overline{1-n} \times zy^{-n}\dot{y} - y^{1-n}\dot{z}}{zz} = \frac{q\dot{z}}{pzz}$, and, by integration, $\frac{y^{1-n}}{z} = \int\frac{q\dot{z}}{pzz}$, that is, $y^{1-n} = z\int\frac{q\dot{z}}{pzz}$. And, becaufe p and q are fuppofed to be given by x; and z alfo, by the fubftitution of $p\dot{x} = \frac{\dot{z}}{\overline{1-n} \times z}$, is given by x; the variables will be feparated, at leaft tranfcendentally.

Refuming, therefore, the equation of the firft example, $a\dot{y} = \frac{fy\dot{x}}{x} + y^2\dot{x}$, that is, $\dot{y} = \frac{fy\dot{x}}{ax} + \frac{yy\dot{x}}{a}$, it will be $p = \frac{f}{ax}$, $q = \frac{1}{a}$, $n = 2$. So that, fubftituting thefe values in the final equation $y^{1-n} = z\int\frac{q\dot{z}}{pzz}$, it will be $\frac{1}{y} = z\int\frac{x\dot{z}}{fzz}$, and the fubftitution $p\dot{x} = \frac{\dot{z}}{\overline{1-n} \times z}$ will be $\frac{f\dot{x}}{ax} = -\frac{\dot{z}}{z}$. And, making $f = 2$, $a = 1$, we fhall have $\frac{2\dot{x}}{x} = -\frac{\dot{z}}{z}$, that is, $z = \frac{1}{xx}$. And therefore $\frac{1}{y} = \frac{1}{xx}\int -xx\dot{x}$. And, by integration, $\frac{1}{y} = \frac{1}{xx} \times \overline{-\frac{1}{3}x^3 + c}$, that is, $3cy - 3xx = x^3y$, as before. And fo we may proceed with the other Examples.

EXAMPLE V.

Let the equation be $ax^4y\dot{y} - bx^4y\dot{y} = ayyx^3\dot{x} - byyx^3\dot{x} + a^6\dot{x} - x^6\dot{x}$, which, divided by $ax^4y - bx^4y$, will be found to be $\dot{y} = \frac{y\dot{x}}{x} + \frac{a^6\dot{x} - x^6\dot{x}}{\overline{a-b} \times x^4y}$, which is a cafe of the canonical equation. Therefore it will be $p = \frac{1}{x}$, $q = \frac{a^6 - x^6}{\overline{a-b} \times x^4}$, $n = -1$. And, by fubftitution, $p\dot{x} = \frac{\dot{z}}{\overline{1-n} \times z}$ will be $\frac{\dot{x}}{x} = \frac{\dot{z}}{2z}$, whence $z = xx$. Then, putting thefe values in the final canonical equation, $y^{1-n} = z\int\frac{q\dot{z}}{pzz}$, we fhall have $yy = xx\int\frac{\overline{a^6 - x^6} \times 2x\dot{x}}{\overline{a-b} \times x^4 \times x^3}$, in which the variables are feparated.

 27. If

27. If the canonical equation were $y^{n-1}\dot{y} = p\dot{x} + qy^n\dot{x}$, where p and q, in a like manner, are any how given by x; the indeterminates may be ſeparated by making $q\dot{x} = \frac{\dot{z}}{nz}$, and $\dot{x} = \frac{\dot{z}}{nqz}$. For, making the ſubſtitutions, it will be $y^{n-1}\dot{y} = \frac{p\dot{z}}{nqz} + \frac{y^n\dot{z}}{nz}$, that is, $\frac{nzy^{n-1}\dot{y} - y^n\dot{z}}{z} = \frac{p\dot{z}}{qz}$; and, dividing by z, $\frac{nzy^{n-1}\dot{y} - y^n\dot{z}}{zz} = \frac{p\dot{z}}{qzz}$; and, by integration, $\frac{y^n}{z} = \int\frac{p\dot{z}}{qzz}$, that is, $y^n = z\int\frac{p\dot{z}}{qzz}$, an equation in which the variables are ſeparated.

For an example, let the equation be $2a^2xy\dot{y} = aayy\dot{x} + 2bx^3\dot{x}$, that is, $y\dot{y} = \frac{bxx\dot{x}}{aa} + \frac{yy\dot{x}}{2x}$. It will be $n = 2$, $p = \frac{bxx}{aa}$, $q = \frac{1}{2x}$, and therefore we ſhall have $\frac{yy}{z} = \int\frac{2bx^3\dot{z}}{aazz}$. But $q\dot{x} = \frac{\dot{x}}{2x} = \frac{\dot{z}}{2x}$, and $x = z$. Therefore it will be $\frac{yy}{x} = \int\frac{2bx\dot{x}}{aa}$, and, by integration, $\frac{yy}{x} = \frac{bxx}{aa} \pm c$; an algebräical curve.

Alſo, the general formula $y^{n-1}\dot{y} = p\dot{x} + qy^n\dot{x}$ might be conſtructed, and conſequently the particular example, by means of the method at § 24.

28. Before I finiſh this Section, I ſhall add one obſervation, that ſometimes the indeterminates are involved and mingled with differential quantities, when it may be allowed to modify the co-efficients; and this ſucceeds eſpecially when the exponents are formed of the co-efficients; and thus making a kind of circuit in the reduction. This artifice chiefly takes place in Phyſico-mathematical Problems, in which magnitudes of very different kinds mingling together, we are more at liberty to make uſe of ſuch conſtant quantities, as beſt ſerve the preſent purpoſe.

For an example, I ſhall propoſe to myſelf this equation, $x^m\dot{x} + \overline{by + yy} \times \frac{c\dot{x}}{x} = y\dot{y}$, which, being prepared according to the method of § 24, will be $x^m\dot{x} + \frac{bcy\dot{x}}{x} = yy \times \overline{\frac{\dot{y}}{y} - \frac{c\dot{x}}{x}}$. Make, then, $\frac{\dot{y}}{y} - \frac{c\dot{x}}{x} = \frac{\dot{p}}{p}$, and we ſhall have the value of $y = px^c$, and $yy = ppx^{2c}$. Theſe values, conveniently ſubſtituted, will give the equation $x^m\dot{x} + bcpx^{c-1}\dot{x} = x^{2c}p\dot{p}$; and, dividing by x^{2c}, it will be $x^{m-2c}\dot{x} + bcpx^{-c-1}\dot{x} = p\dot{p}$. Here it is plain, that, an equality being given between the exponents of the indeterminate x, that is, between $m - 2c$ and $-c - 1$, the variables will be ſeparate, the *homogeneum comparationis* $p\dot{p}$ being only to be divided by the binomial $1 + bcp$. Now, putting

$m - 2c = -c - 1$, it follows $m + 1 = c$; fo that, expounding the conftant c by $m + 1$, we fhall have our defire. If c reprefents unity, which we are at liberty to fuppofe, it will be $m = 0$; and if $c = 2$, it will be $m = 1$. And fo we may go on.

The artifice here explained may be applied to all other equations of a like kind; for example, to this following, $x^m\dot{x} + \frac{cby^n\dot{x}}{x} + \frac{gy^r\dot{x}}{x} = y^t\dot{y}$. For, putting $t = r - 1$, or $= n - 1$, the formula will be thence abbreviated by making ufe of the logarithms.

SECT. III.

Of the Conftruction of more Limited Equations, by the Help of various Subftitutions.

29. In the equation $\overline{x^n\dot{x} \pm ay^n\dot{y}} \times p = \overline{x\dot{y} - y\dot{x}} \times q$, the indeterminates are always feparable; where p and q are promifcuoufly given by y and x after any manner; algebräically, when, in every term of the quantity p, the Sum of the exponents of x and y is the fame, and thus likewife in every term of the quantity q; but it is not required that the fum fhould be the fame in p and q.

The fubftitutions to be made are $y = tz^{\frac{2}{n+1}}$, and $x = t \times \overline{a^3 \mp azz}^{\frac{1}{n+1}}$. Thefe being fubftituted, refpectively, inftead of x, $\dot{x}$, y, $\dot{y}$, and making the neceffary operations, after a very long calculation we fhall come to this equation,

$$t^{n-2}\dot{t} = \frac{\frac{2}{n+1} z^{\frac{1-n}{n+1}} \dot{z} \times \frac{q}{p}}{\overline{a^3 \mp azz}^{\frac{n}{n+1}}}.$$

Now, becaufe it is known, that, in every term of p, the fum of the exponents of x and y is equal, as alfo in every term of q; making in them the fubftitutions of the values given by t and z; in every term of p, t will have the fame power, as alfo in every term of q a fame power; that is to fay, that the *homogeneum comparationis* will be multiplied by a pofitive or negative power of t, or the firft member will be multiplied or divided by that power, and therefore the variables will be feparated.

As,

As, for example, let the equation be $\overline{x\dot{x} + ay\dot{y}} \times \sqrt{y} = \overline{x\dot{y} - y\dot{x}} \times \sqrt{a}$; it will be $n = 1$, $p = \sqrt{y}$, $q = \sqrt{a}$, and therefore $\frac{\dot{t}}{t} = \frac{\dot{z}\sqrt{a}}{\sqrt{a^3 - azz} \times \sqrt{y}}$. But $y = tz$; therefore it will be $\frac{\dot{t}}{\sqrt{t}} = \frac{\dot{z}\sqrt{a}}{\sqrt{a^3z - az^3}}$.

In the ſame equation the indeterminates may be ſeparated, when alſo the exponent n is negative; that is, when the equation is this, $\overline{x^{-n}\dot{x} \pm ay^{-n}\dot{y}} \times p = \overline{x\dot{y} - y\dot{x}} \times q$; and the ſubſtitutions are $y = tz^{\frac{2}{1-n}}$, and $x = t \times \overline{a^3 \mp azz}|^{\frac{1}{1-n}}$. Theſe will give the equation $t^{-n-2}\dot{t} = \frac{\frac{2}{1-n} z^{\frac{1+n}{1-n}}\dot{z} \times \frac{q}{p}}{\overline{a^3 \mp azz}|^{\frac{-n}{1-n}}}$, the ſame as that above, only with the ſigns of n changed. And though the equation were alſo thus expreſſed, $y^n\dot{x} \pm ax^n\dot{y} \times \frac{p}{x^ny^n} = \overline{x\dot{y} - y\dot{x}} \times q$; it follows that this alſo is conſtructible by the ſame ſubſtitutions.

30. Let the equation be more general, $\overline{x^n\dot{x} \pm ay^{\frac{-n-1-c}{c}}\dot{y}} \times p = \overline{x\dot{y} + cy\dot{x}} \times q$. The variables will always be ſeparated by making the ſubſtitutions of $y = t^s z^{\frac{r}{n+1}}$, and $x = t^{-\frac{s}{c}} \times \overline{a \pm acz^{-\frac{r}{c}}}|^{\frac{1}{n+1}}$, where s and r are numbers aſſumed at pleaſure; ſuppoſing, however, this condition, that the quantities p, q, are given algebräically, and in ſuch a manner, that, in every term of the quantity p, the exponent of y, taken as often as the number c denotes, may exceed, or be exceeded by, the exponent of x in the ſame exceſs; and ſo in every term of the quantity q; but it is no matter that the exceſs in p ſhall be the ſame as in q. Thus, for example, if $c = 3$, it may be $p = by^2x^4 + fy^9x^{25}$, &c.; and it may be $q = gy^{\frac{1}{2}}x^3 - hy^{10}x^{\frac{63}{2}}$, &c. It is eaſy to perceive, that it cannot be $c = 0$.

Making the due ſubſtitutions, inſtead of x and y, in the propoſed equation, we ſhall have this following, $-\frac{s}{c}t^{\frac{-sn-c-cs}{c}}\dot{t} = \frac{\frac{r}{n+1} \times z^{\frac{r-n-1}{n+1}}\dot{z} \times \frac{q}{p}}{\overline{a \pm acz^{-\frac{r}{c}}}|^{\frac{n}{n+1}}}$.

For

For example, let it be $\overline{x\dot{x} + ay^{-3}\dot{y}} \times \frac{1}{y} = \overline{x\dot{y} + y\dot{x}} \times x$. Make $s = 1$, $r = 2$; it will be $n = 1$, $c = 1$, $p = \frac{1}{y}$, $q = x$; and, making the fubftitutions in the laft equation found above, we fhall have $-t^{-3}\dot{t} = \frac{\dot{z} \times xy}{\overline{a + az^{-2}})^{\frac{1}{2}}}$.

But, by the fubftitutions made, $x = t^{-1} \times \overline{a + az^{-2}})^{\frac{1}{2}}$, and $y = tz$. Therefore $xy = z \times \overline{a + az^{-2}})^{\frac{1}{2}}$. Whence we fhall have $-\frac{\dot{t}}{t^3} = z\dot{z}$.

31. But let the equation be ftill more general, $\overline{x^n\dot{x} \pm ay^{\frac{-nf-c-f}{c}}\dot{y}} \times p = \overline{fx\dot{y} + cy\dot{x}} \times q$, which comprehends, as particular cafes, the two canonical equations of the foregoing articles; that is, that of § 30, when it is $f = 1$; and that of § 29, when it is $f = 1$, and $c = -1$.

The indeterminates are feparated by means of the fubftitutions $y = t^{\frac{s}{f}} z^{\frac{r}{f \times \overline{n+1}}}$, and $x = t^{-\frac{s}{c}} \times \overline{a \pm \frac{acz^{-\frac{r}{c}}}{f}}\Big)^{\frac{1}{n+1}}$; the condition concerning the quantities p and q being fuch, that, in thefe, the exponent of y being multiplied by c, may exceed, or be exceeded by, the exponent of x multiplied by f, by the fame excefs in each term. The fame quantities p, q, may alfo be fractions, or mixed with fractions, and rational or irrational integers, whatever they may be. And the indeterminates will always be feparable in the equations, provided that p and q are given by x and y in fuch a manner, that, the affigned fubftitutions being made, fuch quantities may arife in their place, that they may be the product of two, one of which fhall contain z, and not t, the other t and not z.

The faid fubftitutions being made, we fhall have this formula,

$$-\frac{s}{c} t^{\frac{-fc-fsn-sc}{cf}} \dot{t} = \frac{\frac{r}{n+1} \times z^{\frac{r-fn-f}{fn+f}} \dot{z} \times \frac{q}{p}}{\overline{a \pm \frac{acz^{-\frac{r}{c}}}{f}}\Big)^{\frac{n}{n+1}}}.$$

EXAMPLE I.

Let the equation be $\overline{xx\dot{x} + ay^8\dot{y}} \times y = -\overline{3x\dot{y} + y\dot{x}} \times ax$. Let it be, as before, $s = 1$, $r = 2$, it will be $f = -3$, $c = 1$, $n = 2$, $q = ax$, $p = y$; and, making the fubftitutions in the laft formula found above, we fhall have

$$-t^{-\frac{8}{3}}\dot{t} = \frac{\frac{2}{3}z^{-\frac{11}{9}}\dot{z} \times \frac{ax}{y}}{\overline{a - \frac{1}{3}az^{-2}}\big|^{\frac{2}{3}}}.$$ But $y = t^{-\frac{1}{3}}z^{-\frac{2}{9}}$, $x = t^{-1} \times \overline{a - \frac{1}{3}az^{-2}}\big|^{\frac{1}{3}}$;

therefore it will be $-\frac{\dot{t}}{tt} = \frac{2a\dot{z}}{3z \times \overline{a - \frac{1}{3}az^{-2}}\big|^{\frac{1}{3}}}$; as was to be found.

EXAMPLE II.

Let the equation be $\overline{x^{\frac{1}{2}}\dot{x} + ay^{-2}\dot{y}} \times \overline{ay^{\frac{1}{2}}x + yyx^{\frac{13}{4}}} = \overline{2x\dot{y} + 3y\dot{x}} \times \overline{y^{\frac{1}{3}}x - yxx}$. Let $s = 1$, $r = 1$; it will be $c = 3$, $f = 2$, $n = \frac{1}{2}$, $p = ay^{\frac{1}{2}}x + yyx^{\frac{13}{4}}$, $q = y^{\frac{1}{3}}x - yxx$. And, making the fubftitutions, it will be

$$\frac{-\frac{1}{3}\dot{t}}{t^{\frac{19}{12}}} = \frac{\frac{2}{3}z^{-\frac{5}{9}}\dot{z} \times \overline{a + \frac{3}{2}az^{-\frac{1}{3}}}\big|^{\frac{1}{3}} - \frac{2}{3}z^{-\frac{1}{3}}\dot{z} \times \overline{a + \frac{3}{2}az^{-\frac{1}{3}}}}{az^{\frac{1}{6}} \times \overline{a + \frac{3}{2}az^{-\frac{1}{3}}}\big|^{\frac{2}{3}} + z^{\frac{2}{3}} \times \overline{a + \frac{3}{2}az^{-\frac{1}{3}}}\big|^{\frac{13}{6}}} *,$$

in which the variables are feparated, as was required.

32. In the equations

$$\begin{aligned}
&(1)\quad pxy^{n-1}\dot{y} = py^n\dot{x} + q\dot{x},\\
&(2)\quad pxy^{n-1}\dot{y} = -py^n\dot{x} + q\dot{x},\\
&(3)\quad apxy^{n-1}\dot{y} = bpy^n\dot{x} + q\dot{x},\\
&(4)\quad apxy^{n-1}\dot{y} = -bpy^n\dot{x} + q\dot{x},
\end{aligned}$$

where p and q are any how given by x; the indeterminates may be feparated, by putting, as to the firft, $y = xz$; as to the fecond, $y = \frac{z}{x}$; as to the third, $y = x^{\frac{b}{a}}z$; as to the fourth, $y = x^{-\frac{b}{a}}z$.

* This equation evidently admits of a fimpler form. EDITOR.

As,

As, for example, let the equation be $2bbxyy\dot{y} - 2x^3yy\dot{y} = bx^4\dot{x} - 3bby^3\dot{x} + 3xxy^3\dot{x}$, which I write thus, $\overline{bb - xx} \times 2xyy\dot{y} = bx^4\dot{x} + \overline{bb - xx} \times -3y^3\dot{x}$. This being referred to the laſt of the four canonical equations, it will be $p = bb - xx$, $a = 2$, $n = 3$, $b = 3$, $q = bx^4$. Therefore we muſt put $y = \frac{z}{x^{\frac{3}{2}}}$, $\dot{y} = \frac{x^{\frac{3}{2}}\dot{z} - \frac{3}{2}zx^{\frac{1}{2}}\dot{x}}{x^3}$, $yy = \frac{zz}{x^3}$, $y^3 = \frac{z^3}{x^{\frac{9}{2}}}$. And, making the ſubſtitutions, we ſhall have $\overline{2bbx - 2x^3} \times \frac{x^{\frac{3}{2}}z^2\dot{z} - \frac{3}{2}x^{\frac{1}{2}}z^3\dot{x}}{x^6} = bx^4\dot{x} + \overline{3bb - 3xx} \times - \frac{z^3\dot{x}}{x^{\frac{9}{2}}}$; that is, $\overline{2bb - 2xx} \times \overline{xzz\dot{z} - \frac{3}{2}z^3\dot{x}} = bx^{\frac{17}{2}}\dot{x} + \overline{3bb - 3xx} \times - z^3\dot{x}$; and, making the uſual multiplications, it will be $2bbxzz\dot{z} - 2x^3zz\dot{z} = bz^{\frac{17}{2}}\dot{x}$, that is, $zz\dot{z} = \frac{bx^{\frac{17}{2}}\dot{x}}{2bbx - 2x^3}$ *.

33. Let the equation be $ax\dot{y} + by\dot{x} + cy^nx^{m-1}\dot{x} + fx^my^{n-1}\dot{y} = 0$. In this the indeterminates may be ſeparated, in general, by putting $x = u^{n-1}z^{n-1}$, and $y = z^{1-m}$; for, making the neceſſary operations, we ſhall come to the equation $\overline{1 - m} \times \overline{a\dot{z} + fu^{mn-m-n+1}\dot{z}} + \overline{n - 1} \times \overline{b\dot{z} + cu^{mn-m-n+1}\dot{z}} = \overline{n - 1} \times \overline{- bzu^{-1}\dot{u} - czu^{mn-m-n}\dot{u}}$, that is, $\frac{\dot{z}}{z} = \frac{\overline{n-1} \times \overline{- bu^{-1}\dot{u} - cu^{mn-m-n}\dot{u}}}{\overline{1-m} \times \overline{a + fu^{mn-m-n+1}} + \overline{n-1} \times \overline{b + cu^{mn-m-n+1}}}$.

As, for example, let the equation be $a^3x\dot{y} - b^3y\dot{x} = cyyx\dot{x} - fxxy\dot{y}$. Then it will be $n = 2$, $m = 2$. Therefore I put $x = \frac{uz}{a}$, and $y = \frac{aa}{z}$, that is, $x = \frac{au}{y}$, and therefore $\dot{x} = \frac{ay\dot{u} - au\dot{y}}{yy}$. Whence, making the due ſubſtitutions, we ſhall have $\frac{a^4u\dot{y}}{y} - b^3 \times \frac{ay\dot{u} - au\dot{y}}{y} = \frac{caay u\dot{u} - caauu\dot{y}}{y} - \frac{faauu\dot{y}}{y}$, that is, $a^4u\dot{y} + ab^3u\dot{y} + aacuu\dot{y} + faauu\dot{y} = ab^3y\dot{u} + aacyu\dot{u}$, and therefore $\frac{\dot{y}}{y} = \frac{ab^3\dot{u} + aacu\dot{u}}{a^4u + ab^3u + aacuu + aafuu}$ *.

34. Let the equation be $\frac{y\dot{x}}{\overline{bx^{\frac{1}{m}} + ay^nx^r}|^m} = \dot{y}$; or, more generally, $\frac{x^{mt-1}y\dot{x}}{\overline{bx^t + ay^nx^r}|^m} = \dot{y}$. The indeterminates will be ſeparated by putting $\overline{bx^t + ay^nx^r}|^m = zx^{mt}$.

* See the Note at the bottom of the preceding page.

Whence $y = \frac{\overline{z^{\frac{1}{m}}x^{t-r} - bx^{t-r}}\big)^{\frac{1}{n}}}{a^{\frac{1}{n}}}$, and therefore $\dot{y} = \frac{\frac{1}{n} \times \overline{z^{\frac{1}{m}}x^{t-r} - bx^{t-r}}\big)^{\frac{1-n}{n}}}{a^{\frac{1}{n}}} \times$

$$\overline{\frac{1}{m}x^{t-r}z^{\frac{1}{m}-1}\dot{z} + \overline{t-r} \times z^{\frac{1}{m}}x^{t-r-1}\dot{x} + \overline{t-r} \times -bx^{t-r-1}\dot{x}} =$$

$\frac{x^{-1}\dot{x} \times \overline{z^{\frac{1}{m}}x^{t-r} - bx^{t-r}}\big)^{\frac{1}{n}}}{a^{\frac{1}{n}}z}$, putting these values of y and x^{mt} in the propofed general equation; and dividing by $\frac{\overline{z^{\frac{1}{m}}x^{t-r} - bx^{t-r}}\big)^{\frac{1}{n}}}{a^{\frac{1}{n}}}$, it will be

$$\frac{\frac{1}{n} \times \overline{\frac{1}{m}x^{t-r}z^{\frac{1}{m}-1}\dot{z} + \overline{t-r} \times z^{\frac{1}{m}}x^{t-r-1}\dot{x} + \overline{t-r} \times -bx^{t-r-1}\dot{x}}}{z^{\frac{1}{m}}x^{t-r} - bx^{t-r}} = \frac{x^{-1}\dot{x}}{z},$$ that is,

$$\frac{1}{m}z^{\frac{1}{m}}\dot{z} + \overline{t-r} \times z^{\frac{1}{m}+1}x^{-1}\dot{x} + \overline{t-r} \times -bzx^{-1}\dot{x} = nz^{\frac{1}{m}}x^{-1}\dot{x} - nbx^{-1}\dot{x};$$

and therefore $$\frac{z^{\frac{1}{m}}\dot{z}}{mnz^{\frac{1}{m}} + \overline{r-t} \times mz^{\frac{1}{m}+1} + \overline{mr-mt} \times bz - mnb} = \frac{\dot{x}}{x}.$$

If you fhould have terms with negative figns, you muft proceed after the fame manner, and in the final equation there would be no other difference, but that of the figns themfelves.

35. Alfo, taking a more univerfal equation, as $\frac{y^{u}\dot{x}}{\overline{bx^{t} + ay^{n}x^{r}}\big)^{m}} = cx^{\frac{ut-mnt-t+r+n-ur}{n}}\dot{y}$; the indeterminates would be feparated by the fame fubftitution.

EXAMPLE I.

Let the equation be $\frac{aay\dot{x}}{\sqrt{bbxx - a^{3}y}} = b\dot{y}$. Make $\sqrt{bbxx - a^{3}y} = xz$, and therefore $y = \frac{bbxx - zzxx}{a^{3}}$, and $\dot{y} = \frac{2bbx\dot{x} - 2zzx\dot{x} - 2xxz\dot{z}}{a^{3}}$. And, making the

the ſubſtitutions, $\frac{aa\dot{x}}{xz} \times \frac{bbxx - zzxx}{a^3} = \frac{2b^3x\dot{x} - 2bzzx\dot{x} - 2bxxz\dot{z}}{a^3}$; that is, $aabbx\dot{x} - aazzx\dot{x} = 2b^3zx\dot{x} - 2bz^3x\dot{x} - 2bxxzz\dot{z}$, or $2bxxzz\dot{z} = 2b^3zx\dot{x} - 2bz^3x\dot{x} + aazzx\dot{x} - aabbx\dot{x}$; and therefore $\frac{2bzz\dot{z}}{2b^3z - 2bz^3 + aazz - aabb} = \frac{\dot{x}}{x}$.

EXAMPLE II.

Let the equation be $\frac{xy\dot{x}}{\sqrt{-bbx^4 + a^3xyy}} = \frac{\dot{y}}{b}$. Make $\sqrt{-bbx^4 + a^3xyy} = zxx$, and therefore $y = \sqrt{\frac{zzx^3 + bbx^3}{a^3}}$, and $\dot{y} = \frac{x^3z\dot{z} + \frac{3}{2}zzxx\dot{x} + \frac{3}{2}bbxx\dot{x}}{a^3\sqrt{\frac{zzx^3 + bbx^3}{a^3}}}$. Wherefore, making the ſubſtitutions, we ſhall have $\frac{x\dot{x}}{zxx}\sqrt{\frac{zzx^3 + bbx^3}{a^3}} = \frac{x^3z\dot{z} + \frac{3}{2}zzxx\dot{x} + \frac{3}{2}bbxx\dot{x}}{a^3b\sqrt{\frac{zzx^3 + bbx^3}{a^3}}}$, that is, $bzzxx\dot{x} + b^3xx\dot{x} = x^3zz\dot{z} + \frac{3}{2}z^3xx\dot{x} + \frac{3}{2}bbzxx\dot{x}$, or $bzzxx\dot{x} + b^3xx\dot{x} - \frac{3}{2}z^3xx\dot{x} - \frac{3}{2}bbzxx\dot{x} = x^3zz\dot{z}$; and therefore $\frac{\dot{x}}{x} = \frac{zz\dot{z}}{bzz - \frac{3}{2}z^3 - \frac{3}{2}bbz + b^3}$.

36. By the ſame ſubſtitution as above, the indeterminates in this equation alſo may be ſeparated.

$\frac{y^u\dot{y}}{\overline{bx^t + ay^nx^r}|^m} = cx^{\frac{tu - n - tmn - ru + t - r}{n}}\dot{x}$. Make $\overline{bx^t + ay^nx^r}|^m = x^{mt}z$, it will be $y = \frac{\overline{x^{t-r}z^{\frac{1}{m}} - bx^{t-r}}|^{\frac{1}{n}}}{a^{\frac{1}{n}}}$; then $\dot{y} = \frac{x^{t-r}z^{\frac{1-m}{m}}\dot{z}}{mn} + \frac{t-r}{n}z^{\frac{1}{m}}x^{t-r-1}\dot{x} + \frac{r-t}{n}bx^{t-r-1}\dot{x}$ into $\frac{\overline{x^{t-r}z^{\frac{1}{m}} - bx^{t-r}}|^{\frac{1-n}{n}}}{a^{\frac{1}{n}}}$; and, making the ſubſtitutions, we ſhall have $\frac{x^{t-r}z^{\frac{1-m}{m}}\dot{z}}{mn} + \frac{t-r}{n}z^{\frac{1}{m}}x^{t-r-1}\dot{x} + \frac{r-t}{n}bx^{t-r-1}\dot{x}$ into

$\dfrac{\overline{x^{t-r}z^{\frac{1}{m}} - bx^{t-r}}\big|^{\frac{u-n+1}{n}}}{a^{\frac{u+1}{n}}x^{tm}z} = cx^{\frac{tu-n-tmn-ru+t-r}{n}}\dot{x}$. Wherefore, dividing the numerator and denominator of the first member of the equation by x^{tm}, and multiplying the whole by $a^{\frac{u+1}{n}}z$; and, instead of $\overline{x^{t-r}z^{\frac{1}{m}} - bx^{t-r}}\big|^{\frac{u+1-n}{n}}$, writing $x^{\frac{tu-tn+t-ru+nr-r}{n}} \times \overline{z^{\frac{1}{m}} - b}\big|^{\frac{u+1-n}{n}}$, which is the same; and, uniting the dimensions of the letter x, we shall find the equation to be divisible by $x^{\frac{ut-n-tmn-ru+t-r}{n}}$, and that being divided accordingly, it will be $\dfrac{xz^{\frac{1-m}{m}}\dot{z}}{mn} + \dfrac{t-r}{n}z^{\frac{1}{m}}\dot{x} + \dfrac{r-t}{n}b\dot{x}$ into $\overline{z^{\frac{1}{m}} - b}\big|^{\frac{u+1-n}{n}} = ca^{\frac{u+1}{n}}z\dot{x}$. And lastly, dividing again by $\overline{z^{\frac{1}{m}} - b}\big|^{\frac{u+1-n}{n}}$, it will be $\dfrac{xz^{\frac{1-m}{m}}\dot{z}}{mn} = \dfrac{r-t}{n} \times z^{\frac{1}{m}}\dot{x} + \dfrac{t-r}{n} \times b\dot{x} + ca^{\frac{u+1}{n}}z\dot{x} \times \overline{z^{\frac{1}{m}} - b}\big|^{\frac{n-u-1}{n}}$, that is, $\dfrac{\dot{x}}{x} =$

$$\frac{z^{\frac{1-m}{m}}\dot{z}}{mnca^{\frac{u+1}{n}}z \times \overline{z^{\frac{1}{m}} - b}\big|^{\frac{n-u-1}{n}} + \overline{mr - mt} \times z^{\frac{1}{m}} + \overline{mt - mr} \times b}.$$

EXAMPLE.

Let the equation be $\dfrac{y^3\dot{y}}{\sqrt{bbxx - aaxy - abxy}} = \dfrac{xx\dot{x}}{c}$. Put $\sqrt{bbxx - aaxy - abxy} = xz$, and therefore $y = \dfrac{bbx - zzx}{aa + ab}$, and $\dot{y} = \dfrac{bb\dot{x} - zz\dot{x} - 2xz\dot{z}}{aa + ab}$. Making, therefore, the substitutions, it will be $\overline{\dfrac{bbx - zzx}{aa + bb}}\Big|^3 \times \dfrac{bb\dot{x} - zz\dot{x} - 2xz\dot{z}}{aa + bb \times xz} = \dfrac{xx\dot{x}}{c}$. And, instead of $\overline{bbx - zzx}|^3$, writing $x^3 \times \overline{bb - zz}|^3$, and multiplying the whole equation by $\overline{aa + ab}|^4 \times zx$, we shall have $x^3 \times \overline{bb - zz}|^3 \times$

$\overline{bb\dot{x} - zz\dot{x} - 2xz\dot{z}} = \overline{aa + ab}|^4 \times \frac{zx^3\dot{x}}{c}$. And, dividing by $x^3 \times \overline{bb - zz}|^3$, it will be $bb\dot{x} - zz\dot{x} - 2xz\dot{z} = \overline{aa + ab}|^4 \times \overline{bb - zz}|^{-3} \times \frac{z\dot{x}}{c}$; that is, $bb\dot{x} - zz\dot{x} + \overline{aa + ab}|^4 \times \overline{bb - zz}|^{-3} \times - \frac{z\dot{x}}{c} = 2xz\dot{z}$. And therefore $\frac{\dot{x}}{x} = \frac{2z\dot{z}}{bb - zz - \frac{z}{c} \times \overline{bb - zz}|^{-3} \times \overline{aa + ab}|^4}$.

37. The ſame ſubſtitution will ſerve, in like manner, for a more general equation, $\frac{\overline{bx^t + fy^n x^r}|^v \times y^u \dot{y}}{\overline{bx^t + ay^n x^r}|^m} = cx^{\frac{ut - n - tmn - ru + t - r + ntv}{n}} \dot{x}$. Alſo, it will ſerve for the equation $\frac{y^{n-1}\dot{y}}{\overline{bx^t + cx^r + ay^n x^r}|^m} = fx^{t - r - 1 - mt}\dot{x}$, by making $\overline{bx^t + cx^r + ay^n x^r}|^m = x^{mt}z$; which, if $m = 1$, will be a particular caſe of § 27; and if it be $c = 0$, will be a particular caſe of § 36. Moreover, we may alſo conſtruct the equation $\frac{\overline{gx^t + hx^r + ky^n x^r}|^e \times y^{n-1}\dot{y}}{\overline{ax^t + bx^r + cy^n x^r}|^m} = fx^{t - r - 1 + et - mt}\dot{x}$, when it is $ch = bk$, making uſe of the ſame ſubſtitution, $\overline{ax^t + bx^r + cy^n x^r}|^m = x^{mt}z$.

Now, if it ſhould be alſo $h = 0$, $b = 0$, the equation will be a particular caſe of the firſt equation of this article.

38. Theſe equations may be conſtructed, $\frac{a\dot{y}}{b + \overline{cy^n + fx}|^u} = gy^{1-n}\dot{x}$, and $\frac{ay^{n-1}\dot{y}}{b + \overline{cy^n + fx^m}|^u} = gx^{m-1}\dot{x}$, by putting, for the firſt, $\overline{cy^n + fx}|^u = z$, and for the ſecond, $\overline{cy^n + fx^m}|^u = z$. And, as for the firſt, it will be then $y = \frac{\overline{z^{\frac{1}{u}} - fx}|^{\frac{1}{n}}}{c^{\frac{1}{n}}}$, and $\dot{y} = \frac{1}{n} \times \frac{\overline{z^{\frac{1}{u}} - fx}|^{\frac{1-n}{n}}}{c^{\frac{1}{n}}} \times \overline{\frac{1}{u} z^{\frac{1-u}{u}} \dot{z} - f\dot{x}}$; and therefore, making the ſubſtitutions, we ſhall have $az^{\frac{1-u}{u}}\dot{z} = nubcg\dot{x} + nucgz\dot{x} + auf\dot{x}$, that

that is, $\dfrac{az^{\frac{1-u}{u}}\dot{z}}{nubcg + nucgz + auf} = \dot{x}$. As to the second, we shall have $y = \dfrac{\overline{z^{\frac{1}{u}} - fx^m}\Big|^{\frac{1}{n}}}{c^{\frac{1}{n}}}$, and therefore $\dot{y} = \frac{1}{n} \times \dfrac{\overline{z^{\frac{1}{u}} - fx^m}\Big|^{\frac{1-n}{n}}}{c^{\frac{1}{n}}} \times \overline{\frac{1}{u} z^{\frac{1-u}{u}} \dot{z} - mfx^{m-1}\dot{x}}$;

and, making the substitutions, $x^{m-1}\dot{x} = \dfrac{az^{\frac{1-u}{u}}\dot{z}}{bcgnu + cgnuz + mafu}$.

Likewise, if we take a more general equation, $\dfrac{ay^{n-1}\dot{y}}{b + \overline{cy^n + p}\Big|^u} = gq\dot{x}$, where p and q are any how given by x and constants; if it be $q = \frac{\dot{p}}{\dot{x}}$, the indeterminates may be separated, by putting, in like manner, $\overline{cy^n + p}\Big|^u = z$. For it will be $y = \dfrac{\overline{z^{\frac{1}{u}} - p}\Big|^{\frac{1}{n}}}{c^{\frac{1}{n}}}$, and $\dot{y} = \dfrac{\frac{1}{n} \times \overline{z^{\frac{1}{u}} - p}\Big|^{\frac{1-n}{n}}}{c^{\frac{1}{n}}} \times \overline{\frac{1}{u} z^{\frac{1-u}{u}}\dot{z} - \dot{p}}$; and, making the substitutions, the equation will be $nbcguq\dot{x} + ncguzq\dot{x} + au\dot{p} = az^{\frac{1-u}{u}}\dot{z}$. But if we suppose $\dot{p} = q\dot{x}$, then it will be $\dfrac{az^{\frac{1-u}{u}}\dot{z}}{nbcgu + ncguz + au} = q\dot{x}$.

EXAMPLE I.

Let the equation be $a^3\dot{y} = 6b^3\dot{x} - 3bb\dot{x}\sqrt{cy + bx}$, or $\dfrac{a^3\dot{y}}{2b - \sqrt{cy+bx}} = 3bb\dot{x}$. Make $\sqrt{cy + bx} = z$, it will be $y = \dfrac{zz - bx}{c}$, $\dot{y} = \dfrac{2z\dot{z} - b\dot{x}}{c}$; and, making the substitutions, $\dfrac{2a^3z\dot{z} - a^3b\dot{x}}{2bc - cz} = 3bb\dot{x}$, or $2a^3z\dot{z} = 6b^3c\dot{x} - 3bbcz\dot{x} + a^3b\dot{x}$, and therefore $\dfrac{2a^3z\dot{z}}{6b^3c - 3bbcz + a^3b} = \dot{x}$.

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EXAMPLE II.

Let the equation be $\frac{ay\dot{y}}{b + \sqrt[3]{y^3 + aax - bxx}} = aa\dot{x} - 2bx\dot{x}$. Make $\sqrt[3]{y^3 + a^2x - bx^2} = z$; it will be $y = \overline{z^3 - aax + bxx}|^{\frac{1}{3}}$, and $\dot{y} = \frac{zz\dot{z} - \frac{1}{3}aa\dot{x} + \frac{2}{3}bx\dot{x}}{\overline{z^3 - aax + bxx}|^{\frac{2}{3}}}$; whence, making the fubftitutions, the equation will be $\frac{azz\dot{z} - \frac{1}{3}a^3\dot{x} + \frac{2}{3}abx\dot{x}}{b + z} = aa\dot{x} - 2bx\dot{x}$; that is, $3azz\dot{z} = a^3\dot{x} - 2abx\dot{x} + 3aab\dot{x} - 6bbx\dot{x} + 3aaz\dot{x} - 6bzx\dot{x}$; and, dividing by $a + 3b + 3z$, it will be $\frac{3azz\dot{z}}{a + 3b + 3z} = aa\dot{x} - 2bx\dot{x}$.

39. The equation, or canonical formula, $ax^m\dot{x} + cyyx^n\dot{x} = \dot{y}$, has not it's indeterminates feparable in general, whatever the exponent m may be; yet they are feparable in an infinite number of cafes; that is, the exponent m may receive infinite values, in which the defired feparation will fucceed.

To determine which I make ufe of a method not unlike to that of § 23. Make $y = Ax^p + x^r t$; where the quantity A, and the exponents p, r, are arbitrary conftants, to be determined as exigence may require, and t in a new indeterminate quantity. Therefore it will be $\dot{y} = pAx^{p-1}\dot{x} + rtx^{r-1}\dot{x} + x^r\dot{t}$, and $yy = AAx^{2p} + 2Ax^{p+r}t + ttx^{2r}$. Wherefore, fubftituting thefe values in the propofed formula, they will give this following, $ax^m\dot{x} + cAAx^{2p+n}\dot{x} + 2cAtx^{p+r+n}\dot{x} + cttx^{2r+n}\dot{x} = pAx^{p-1}\dot{x} + rtx^{r-1}\dot{x} + x^r\dot{t}$. Let us fuppofe $cAA = pA$, $2p + n = p - 1$, $r = 2cA$; that is, $p = -n - 1$, $A = \frac{-n-1}{c}$, $r = -2n - 2$. By thefe, in the laft formula, will vanifh the fecond, third, fifth, and fixth terms, and it will be reduced to $ax^m\dot{x} + cttx^{-3n-4}\dot{x} = x^{-2n-2}\dot{t}$. That is, (dividing by x^{-2n-2},) $ax^{m+2n+2}\dot{x} + cttx^{-n-2}\dot{x} = \dot{t}$; or (D) $ax^K\dot{x} + cttx^X\dot{x} = \dot{t}$, making $m + 2n + 2 = K$, and $-n - 2 = X$.

I refume

I resume the proposed equation $ax^m\dot{x} + cyyx^n\dot{x} = \dot{y}$, which, putting $y = \frac{1}{z}$, is transformed into this other, $azzx^m\dot{x} + cx^n\dot{x} = -\dot{z}$; in which is put, as above, $z = Bx^q + x^v u$, where B, q, v, are constants, to be determined as before, and u is a new indeterminate quantity. Therefore it will be $\dot{z} = qBx^{q-1}\dot{x} + vux^{v-1}\dot{x} + x^v\dot{u}$, $zz = BBx^{2q} + 2Bx^{q+v}u + uux^{2v}$. And these values being substituted, we shall have $aBBx^{2q+m}\dot{x} + 2aBux^{q+v+m}\dot{x} + auux^{2v+m}\dot{x} + cx^n\dot{x} = -qBx^{q-1}\dot{x} - vux^{v-1}\dot{x} - x^v\dot{u}$. Now, if we suppose $aBB = -Bq$, $2q + m = q - 1$, $-v = 2aB$; that is, $q + m = -1$, $B = \frac{m+1}{a}$, $v = -2m - 2$; with these in this last formula will vanish the first, second, fifth, and sixth terms, and it will be reduced to $auux^{-3m-4}\dot{x} + cx^n\dot{x} = -x^{-2m-2}\dot{u}$; that is, (dividing by x^{-2m-2},) $cx^{2m+n+2}\dot{x} + auux^{-m-2}\dot{x} = -\dot{u}$, or (G) $cx^\delta\dot{x} + auux^\omega\dot{x} = -\dot{u}$; making $2m + n + 2 = \delta$, and $-m - 2 = \omega$.

Now, in the proposed equation, the indeterminates are separable when $m = n$. Wherefore, also, in the formulæ marked D, G, the indeterminates will be separable, when it is $m + 2n + 2 = -n - 2$, $2m + n + 2 = -m - 2$, because m obtains two values, that is, $m = -3n - 4$, $m = \frac{-n-4}{3}$; which being substituted, the separation of the indeterminates will succeed. For then, in the proposed equation, the indeterminates will be separated when it is $m = \frac{-n-4}{3}$; also, they will be separated in the formulæ D, G, when it is $K = \frac{-X-4}{3}$, $\delta = \frac{-\omega-4}{3}$, because there are other two values of m, that is, $m = \frac{-5n-8}{3}$, $m = \frac{-3n-8}{5}$.

By the same way of argumentation, we may have infinite other values of m; as $m = \frac{-7n-12}{5}$, $m = \frac{-5n-12}{7}$, $m = \frac{-9n-16}{7}$, $m = \frac{-7n-16}{9}$, &c.; and, in general, $m = \frac{\overline{2h \pm 1} \times -n - 4h}{2h \mp 1}$, taking h any integer positive number, beginning from unity. Putting any of these values in the proposed equation, we shall have the indeterminates separable.

It

It may be added, that the indeterminates will alſo be ſeparable in the propoſed equation, when the exponent m is ſuch, that, by the method of § 19, it may be reduced to a caſe of § 14.

This would be the place to make uſe of two Diſſertations of the very learned Mr. *Euler*, inſerted in the Memoirs of the Academy of *Peterſburg*, Tom. VI. But, becauſe of the ſubtile manner in which that author proceeds, I ſhould be obliged to exceed thoſe limits which I had fixed to myſelf, intending only a plain and ſimple Inſtitution. I ſhall therefore leave the curious reader to ſeek them in the book itſelf.

PROBLEM I.

40. To find the curve, the ſubtangent of which is equal to the ſquare of the ordinate, divided by a conſtant quantity.

Making the abſciſs equal to x, the ordinate equal to y, the ſubtangent is always $\frac{y\dot{x}}{\dot{y}}$, which therefore ought to be equal to $\frac{yy}{a}$. Therefore we ſhall have the equation $\frac{y\dot{x}}{\dot{y}} = \frac{yy}{a}$, or $a\dot{x} = y\dot{y}$, and, by integration, $ax = \frac{1}{2}yy$, or $2ax = yy$, which is the *Apollonian* parabola.

If the ſubtangent ought to be equal to twice the abſciſs, we ſhould have the equation $\frac{y\dot{x}}{\dot{y}} = 2x$, and therefore $\frac{\dot{x}}{2x} = \frac{\dot{y}}{y}$, and, by integration, $\frac{1}{2}lx + \frac{1}{2}la = ly$, (where the conſtant $\frac{1}{2}la$ is added, to fulfil the law of homogeneity,) that is, $l\sqrt{ax} = ly$; and, returning from the logarithms, $\sqrt{ax} = y$, or $ax = yy$, which is alſo the ſame parabola.

If the ſubnormal is to be conſtant, it will be $\frac{y\dot{y}}{\dot{x}} = a$, that is, $y\dot{y} = a\dot{x}$, and, by integration, $\frac{1}{2}yy = ax$, or $yy = 2ax$, which is again the ſame parabola.

Let the ſubtangent be triple of the abſciſs; it will be $\frac{y\dot{x}}{\dot{y}} = 3x$, or $\frac{\dot{x}}{3x} = \frac{\dot{y}}{y}$, and, by integration, $l\sqrt[3]{aax} = ly$, or $aax = y^3$, which is the firſt cubical parabola.

Let

Let the ſubtangent be a multiple of the abſciſs, according to any number m; it will be $\frac{y\dot{x}}{\dot{y}} = mx$, that is, $\frac{\dot{x}}{mx} = \frac{\dot{y}}{y}$, and, by integration, $l\sqrt[m]{a^{m-1}x} = ly$, or $a^{m-1}x = y^{m}$, a curve of the parabolic kind.

Let the ſubtangent be $\frac{2ax + xx}{a + x}$; then the equation is $\frac{y\dot{x}}{\dot{y}} = \frac{2ax + xx}{a + x}$, that is, $ay\dot{x} + yx\dot{x} = 2ax\dot{y} + xx\dot{y}$, or $\frac{a\dot{x} + x\dot{x}}{2ax + xx} = \frac{\dot{y}}{y}$. And, by integration, it will be $ly = \frac{1}{2}l\,\overline{2ax + xx}$, and therefore $xx + 2ax = yy$, an equation to the hyperbola.

Let the ſubtangent be $\frac{2axy - 3x^3}{ay + 3xx}$; then the equation will be $\frac{y\dot{x}}{\dot{y}} = \frac{2axy - 3x^3}{ay + 3xx}$, that is, $ayy\dot{x} + 3yxx\dot{x} = 2axy\dot{y} - 3x^3\dot{y}$. According to what has been already delivered at § 18, I endeavour to reduce this equation to a caſe of § 14. Therefore I make $y = \frac{zz}{a}$, $\dot{y} = \frac{2z\dot{z}}{a}$; and, making the ſubſtitutions, it will be $z^4\dot{x} + 3zzxx\dot{x} = 4xz^3\dot{z} - 6x^3z\dot{z}$, where now it is reduced to the ſaid caſe. Wherefore the indeterminates will be ſeparated, if we put $z = \frac{xp}{a}$, $\dot{z} = \frac{x\dot{p} + p\dot{x}}{a}$; and, making the ſubſtitutions, it will be $\frac{p^4x^4\dot{x}}{a^4} + \frac{3ppx^4\dot{x}}{aa} = \frac{4x^4p^3}{a^3} \times \frac{x\dot{p} + p\dot{x}}{a} - \frac{6x^4p}{a} \times \frac{x\dot{p} + p\dot{x}}{a}$, that is, $9aap\dot{x} - 3p^3\dot{x} = 4xpp\dot{p} - 6aax\dot{p}$, and therefore $\frac{\dot{x}}{x} = \frac{4pp\dot{p} - 6aa\dot{p}}{9aap - 3p^3}$; and, by integration, $lx = \frac{lm}{l\sqrt[3]{p^4 - 3aapp}}$. And, reſtoring the value of p, that is, $a\sqrt{\frac{ay}{xx}}$, it will be $x = \frac{m}{\sqrt[3]{\frac{a^6yy - 3a^5yxx}{x^4}}}$, that is, finally, $a^6y^2 - 3a^5yx^2 = mx$.

The two ſubſtitutions made of $y = \frac{zz}{a}$, and $z = \frac{xp}{a}$, in order to ſeparate the indeterminates, plainly ſhow us that it would have been ſufficient if, at firſt, we had made but one of them, or $y = \frac{xxpp}{a^3}$.

But we might have obtained our deſire ſomething more expeditiouſly, by writing the equation thus: $3yxx\dot{x} + 3x^3\dot{y} = 2axy\dot{y} - ayy\dot{x}$; which, divided by xx, will be $3y\dot{x} + 3x\dot{y} = \frac{2axy\dot{y} - ay^2\dot{x}}{xx}$; and, by integration, $3xy = \frac{ayy}{x}$, that is, $\frac{1}{3}ay = xx$, the *Apollonian* parabola, when we omit the conſtant m.

Let

Let the ſubtangent be $\frac{4x^3 - axy}{3xx - ay}$; the equation will be $\frac{4x^3 - axy}{3xx - ay} = \frac{y\dot{x}}{\dot{y}}$, that is, $4x^3\dot{y} - axy\dot{y} = 3xxy\dot{x} - ayy\dot{x}$, which I write in another manner, thus: $4x^3\dot{y} - 3yxx\dot{x} = axy\dot{y} - ayy\dot{x}$. I obſerve that the ſecond member would be integrable, if it were divided by xxy; I divide, therefore, the whole equation, whence it is $\frac{4x\dot{y} - 3y\dot{x}}{y} = \frac{ax\dot{y} - ay\dot{x}}{xx}$. I ſuppoſe the integral of this ſecond member $\frac{ay}{x} = z$; and, making y to vaniſh out of the equation, it will be $\frac{4x \times \overline{x\dot{z} + z\dot{x}} - 3zx\dot{x}}{zx} = \dot{z}$, that is, $\frac{4x\dot{z} + z\dot{x}}{z} = \dot{z}$, which may be conſtructed by the method of § 14, or elſe prepared according to the method of § 24, it will be $x \times \overline{\frac{4\dot{z}}{z} + \frac{\dot{x}}{x}} = \dot{z}$. Therefore I make $\frac{4\dot{z}}{z} + \frac{\dot{x}}{x} = \frac{\dot{p}}{p}$, and, by integration, $lz^4x = la^4p$, or $z^4x = a^4p$; and therefore, making x to vaniſh out of the final equation, we ſhall have, laſtly, $\frac{a^4p}{z^4} \times \frac{\dot{p}}{p} = \dot{z}$, that is, $a^4\dot{p} = z^4\dot{z}$, and, by integration, $a^4p = \frac{1}{5}z^5$; in which, reſtoring the value of p, then that of z, it will be $xx = \frac{1}{5}ay$, which is the *Apollonian* parabola.

Let the ſubtangent be $\frac{\overline{a + x} \times l\overline{a + x}}{a + l\overline{a+x}}$; the equation will be $\frac{\overline{a+x} \times l\overline{a+x}}{a + l\overline{a+x}} = \frac{y\dot{x}}{\dot{y}}$, that is, $\frac{\dot{y}}{y} = \frac{a\dot{x} + \dot{x}\,l\overline{a+x}}{\overline{a+x} \times l\overline{a+x}}$. In order to proceed to the integration, I make $\overline{a + x} \times l\overline{a + x} = z$, and therefore $\dot{z} = \dot{x} \times l\overline{a + x} + a\dot{x}$; (ſuppoſing the logarithmic with the ſubtangent $= a$.) Theſe values being ſubſtituted in the equation, it will be $\frac{\dot{y}}{y} = \frac{\dot{z}}{z}$, and integrating, it is $y = z$, that is, $y = \overline{a + x} \times l\overline{a + x}$, a tranſcendent curve, but which is eaſily deſcribed, ſuppoſing the logarithmic.

PROBLEM II.

41. To find the curve, the area of which is equal to two third parts of the rectangle of the co-ordinates.

The formula for the area is $y\dot{x}$, and therefore we ſhall have $\int y\dot{x} = \frac{2}{3}xy$; whence $y\dot{x} = \frac{2}{3}x\dot{y} + \frac{2}{3}y\dot{x}$, that is, $y\dot{x} = 2x\dot{y}$, or $\frac{\dot{x}}{2x} = \frac{\dot{y}}{y}$; and, by integra-

tion,

tion, as before, it is $l\sqrt{ax} = ly$, $ax = yy$. The curve is the ſame *Apollonian* parabola.

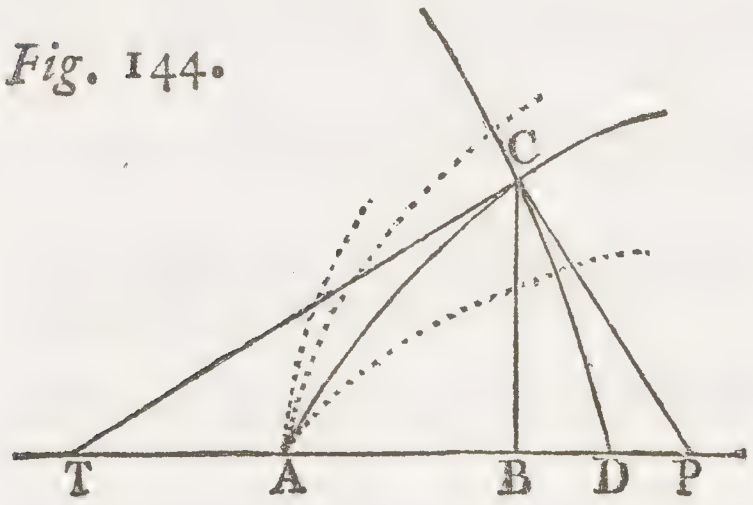
Let the area be equal to the fourth power of the ordinate, divided by a conſtant ſquare; then it will be $\int y\dot{x} = \frac{y^4}{aa}$, that is, $y\dot{x} = \frac{4y^3\dot{y}}{aa}$, or $aa\dot{x} = 4yy\dot{y}$; and, by integration, $\frac{3}{4}aax = y^3$, the firſt cubic parabola.

Let the area be equal to the power denoted by m of the ordinate, divided by a conſtant; it will be $\int y\dot{x} = \frac{y^m}{a^{m-2}}$, that is, $y\dot{x} = \frac{my^{m-1}\dot{y}}{a^{m-2}}$, a curve of the parabolic or hyperbolic kind, according as $m - 1$ ſhall be poſitive or negative.

PROBLEM III.

42. In infinite number of parabolas being given, of any the ſame kind; to find what that curve is, which cuts them all at right angles.

Fig. 144.



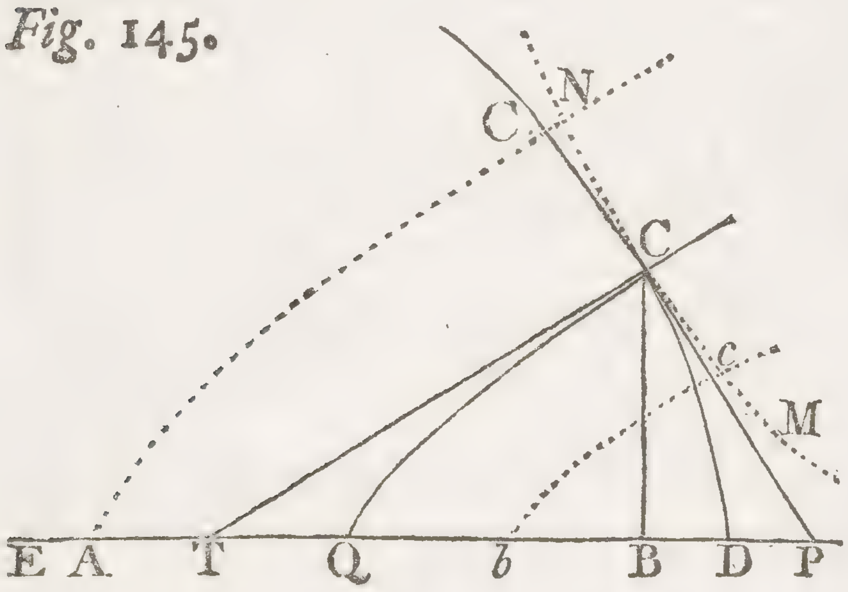
Let the equation of the curve required be $p^{m-n}x^n = y^m$, which, (p being conſidered as arbitrary, and ſuſceptible of infinite values,) expreſſes infinite parabolas; and (conſidering m and n in the ſame manner,) expreſſes any kind of parabolas. And, firſt, let them all belong to the ſame axis AB, (Fig. 144.) with vertex A, and different only in their parameters. Let AC be one of theſe infinite parabolas, in which $AB = x$, $BC = y$.

From any point C let the tangent CT be drawn, and the normal CP. It is known already, that it will be $BT = \frac{mx}{n}$. Let DC be the curve required; and, becauſe this ought to cut the parabola perpendicularly in the point C, in an infiniteſimal portion it muſt coincide with the normal CP in the point C. Therefore CT, the tangent of the parabola AC, will be likewiſe perpendicular to the curve DC in the point C, and conſequently, at the ſame time, BT will be both a ſubtangent to the parabola, and a ſubnormal of the curve required, DC. What is ſaid of the parabola AC agrees with any other of the ſame kind. Therefore the problem conſiſts in finding, of what kind is the curve DC,

DC, whofe fubnormal is $= \frac{mx}{n}$. Now the general expreffion of the fubnormal is $\frac{y\dot{y}}{\dot{x}}$, which, in this cafe, ought to be taken negative, becaufe, in the curve DC, as AB, or x, increafes, at the fame time BC, or y, decreafes; and therefore the differential equation will be $\frac{mx}{n} = -\frac{y\dot{y}}{\dot{x}}$; and, feparating the variables, $\frac{mx\dot{x}}{n} = -y\dot{y}$; and, by integration, $\frac{mxx}{2n} = -\frac{1}{2}yy + aa$, or $\frac{nyy}{m} = \frac{2naa}{m} - xx$, which is an equation to the ellipfis. And, becaufe the parameter p does not at all enter here, the folution will be general for the infinite parabolas that may be thus defcribed.

If the exponent n of the equation $p^{m-n}x^n = y^m$ is fuppofed to be negative, fo that the equation may be $x^n y^m = p^{m+n}$, in which now it is pofitive; it will belong to infinite hyperbolas of the fame kind between the afymptotes, the fubtangents of which are $-\frac{mx}{n}$, and the fubnormal of the curve DC ought alfo to be equal to thefe. Then it will be $-\frac{mx}{n} = -\frac{y\dot{y}}{\dot{x}}$, or $\frac{mx\dot{x}}{n} = y\dot{y}$. And, by integration, $\frac{mxx}{2n} = \frac{1}{2}yy + aa$, or $xx - \frac{2naa}{m} = \frac{nyy}{m}$, an equation to the hyperbola.

Fig. 145.



If the infinite parabolas AC, QC, &c. of the equation $p^{m-n}z^n = y^m$, fhall have all the fame parameter, but each a different vertex in the fame axis; that is to fay, if one of them be conceived to move always upon the axis parallel to itfelf; from a fixed point A (Fig. 145.) making any abfcifs AB $= x$, and taking any curve QC, whofe abfcifs is QB $= z$, and ordinate BC $= y$; then will alfo $-\frac{y\dot{y}}{\dot{x}}$ be the fubnormal of the curve DC required, and therefore equal to the fubtangent BT of the parabola QC. Whence the equation $-\frac{y\dot{y}}{\dot{x}} = \frac{mz}{n}$; but, by the equation of the parabola we have $z = \frac{y^{\frac{m}{n}}}{p^{\frac{m-n}{n}}}$, and therefore $-\frac{y\dot{y}}{\dot{x}} =$

$\frac{my^{\frac{m}{n}}}{np^{\frac{m-n}{n}}}$, that is, $\dot{x} = -\frac{n}{m}p^{\frac{m-n}{n}}y^{1-\frac{m}{n}}\dot{y}$; and, by integration, $x = -\frac{nnp^{\frac{m-n}{n}}y^{\frac{2n-m}{n}}}{m \times \overline{2n-m}}$, the equation of the curve required, DC.

If the parabolas are the *Apollonian*, that is, $m = 2$, $n = 1$, the integrated equation would not be of ufe in this cafe; for, making the fubftitutions of the values of m and x, we fhould have $x = -\frac{p}{0}$. But, taking the differential equation, it would be $\dot{x} = -\frac{1}{2}p \times \frac{\dot{y}}{y}$, an equation to the logarithmic. Therefore the curve which cuts the infinite *Apollonian* parabolas at right angles will be the logarithmic MCN, the fubtangent of which is equal to half the parameter of the parabola.

Let the parabolas be the firft cubics, that is, $m = 3$, $n = 1$; it will be $x = -\frac{ppy^{-1}}{-3}$, or $xy = \frac{1}{3}pp$, and the curve DC will be the hyperbola between it's afymptotes.

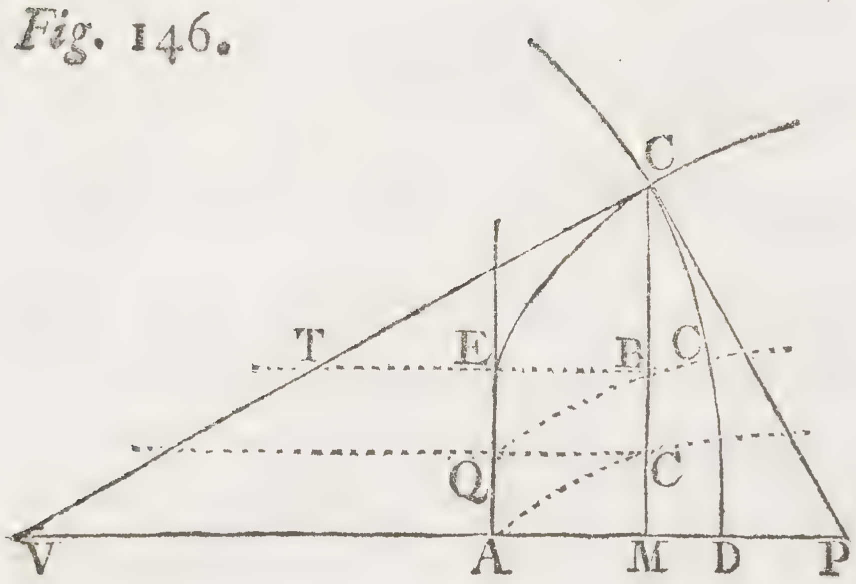
Let the parabolas be the fecond cubics, that is, $m = 3$, $n = 2$; it will be $x = -\frac{4}{3}\sqrt{py}$, or $xx = \frac{16}{9}py$, and the curve DC will be the common parabola. Taking other values for m and n, we fhall have other curves.

If the parabolas AC, QC, &c. befides having a different vertex on the fame axis, fhould have their parameter variable, that is, equal in each to the refpective diftances of the vertex from the fixed point E; taking any one of them, QC, make EB $= x$ the abfcifs of the curve required DC, the ordinate BC $= y$, EQ $= p =$ parameter; it will be QB $= x - p$, and the equation of the infinite parabolas $p^{m-n} \times \overline{x-p}|^{n} = y^{m}$, and the fubtangent BT $= \frac{m}{n} \times \overline{x-p}$, and therefore the equation $-\frac{y\dot{y}}{\dot{x}} = \frac{m}{n} \times \overline{x-p}$.

If the parabolas be *Apollonian*, that is, $m = 2$, $n = 1$, it will be $p = \frac{1}{2}x \pm \sqrt{\frac{1}{4}xx - yy}$; whence, making the fubftitutions in the equation $-\frac{y\dot{y}}{\dot{x}} = \frac{m}{n} \times \overline{x-p}$, it will be $-\frac{y\dot{y}}{\dot{x}} = x \mp 2\sqrt{\frac{1}{4}xx - yy}$, which may be reduced to a feparation of the indeterminates by the method of § 14; and then we may go on to the integral, which will be algebräical.

If

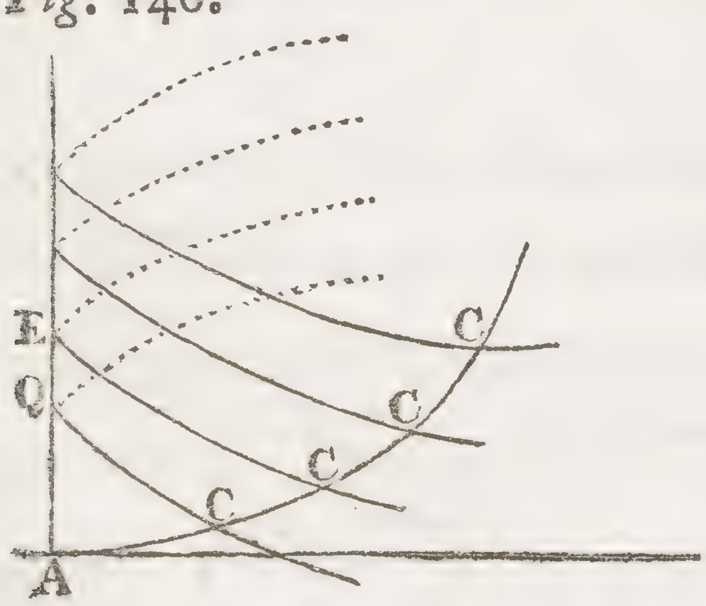
Fig. 146.



If the infinite parabolas AC, QC, &c. of the equation $p^{m-n}z^n = y^m$ shall have the same constant parameter, the axes parallel, and the vertices variable in the perpendicular to the axes; that is to say, if one of them be moved in such a manner, as that every one of it's points may describe perpendiculars to the axes: Taking any one of them, EC, (Fig. 146.) and calling AM = EB = z, BC = y, MC = x; and, drawing to the parabola EC the tangent CT, produced to V, then MV will be the subnormal of the curve DC required. Now, because it is BT = $\frac{mz}{n}$, it will be MV = $\frac{mzx}{ny}$; whence we should have the equation $\frac{mzx}{ny} = -\frac{x\dot{x}}{\dot{z}}$; and, instead of y, substituting it's value $p^{\frac{m-n}{m}} z^{\frac{n}{m}}$, given by the equation $p^{m-n}z^n = y^m$, it will be, finally, $\frac{mzx}{np^{\frac{m-n}{m}} z^{\frac{n}{m}}} = -\frac{x\dot{x}}{\dot{z}}$, that is, $-\frac{mz\dot{z}}{np^{\frac{m-n}{m}} z^{\frac{n}{m}}} = \dot{x}$, and, by integration, $x = -\frac{mmz^{\frac{2m-n}{m}}}{n \times \overline{2m-n} \times p^{\frac{m-n}{m}}}$, the equation of the curve required, DC.

Let the parabolas be the *Apollonian*, that is, $m = 2$, $n = 1$; it will be $x = -\frac{4z^{\frac{3}{2}}}{3p^{\frac{1}{2}}}$, or $\frac{9}{16}pxx = z^3$; and therefore the curve DC will be the second cubic parabola, of which the *latus rectum* will be to that of the parabola AC as 9 to 16.

Fig. 146.

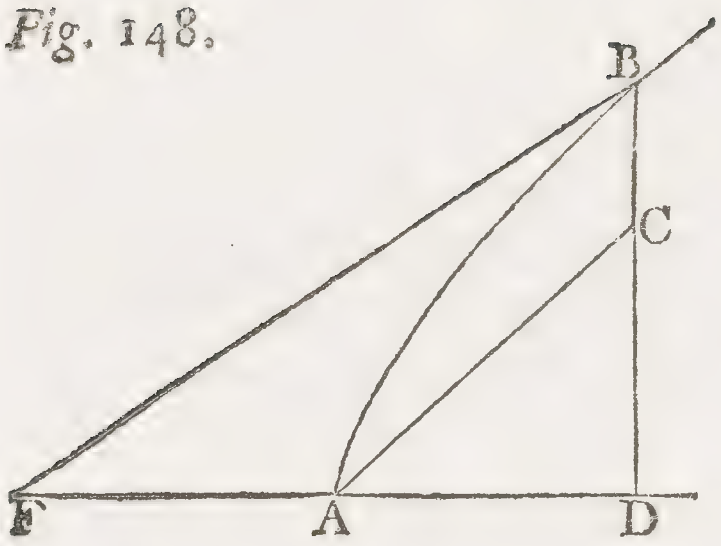


It is to be observed, that, in this case, the position of the curve DC will not be that marked in Fig. 146, but will have it's vertex in A, cutting the inferior part of the *Apollonian* parabola at right angles; that is, meeting the convexity, as in Fig. 147.

Another kind being pitched upon for the parabolas AC, also the curve DC will be a parabola of another kind.

PROBLEM IV.

Fig. 148.



43. Upon the right line AD let the right line AC insist at half a right angle; the equation of the curve AB is required, the property of which is, that the ordinate BD may have to the subtangent DF, the ratio of a constant line a, to BC.

Make AD $= x$, DB $= y$; it will be CB $= y - x$. Whence, by the condition of the problem, we shall have $y . \frac{y\dot{x}}{\dot{y}} :: a . y - x$; and therefore the equation $a\dot{x} = y\dot{y} - x\dot{y}$. Now, to separate the indeterminates, I make use of the method of § 23. Wherefore, putting $x = Ay + p + B$, and $\dot{x} = A\dot{y} + \dot{p}$; and, making the substitutions, it will be $aA\dot{y} + a\dot{p} = y\dot{y} - Ay\dot{y} - p\dot{y} - B\dot{y}$. Now, in this equation, the indeterminates will be separated, if the first and second terms of the *homogeneum comparationis* be made to vanish; that is, if A $= 1$, and B remains arbitrary, which, for brevity-sake, I will make B $= 0$. Therefore the substitutions to be made will be $x = y + p$, $\dot{x} = \dot{y} + \dot{p}$, and the equation will be $a\dot{p} = -a\dot{y} - p\dot{y}$, that is, $\frac{a\dot{p}}{a + p} = -\dot{y}$, a transcendent curve, and which depends on the logarithmic.

PROBLEM V.

44. To find the curve, the area of which is $axy + bx^c y^e$; where the absciss is x, and the ordinate y, as usual.

Therefore it ought to be $\int y\dot{x} = axy + bx^c y^e$; and therefore $y\dot{x} = ax\dot{y} + ay\dot{x} + cby^e x^{c-1}\dot{x} + ebx^c y^{e-1}\dot{y}$; or, making $a - 1 = m$, it is $my\dot{x} + ax\dot{y} + cby^e x^{c-1}\dot{x} + ebx^c y^{e-1}\dot{y} = 0$. To separate the indeterminates in this equation, we may make use of the method of § 33, putting $x = u^{e-1} z^{e-1}$, and $y = z^{1-c}$; whence $\dot{x} = \overline{e-1}\, z^{e-1} u^{e-2}\dot{u} + \overline{e-1}\, u^{e-1} z^{e-2}\dot{z}$, and $\dot{y} = \overline{1-c}\, z^{-c}\dot{z}$. Now, making

making the fubftitutions, we fhould obtain an equation much compounded, and which would require a very long calculation.

To come, then, to the point with brevity; refuming the equation $\int y\dot{x} = bx^c y^e + axy$, put $x^c y^e = q$, whence the equation will be $\int y\dot{x} = bq + axy$, and therefore $y\dot{x} = b\dot{q} + ax\dot{y} + ay\dot{x}$. This fuppofed, I make ufe of the method of § 24, in the form of which I write the equation thus, $axy \times \overline{\frac{1-a}{a} \times \frac{\dot{x}}{x} - \frac{\dot{y}}{y}} = b\dot{q}$; then I put $\frac{1-a}{a} \times \frac{\dot{x}}{x} - \frac{\dot{y}}{y} = \frac{\dot{p}}{p}$; and then integrating, it will be $\frac{1-a}{a} l \frac{x}{y} = lp$, or $\frac{x^{\frac{1-a}{a}}}{y} = p$. Wherefore, making the neceffary fubftitutions, we fhall have the equation $\frac{ax^{\frac{1}{a}}\dot{p}}{pp} = b\dot{q}$. Now, to exprefs the quantity $x^{\frac{1}{a}}$ by the affumed quantities p, q, we muft confider, that $x^c y^e = q$, that is, $y^e = \frac{q}{x^c}$, or $y = \frac{q^{\frac{1}{e}}}{x^{\frac{c}{e}}}$. But we have alfo $\frac{x^{\frac{1-a}{e}}}{p} = y$; therefore $\frac{x^{\frac{1-a}{a}}}{p} = \frac{q^{\frac{1}{c}}}{x^{\frac{c}{e}}}$, or $x^{\frac{e-ae+ac}{ae}} = q^{\frac{1}{e}}p$; and, laftly, $x^{\frac{1}{a}} = q^{\frac{1}{e-ae+ac}} \times p^{\frac{e}{e-ae+ac}}$. Then, making this fubftitution inftead of $x^{\frac{1}{a}}$, we fhall have the equation $ap^{\frac{e}{e-ae+ac} - 2}\dot{p} = \frac{b\dot{q}}{q^{\frac{1}{e-ae+ac}}}$; that is, $ap^{\frac{2ae-2ac-e}{e-ae+ac}}\dot{p} = bq^{\frac{-1}{e-ae+ac}}\dot{q}$; and, by integration, $\frac{ae - aae + aac}{ae - ac} \times p^{\frac{ae-ac}{e-ae+ac}} = \frac{be - bae + bac}{e - ae + ac - 1} \times q^{\frac{e-ae+ac-1}{e-ae+ac}} + g$; which is the equation of the curve required.

It is plain that this curve will be algebräical, at leaft when the quantities a, c, e, fhall be rational; and, on the contrary, it will be tranfcendental when one of thefe fhall be irrational. I fay at leaft, becaufe, making a, c, e, rational, the curve, however, will be tranfcendental if $e = c$; or if $a = \frac{1-e}{c-e}$; or if $c = 1$, and at the fame time $a = 1$; or $a = 0$, and alfo $e = 1$. And in feveral other cafes, which it is not neceffary to enumerate.

SECT. IV.

Of the Reduction of Fluxional Equations, of the Second Degree, &c.

45. When the differential equations of the second degree are fuch, that the rules here explained for integrations may be adapted to them, as well in cafes of feparate variables, as in thofe that are mixed; nothing elfe remains to be done, but to apply the faid rules, and thus, by means of integration, to reduce them to firft differentials; therefore there is no need to add any thing further about this matter. If, after the formulæ thus reduced to the firft degree, the indeterminates will not then be feparable, as is often the cafe, nor fhall be in any wife conftructible; it is not the method that is in fault, by which the fecond differences are refolved, but rather that by which the firft differences are managed.

Therefore we ought to employ our induftry about the reduction of the differentio-differential equations, that, by the rules already taught, they may be made fit for integration, which may be attempted feveral ways.

46. One way will be, to make ufe of the common expedients of vulgar Algebra, by tranfpofing the terms, by multiplying or dividing them by fome quantity, and fuch like. But, firft, before any other thing, it is neceffary to recollect, or to know, if, from paffing from firft to fecond fluxions, there be any fluxion that was taken for conftant, and what it was. And befides, that as, in the integration of firft differences to finite quantities, there is always added fome conftant quantity; fo, likewife, in the integrations of fecond to firft differences, fome conftant quantities fhould be added. This fuppofed, let us proceed to fome Examples.

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EXAMPLE I.

Let this equation be propofed, $\frac{by^m}{c^m} = \frac{2ay\ddot{x} + a\dot{x}\dot{y}}{\dot{u}\dot{y}}$, in which $\dot{u} = \sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}$ is the element of a curve, and is fuppofed conftant. I write it thus, $\frac{by^m \dot{y}\dot{u}}{c^m} = 2ay\ddot{x} + a\dot{x}\dot{y}$.

As $\dot{u}$ is conftant, the firft member will be integrable, even though it fhould be multiplied or divided by any function of y; and I obferve, that the fecond would be fo alfo, if it were divided by $2\sqrt{y}$. Therefore I divide the whole equation by $2\sqrt{y}$, and it will be $\frac{by^m \dot{y}\dot{u}}{2c^m\sqrt{y}} = \frac{2ay\ddot{x} + a\dot{x}\dot{y}}{2\sqrt{y}}$; and, by integration, it will be $\frac{by^{m+\frac{1}{2}}\dot{u}}{\overline{m+\frac{1}{2}} \times 2c^m} = a\dot{x}\sqrt{y} + a\dot{u}\sqrt{a}$, which equation is now reduced to firft fluxions.

In the integration I have added $\dot{u}$ for this reafon, becaufe it is conftant; and I have multiplied it by $a\sqrt{a}$, to preferve the law of homogeneity.

EXAMPLE II.

Let the equation be $f = \frac{\dot{x}\dot{x} - y\ddot{y}}{y^3\dot{x}\dot{x}}$, in which $y\dot{x}$ is taken for a conftant. I multiply it by $2\dot{y}$, and it will be $2f\dot{y} = \frac{2\dot{x}\dot{x}\dot{y} - 2y\dot{y}\ddot{y}}{y^3\dot{x}\dot{x}}$, that is, $2f\dot{y} = \frac{2\dot{y}}{y^3} - \frac{2y\ddot{y}}{yy\dot{x}\dot{x}}$; and, by integration, becaufe of $y\dot{x}$ being conftant, it will be $\int 2f\dot{y} = -\frac{1}{yy} - \frac{\dot{y}\dot{y}}{yy\dot{x}\dot{x}} + nyy\dot{x}\dot{x}$.

EXAMPLE III.

Let the equation be $f = \frac{\dot{u}\dot{u} - y\ddot{y}}{y^3\dot{x}\dot{x}}$, in which let $\dot{x}$ be conftant, and $\dot{u}$ the element of a curve, that is, $\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}} = \dot{u}$. Therefore, becaufe $\dot{x}$ is conftant, it will be $\dot{y}\ddot{y} = \dot{u}\ddot{u}$; and therefore, fubftituting the value of $\ddot{y}$ in the equation, it will be $f = \frac{\dot{y}\dot{u}\dot{u} - y\dot{u}\ddot{u}}{y^3\dot{y}\dot{x}\dot{x}}$; and, multiplying by $2y$, it is $2fy = \frac{2y\dot{y}\dot{u}\dot{u} - 2yy\dot{u}\ddot{u}}{y^3\dot{y}\dot{x}\dot{x}}$, that is, $2f\dot{y} = \frac{2y\dot{y}\dot{u}\dot{u} - 2yy\dot{u}\ddot{u}}{y^4\dot{x}\dot{x}}$; and, by integration, $2\int f\dot{y} = -\frac{\dot{u}\dot{u}}{yy\dot{x}\dot{x}} + n\dot{x}\dot{x}$.

Again, after another manner. Inftead of $\dot{u}$, putting it's value in the equation, it will be $f = \frac{\dot{x}\dot{x} + \dot{y}\dot{y} - y\ddot{y}}{y^3\dot{x}\dot{x}}$; and, multiplying by $2y\dot{y}$, it is $2fy\dot{y} = \frac{2y\dot{y}\dot{x}\dot{x} + 2y\dot{y}^3 - 2yy\dot{y}\ddot{y}}{y^3\dot{x}\dot{x}}$, that is, $2f\dot{y} = \frac{2y\dot{y}\dot{x}\dot{x} + 2y\dot{y}^3 - 2yy\dot{y}\ddot{y}}{y^4\dot{x}\dot{x}}$; and, by integration, $2\int f\dot{y} = -\frac{\dot{x}\dot{x} + \dot{y}\dot{y}}{yy\dot{x}\dot{x}} \pm n\dot{x}\dot{x}$.

EXAMPLE IV.

Let the equation be $a\dot{x} = \frac{xy\ddot{y} + x\dot{y}\dot{y}}{\dot{x}}$, in which let $\dot{x}$ be conftant. Multiplying by $\dot{x}$, and dividing by x, it will be $\frac{a\dot{x}\dot{x}}{x} = y\ddot{y} + \dot{y}\dot{y}$; and, by integration, becaufe $\dot{x}$ is conftant, it is $a\dot{x}lx + A\dot{x} = y\dot{y}$. Now, if we fhould make the affumed conftant $A = a$, we fhould have $a\dot{x}lx + a\dot{x} = y\dot{y}$; and, proceeding to integration, $axlx = \frac{1}{2}yy$.

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EXAMPLE V.

Let the equation be $f = \frac{\dot x\dot y\dot u\ddot u + y\dot u\dot u\ddot x - y\dot x\dot u\ddot u}{y\dot x\dot y\dot t\dot t}$, in which $\dot u$ is the little arch or element of a curve, $\dot t$ is given by x and y, and no firſt fluxion is yet taken for conſtant. I divide it by $y^3\dot x^3$, and multiply it by 2, and it will be $\frac{2f}{y^3\dot x^3} = \frac{2\dot x\dot y\dot u\ddot u + 2y\dot u\dot u\ddot x - 2y\dot x\dot u\ddot u}{y^4\dot x^4\dot y\dot t\dot t}$, or $\frac{2f\dot y\dot t\dot t}{yy\dot x\dot x} = \frac{2y\dot x\dot x\dot y\dot u\ddot u + 2yy\dot u\dot u\dot x\ddot x - 2yy\dot x\dot x\dot u\ddot u}{y^4\dot x^4}$; and, by integration, $2\int\frac{f\dot y\dot t\dot t}{yy\dot x\dot x} = -\frac{\dot u\dot u}{yy\dot x\dot x} \pm n$.

But it may truly be ſaid to be a thing impoſſible, to make uſe of this method in ſuch equations, in which the quantities are intricate and compounded, when we do not know the integrations pretty nearly before-hand, which we are to make. Wherefore I ſhall go on to other methods.

47. In the ſolution of problems, when we are to proceed from firſt to ſecond fluxions, it may be much more convenient not to aſſume any fluxion for conſtant, though we are at liberty to do it: that we may be able the better, when the formula is under our inſpection, to determine that to be ſuch conſtant, by which the expreſſion may be much abbreviated, and moſt readily integrable. The Examples will beſt make this method to be underſtood.

EXAMPLE I.

Let the equation be $f = \frac{\dot y^3 + \dot x^2\ddot y - x\dot y\ddot x + x\dot x\ddot y}{2x^3\dot y^3}$, which may ariſe without having taken any fluxion for conſtant. To ſhorten this formula, I conſider, what may be that fluxion which, taken for conſtant, will deſtroy two terms of the *homogeneum comparationis*, and leave only two in the equation; and I find there may be two, that is, $x\dot y$ and $\frac{\dot x}{x}$. Therefore make $x\dot y = c$, and taking the difference, it is $x\ddot y + \dot x\dot y = 0$. Then multiplying by $\dot x$, it is $x\dot x\ddot y + \dot x\dot x\dot y = 0$, by which means the ſecond and fourth terms of the *homogeneum* diſappear out of the principal equation, ſo that we ſhall have $f = \frac{\dot y^3 - x\dot y\ddot x}{2x^3\dot y^3}$. But, as it is

$x\ddot{y} + \dot{x}\dot{y} = 0$, it will be $\dot{y} = -\frac{x\ddot{y}}{\dot{x}}$; whence, by fubftitution, $f = -\frac{x\dot{y}\ddot{y}}{2x^3\dot{x}\dot{y}^3} - \frac{x\dot{y}\ddot{x}}{2x^3\dot{y}^3}$, that is, $f = -\frac{x\dot{y}\ddot{y} + x\dot{x}\dot{y}\ddot{x}}{2x^3\dot{x}\dot{y}^3}$, or $f = -\frac{\dot{y}\ddot{y} + \dot{x}\ddot{x}}{2x^2\dot{x}\dot{y}^2}$. But $x\dot{y} = c$, and therefore $f = -\frac{\dot{y}\ddot{y} + \dot{x}\ddot{x}}{2cc\dot{x}}$; and, laftly, $f\dot{x} = -\frac{\dot{y}\ddot{y} + \dot{x}\ddot{x}}{2cc}$; and, by integration, $\int f\dot{x} = -\frac{\dot{y}\dot{y} + \dot{x}\dot{x}}{4cc} \pm n$, or $\int f\dot{x} = -\frac{\dot{y}\dot{y} + \dot{x}\dot{x}}{4xx\dot{y}\dot{y}} \pm n$. When I came to the equation $f = \frac{\dot{y}^3 - x\dot{y}\ddot{x}}{2x^3\dot{y}^3}$, we might more briefly have gone on to the integration, by multiplying by $\dot{x}$, and difpofing it thus, $f\dot{x} = \frac{\dot{x}}{2x^3} - \frac{\dot{x}\ddot{x}}{2xx\dot{y}\dot{y}}$, where, becaufe $x\dot{y}$ is conftant, it will be $\int f\dot{x} = -\frac{1}{4xx} - \frac{\dot{x}\dot{x}}{4xx\dot{y}\dot{y}} \pm n$, as before.

Now let us make conftant the quantity $\frac{\dot{x}}{x}$. Such a fuppofition giving $\frac{x\ddot{x} - \dot{x}\dot{x}}{xx} = 0$, and alfo $-x\dot{y}\ddot{x} + \dot{x}\dot{x}\dot{y} = 0$, takes away the fecond and third terms from the principal equation, and changes it into this, $f = \frac{\dot{y}^3 + x\dot{x}\ddot{y}}{2x^3\dot{y}^3}$; and, multiplying by $\dot{x}$, it is $f\dot{x} = \frac{\dot{x}\dot{y}^3 + x\dot{x}^2\ddot{y}}{2x^3\dot{y}^3}$, the integral of which, (becaufe of $\frac{\dot{x}}{x}$, or $\frac{\dot{x}\dot{x}}{xx}$ conftant,) will be found to be $\int f\dot{x} = -\frac{1}{4xx} - \frac{\dot{x}\dot{x}}{4xx\dot{y}\dot{y}} \pm n$, as above.

48. But, to know nearly what fluxion may be taken for conftant, it may be obferved, if, in the propofed equation, there be two, three, or more terms, which, being multiplied or divided by a quantity which is common to them, they may be reduced to be integrable; then making the integration, their integral may be taken as conftant, and fo proceed in the manner fpecified. If not always, yet fometimes, at leaft, we fhall fucceed in our attempt.

I refume the equation $f = \frac{\dot{y}^3 + \dot{x}^2\dot{y} - x\dot{y}\ddot{x} + x\dot{x}\ddot{y}}{2x^3\dot{y}^3}$, and obferve, that the two terms $\dot{x}^2\dot{y} + x\dot{x}\ddot{y}$, being divided by $\dot{x}$, will become $\dot{x}\dot{y} + x\ddot{y}$, which is an integrable quantity, and that it's integral is $x\dot{y}$. See, then, upon what account we may take this quantity for conftant. In like manner, I obferve, that the two terms $\dot{x}^2\dot{y} - x\dot{y}\ddot{x}$, if they be divided by $-xx\dot{y}$, will give us $\frac{-\dot{x}\dot{x} + x\ddot{x}}{xx}$, an integrable quantity, the integral of which is $\frac{\dot{x}}{x}$; therefore the fluxion $\frac{\dot{x}}{x}$ might alfo be taken as conftant.

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For example, let the formula $xy \times \overline{\dot x\ddot y - \dot y\ddot x} = y\dot y\dot x^2 - y^2\dot z\dot y^2 - x\dot x\dot y^2$ be proposed, in which the variable z is any how given by y. I dispose it thus, $xy\dot x\ddot y + yy\dot z\dot y^2 = yxy\ddot x + y\dot y\dot x^2 - x\dot x\dot y^2$, and observe, that, if the *homogeneum comparationis* be divided by $yy\dot y$, it will be $\frac{yx\ddot x + y\dot x^2 - x\dot x\dot y}{yy}$, the integral of which is $\frac{x\dot x}{y}$. Therefore I take $\frac{x\dot x}{y}$ for constant, and make $\frac{x\dot x}{y} = c$, and thence $\frac{xy\ddot x + y\dot x^2 - x\dot x\dot y}{yy} = 0$. Whence the proposed equation will become $xy\dot x\ddot y + yy\dot z\dot y^2 = 0$, that is, $\dot z = -\frac{x\dot x\ddot y}{y\dot y^2}$; and, by integration, because of $\frac{x\dot x}{y}$ constant, it will be $z = \frac{x\dot x}{y\dot y} \pm n$.

49. In an equation of the second degree, when either of the two indeterminates are wanting with all it's functions, and only it's first or second differences enter in the formula, any how compounded and raised to any dignity; the integration, or reduction to first fluxions, will always be in our power, by help of a substitution. This will be, to make the first fluxion, which is flowing or indeterminate, equal to a new variable multiplied into a constant assumed fluxion, or which may be assumed at pleasure, in case that no other be appointed constant. For example, in a given equation, let $\dot x$, at first, be supposed variable, and $\dot y$ constant; make $\dot x = p\dot y$, and taking the fluxions, on the supposition of $\dot y$ being constant, it will be $\ddot x = \dot p\dot y$. Making this substitution instead of $\ddot x$, and the equation being managed by substituting the values taken from the equation $\dot x = p\dot y$, it will always be reduced to first fluxions.

Or, perhaps, it may be more convenient to make the first fluxion of the variable, which is wanting in the equation, equal to a new indeterminate, multiplied into the first fluxion of the other. Making the necessary substitutions, and having a due regard to the fluxion which, at first, was taken for constant, we shall have the proposed equation reduced to first fluxions.

EXAMPLE I.

Let us take again the equation of the first example of § 46, $\frac{by^m}{c^m} = \frac{2ay\ddot x + a\dot x\dot y}{\dot u\dot y}$, in which $\dot u$ is supposed constant. Make, therefore, $\dot x = p\dot u$, and by differencing, $\ddot x = \dot p\dot u$. Then, substituting this value, we shall have $\frac{by^m}{c^m} =$

$\frac{2a\dot{y}\dot{p}\dot{u} + ap\dot{y}\dot{y}}{\dot{y}\dot{u}}$, that is, $\frac{by^m}{c^m} = \frac{2ay\dot{p} + ap\dot{y}}{\dot{y}}$, and therefore $\frac{by^m\dot{y}}{c^m} = 2ay\dot{p} + ap\dot{y}$, which equation, divided by $2\sqrt{y}$, is integrable, and the integral is $\frac{by^{m+\frac{1}{2}}}{m+\frac{1}{2} \times 2c^m} = ap\sqrt{y} \pm g$. But $p = \frac{\dot{x}}{\dot{u}}$, therefore $\frac{by^{m+\frac{1}{2}}\dot{u}}{m+\frac{1}{2} \times 2c^m} = a\dot{x}\sqrt{y} \pm g\dot{u}$.

EXAMPLE II.

Let the equation be $fyy\dot{y}\dot{x}\dot{x} = -\dot{u}\ddot{u}$, where f is given by y, $\dot{u}$ is the element of a curve, and $y\dot{x}$ is the fluxion taken for conſtant. Therefore I make $\dot{u} = py\dot{x}$, and, by differencing, it is $\ddot{u} = y\dot{p}\dot{x}$; and therefore, making the ſubſtitutions, it is $fy^2\dot{y}\dot{x}^2 = -y^2p\dot{p}\dot{x}^2$, that is, $f\dot{y} = -p\dot{p}$. Whence, by integration, $2\int f\dot{y} = -pp + 2m$. But $pp = \frac{\dot{u}\dot{u}}{yy\dot{x}\dot{x}} = \frac{\dot{x}\dot{x} + \dot{y}\dot{y}}{yy\dot{x}\dot{x}}$. Wherefore, making the ſubſtitutions and the reduction, we ſhall have $\dot{x} = \frac{\dot{y}}{\sqrt{2myy - 1 - 2yy\int f\dot{y}}}$.

Now I reduce the ſame equation by means of the other ſubſtitution mentioned before. Make, therefore, $\dot{x} = p\dot{u}$, and $\ddot{x} = \dot{p}\dot{u} + p\ddot{u}$, whence $\ddot{u} = \frac{\ddot{x} - \dot{p}\dot{u}}{p}$. Making the ſubſtitutions, the equation will be $fyyp\dot{p}\dot{y}\dot{u}\dot{u} = \frac{-\dot{u}\ddot{x} + \dot{p}\dot{u}\dot{u}}{p}$. But the fluxion $y\dot{x}$ is aſſumed as conſtant, whence we ſhall have $y\ddot{x} + \dot{y}\dot{x} = 0$, that is, $\ddot{x} = -\frac{\dot{x}\dot{y}}{y}$, or $\ddot{x} = -\frac{p\dot{u}\dot{y}}{y}$. And, ſubſtituting this value again in in the equation, it will be $fppyy\dot{y} = \frac{\dot{y}}{y} + \frac{\dot{p}}{p}$. This ſuppoſed, we may go on, and make $\frac{\dot{y}}{y} + \frac{\dot{p}}{p} = \frac{\dot{q}}{q}$, whence $py = q$, and therefore $fqq\dot{y} = \frac{\dot{q}}{q}$, or $f\dot{y} = \frac{\dot{q}}{q^3}$. And, by integration, $\int f\dot{y} = -\frac{1}{2qq} + m$. But $qq = ppyy = \frac{yy\dot{x}\dot{x}}{\dot{u}\dot{u}} = \frac{yy\dot{x}\dot{x}}{\dot{x}\dot{x} + \dot{y}\dot{y}}$. Therefore it will be $2\int f\dot{y} = -\frac{\dot{x}\dot{x} + \dot{y}\dot{y}}{yy\dot{x}\dot{x}} + 2m$; from whence we may derive, as above, $\dot{x} = \frac{\dot{y}}{\sqrt{2myy - 1 - 2yy\int f\dot{y}}}$.

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EXAMPLE III.

I resume the equation of Example III, § 46, $fy^3\dot{x}\dot{x} = \dot{x}\dot{x} + \dot{y}\dot{y} - y\ddot{y}$, in which $\dot{x}$ is constant; and make $\dot{y} = p\dot{x}$, and therefore $\ddot{y} = \dot{p}\dot{x}$. Making the substitutions, it will be $fy^3\dot{x}\dot{x} = \dot{x}\dot{x} + \dot{y}\dot{y} - y\dot{p}\dot{x}$; and, making $\dot{x}$ to vanish by it's value $\frac{\dot{y}}{p}$, we shall have $\frac{fy^3\dot{y}\dot{y}}{pp} = \frac{\dot{y}\dot{y}}{pp} + \dot{y}\dot{y} - \frac{y\dot{y}\dot{p}}{p}$; that is, $fy^3\dot{y}\dot{y} = \dot{y}\dot{y} + pp\dot{y}\dot{y} - yp\dot{y}\dot{p}$. And, dividing by $y^3\dot{y}$, it will be $f\dot{y} = \frac{\dot{y}}{y^3} + \frac{pp\dot{y} - yp\dot{p}}{y^3}$. And, by integration, $\int f\dot{y} = -\frac{1}{2yy} - \frac{pp}{2yy} + m$. And, instead of p, substituting it's value $\frac{\dot{y}}{\dot{x}}$, it is $\int f\dot{y} = -\frac{1}{2yy} - \frac{\dot{y}\dot{y}}{2yy\dot{x}\dot{x}} + m$, that is, $2\int f\dot{y} = -\frac{\dot{x}\dot{x} + \dot{y}\dot{y}}{yy\dot{x}\dot{x}} + 2m$; and therefore $\dot{x} = \frac{\dot{y}}{\sqrt{2myy - 1 - 2yy\int f\dot{y}}}$.

50. If, in the proposed equation, no fluxion has been taken for constant, one may be taken at pleasure, and the operation may be performed, as is done at § 48.

As, for example, the equation of Example V, § 46, being given, in which no fluxion is assumed as constant, that is, $fy^3\dot{y}\dot{x}^3 = \dot{x}\dot{y}\dot{u}\dot{u} + y\dot{u}\dot{u}\ddot{x} - y\dot{x}\dot{u}\ddot{u}$, (putting $y\dot{x}$ instead of $\dot{t}$,) if $\dot{x}$ be made constant, it will expunge the term $y\dot{u}\dot{u}\ddot{x}$, and the equation will become $fy^3\dot{y}\dot{x}^2 = \dot{y}\dot{u}^2 - y\dot{u}\ddot{u}$. Now, to reduce it, we must put $\dot{u} = p\dot{x}$, whence $\ddot{u} = \dot{p}\dot{x}$. These values being substituted, we shall have $fy^3\dot{y}\dot{x}\dot{x} = pp\dot{y}\dot{x}\dot{x} - yp\dot{p}\dot{x}\dot{x}$, that is, $fy^3\dot{y} = pp\dot{y} - yp\dot{p}$; which equation, in order to proceed to integration, I write thus, $fy^3\dot{y} = ppy \times \overline{\frac{\dot{y}}{y} - \frac{\dot{p}}{p}}$. Therefore, integrating by the method of § 24 aforegoing, $\int f\dot{y} = -\frac{pp}{2yy} + m$; and, restoring the value of p, $\int f\dot{y} = -\frac{\dot{u}\dot{u}}{2yy\dot{x}\dot{x}} + m$.

If $\dot{u}$ be taken as constant, the term $y\dot{x}\dot{u}\ddot{u}$ will be expunged, and the equation will be $fy^3\dot{y}\dot{x}^3 = \dot{x}\dot{y}\dot{u}\dot{u} + y\dot{u}\dot{u}\ddot{x}$, and therefore we must put $\dot{x} = p\dot{u}$, $\ddot{x} = \dot{p}\dot{u}$. These values being substituted, we shall have $fy^3\dot{y} \times p^3\dot{u}^3 = p\dot{y}\dot{u}^3 + y\dot{p}\dot{u}^3$, that is, $fy^3\dot{y} = \frac{p\dot{y} + y\dot{p}}{p^3}$; then, by integration, it will be $\int f\dot{y} = -\frac{1}{2ppyy} + m$; and restoring the value of p, it will be $\int f\dot{y} = -\frac{\dot{u}\dot{u}}{2yy\dot{x}\dot{x}} + m$.

51. To aſſume at pleaſure any fluxion as conſtant, in equations wherein there is none already ſo taken, may make ſome equations ſubject to the method of § 49, which are not ſo already, becauſe of having both the indeterminates finite quantities. And this by aſſuming ſuch a fluxion for conſtant, as may make all the terms to vaniſh, in which is found one of the finite indeterminates, thoſe only remaining which include the other.

For example, let the equation be $\dot{x}^3 - \dot{x}\dot{y}\dot{y} = y\dot{x}\ddot{x} + 2x\dot{y}\ddot{y}$, in which no fluxion is taken as conſtant. If we make $\dot{x}$ conſtant, the firſt term of the *homogeneum comparationis* will vaniſh; and if we make $\dot{y}$ conſtant, the laſt term will vaniſh; and, in either caſe, there remains only one of the indeterminates. Therefore, appointing $\dot{x}$ to be conſtant, the equation will be $\dot{x}^3 - \dot{x}\dot{y}\dot{y} = 2x\dot{y}\ddot{y}$. Put $\dot{y} = \frac{p\dot{x}}{a}$, $\ddot{y} = \frac{\dot{p}\dot{x}}{a}$, and making the ſubſtitutions, it will be $\dot{x}^3 - \frac{pp\dot{x}^3}{aa} = \frac{2xp\dot{p}\dot{x}\dot{x}}{aa}$, that is, $aa\dot{x} - pp\dot{x} = 2xp\dot{p}$, or $\frac{\dot{x}}{x} = \frac{2p\dot{p}}{aa - pp}$; then, by integration, it will be $lx = -l\overline{aa - pp} + lm$, and therefore $x = \frac{m}{aa - pp}$. And, inſtead of p, reſtoring it's value $\frac{a\dot{y}}{\dot{x}}$, it will be $x = \frac{m}{aa - \frac{aa\dot{y}\dot{y}}{\dot{x}\dot{x}}}$, that is, $x = \frac{m\dot{x}\dot{x}}{aa\dot{x}^2 - aa\dot{y}^2}$, or $m\dot{x}^2 = aax\dot{x}^2 - a^2x\dot{y}^2$.

52. But when the taking at pleaſure a fluxion for conſtant, does not ſucceed in eliminating one of the two finite indeterminates, or if the conſtant fluxion be already fixed, ſo that both the indeterminates remain in the equation; there is no general method as yet diſcovered, how to proceed further.

The methods here explained may ſometimes have their uſe, as alſo the uſual expedients of common Algebra, ſuch as multiplication, diviſion, &c. As, for example, in the equation $xxy\dot{y}\ddot{y} = x\ddot{x} - \dot{x}\dot{x}$, which, being divided by xx, will be $y\dot{y}\ddot{y} = \frac{x\ddot{x} - \dot{x}\dot{x}}{xx}$, and therefore is integrable, (ſuppoſing $\dot{y}$ to be conſtant,) and the integral is $\frac{1}{2}yy\dot{y} = \frac{\dot{x}}{x} + m\dot{y}$.

Sometimes a ſubſtitution may make the propoſed equation within the reach of the method of § 49. And, indeed, the equation $x^m\ddot{x} = y\ddot{y} + \dot{y}\dot{y} + yy\dot{y}\dot{y}$, which is not ſubject to the canon of the aforeſaid article, will however be ſo, if we make $y\dot{y} = \dot{z}$; whence it will be $x^m\ddot{x} = \ddot{z} + \dot{z}\dot{z}$.

53. Wherefore, in caſe that in the equation there ſhould be already a conſtant fluxion, it may be of good uſe to change the propoſed equation into another equivalent

equivalent to it, in which no fluxion is conſtant. To do which, let there be a general equation $\dot{y} = p\dot{x}$, where p is a quantity any how given by x and y, and let $\dot{x}$ be conſtant. By taking the difference, it will be $\ddot{y} = \dot{p}\dot{x}$. But it is $p = \frac{\dot{y}}{\dot{x}}$; then, by differencing, without making any conſtant fluxion, it will be $\dot{p} = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}\dot{x}}$. Wherefore, the value of $\dot{p}$ being ſubſtituted in the equation $\ddot{y} = \dot{p}\dot{x}$, we ſhall have $\ddot{y} = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}}$. So that, in any propoſed equation in which $\dot{x}$ is conſtant, inſtead of $\ddot{y}$, if we put it's value, $\frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}}$, it will be changed into another that is equivalent to it, in which there is no conſtant fluxion.

But, becauſe often other more compound fluxions may be aſſumed as conſtant, or have been at firſt aſſumed, it may be of uſe to render this method more univerſal.

Let us take this general equation $\dot{y} = mp\dot{x}$, where p is likewiſe given, in any manner, by x and y, and m is any function whatever of x or of y, or of both together. Let $m\dot{x}$ be conſtant; then, by differencing, it will be $\ddot{y} = m\dot{x}\dot{p}$. But $p = \frac{\dot{y}}{m\dot{x}}$; and by differencing, without aſſuming any conſtant, it is $\dot{p} = \frac{m\dot{x}\ddot{y} - \dot{m}\dot{x}\dot{y} - m\dot{y}\ddot{x}}{mm\dot{x}\dot{x}}$. Wherefore, ſubſtituting this value in the equation $\ddot{y} = m\dot{x}\dot{p}$, inſtead of $\dot{p}$, we ſhall have $\ddot{y} = \frac{m\dot{x}\ddot{y} - \dot{m}\dot{x}\dot{y} - m\dot{y}\ddot{x}}{m\dot{x}}$. Wherefore in any propoſed equation, in which $m\dot{x}$ is conſtant, if, inſtead of $\ddot{y}$, we put it's value now found, it will be changed into another which is equivalent, in which no fluxion is conſtant.

After this manner equations being made complete, that is, ſuch as may have no conſtant fluxion, in proceeding to the reduction, we ſhall be at liberty to take that for conſtant, by the aſſiſtance of which we may beſt attain our purpoſe.

EXAMPLE I.

Let it be propoſed to reduce this equation, $\dot{x}x\ddot{y} - \dot{y}^3 = a\dot{x}\ddot{y} + x\dot{x}\ddot{y}$, in which $\dot{x}$ is conſtant. Therefore, inſtead of $\ddot{y}$, putting it's value $\frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}}$, (for

in this cafe $m = 1$, and $\dot{m} = 0$,) it will be $\dot{x}x\ddot{y} - \dot{y}^3 = a\dot{x}\ddot{y} - a\dot{y}\ddot{x} + x\dot{x}\ddot{y} - x\dot{y}\ddot{x}$, in which no fluxion is conftant. Whence, making $\dot{y}$ conftant, it will be found to be $\dot{x}\dot{x} + x\ddot{x} + a\ddot{x} = \dot{y}\dot{y}$; and, by integration, $x\dot{x} + a\dot{x} = y\dot{y}$, which is an equation to the hyperbola.

EXAMPLE II.

Let the equation be $-\frac{x\dot{y}\dot{y} + xy\ddot{y} + y\dot{y}\dot{x}}{y\dot{y}} = \frac{aa\dot{x} - xx\dot{x}}{aa + xx}$, in which the fluxion $y\dot{x}$ is affumed as conftant. To transform it into another, in which there is no conftant fluxion, becaufe in this cafe it is $m = y$, the value of $\ddot{y}$ to be fubftituted will be $\frac{y\dot{x}\ddot{y} - \dot{x}\dot{y}\dot{y} - y\dot{y}\ddot{x}}{y\dot{x}}$, and therefore the equation is $-\frac{x\dot{y}}{y} - \dot{x} - \frac{xy\dot{x}\ddot{y} - x\dot{x}\dot{y}\dot{y} - xy\dot{y}\ddot{x}}{y\dot{x}\dot{y}} = \frac{aa\dot{x} - xx\dot{x}}{aa + xx}$. To reduce this, making $x\dot{y}$ a conftant fluxion, in confequence of which it will be $x\ddot{y} + \dot{x}\dot{y} = 0$, that is, $-\ddot{y} = \frac{\dot{x}\dot{y}}{x}$; then making the fubftitution, it is $-\frac{x\dot{y}}{y} - \dot{x} + \dot{x} + \frac{x\dot{y}}{y} + \frac{x\ddot{x}}{\dot{x}} = \frac{aa\dot{x} - xx\dot{x}}{aa + xx}$, that is, $-\frac{\ddot{x}}{\dot{x}} = \frac{xx\dot{x} - aa\dot{x}}{aax + x^3}$; and, by integration, $-l\dot{x} = l\frac{aa + xx}{x} - lx\dot{y}$. Here I fubtract $lx\dot{y}$, becaufe it is a conftant quantity. And, taking away the logarithms, $\frac{1}{\dot{x}} = \frac{aa + xx}{xx\dot{y}}$, that is, $x^2\dot{y} = a^2\dot{x} + x^2\dot{x}$.

EXAMPLE III.

Let the equation be $-\frac{\dot{x}\ddot{y}}{\dot{y}} - \frac{\dot{y}\dot{x}}{y} = \frac{\dot{x}\dot{x} + \dot{y}\dot{y}}{x}$, and $y\dot{x}$ a conftant fluxion. Therefore, inftead of $\ddot{y}$, I put it's correfponding value, $\frac{y\dot{x}\ddot{y} - \dot{x}\dot{y}\dot{y} - y\dot{y}\ddot{x}}{y\dot{x}}$, and it will be $-\frac{\dot{x}\ddot{y}}{\dot{y}} + \frac{\dot{y}\ddot{x}}{\dot{y}} = \frac{\dot{x}\dot{x} + \dot{y}\dot{y}}{x}$, in which there is no conftant fluxion. Wherefore, taking $\dot{y}$ conftant, it will be $x\ddot{x} = \dot{x}\dot{x} + \dot{y}\dot{y}$. Which equation is the cafe of § 49, and therefore it's reduction is known.

54. The

54. The method explained in the foregoing Section, at § 24, may be also of use in differentio-differential equations, by proceeding nearly in the manner there pursued. Here is the practice in some Examples.

EXAMPLE I.

I resume the formula of the first Example of this Section, $\frac{by^m}{c^m} = \frac{2ay\ddot{x} + a\dot{x}\dot{y}}{\dot{u}\dot{y}}$, in which $\dot{u} = \sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}$ is assumed constant. It will be $\frac{by^m\dot{y}\dot{u}}{ac^m} = 2y\ddot{x} + \dot{x}\dot{y}$. I prepare it after the following manner, $\overline{\frac{\ddot{x}}{\dot{x}} + \frac{\dot{y}}{2y}} \times \dot{x} = \frac{by^m\dot{y}\dot{u}}{ac^m \times 2y}$, where I observe, that the two quantities under the vinculum are integrable, by means of the logarithms. Therefore I make $\frac{\ddot{x}}{\dot{x}} + \frac{\dot{y}}{2y} = \frac{\dot{p}}{p}$; and therefore $l\dot{x} + l\sqrt{y} = lp + l\dot{u}$; (I add $l\dot{u}$, because of $\dot{u}$ constant,) that is, $\dot{x}\sqrt{y} = p\dot{u}$. Wherefore, in the proposed equation, instead of $\frac{\ddot{x}}{\dot{x}} + \frac{\dot{y}}{2y}$, substituting it's value $\frac{\dot{p}}{p}$, and, instead of $\dot{x}$, it's value $\frac{p\dot{u}}{\sqrt{y}}$, it will be $\frac{\dot{p}\dot{u}}{\sqrt{y}} = \frac{by^{m-1}\dot{y}\dot{u}}{2ac^m}$, or $\dot{p} = \frac{by^{m-\frac{1}{2}}\dot{y}}{2ac^m}$; and, by integration, $h + p = \frac{by^{m+\frac{1}{2}}}{\overline{m+\frac{1}{2}} \times 2ac^m}$. But $p = \frac{\dot{x}\sqrt{y}}{\dot{u}}$, and therefore, lastly, $h\dot{u} + \dot{x}\sqrt{y} = \frac{by^{m+\frac{1}{2}}\dot{u}}{\overline{m+\frac{1}{2}} \times 2ac^m}$, as in the Example quoted above.

EXAMPLE II.

Let the equation be $\frac{-\ddot{x}\sqrt{xx+yy}}{x} = \frac{\overline{y\dot{x} - x\dot{y}}^2}{xx+yy}$, in which $y\dot{x} - x\dot{y}$ is constant. The second fluxion $\ddot{x}$, divided by the constant $\dot{x}y - x\dot{y}$, will give us an integrable quantity, and therefore I write the equation thus, $\frac{-\ddot{x}}{y\dot{x} - x\dot{y}} =$

$\frac{x \times \overline{y\dot{x} - x\dot{y}}}{xx + yy \times \sqrt{xx + yy}}$. But I obſerve, that, in the ſecond member, the quantity $y\dot{x} - x\dot{y}$ is ſummable when it is divided by yy; therefore I prepare the equation according to this method, and it will be $\frac{-\ddot{x}}{y\dot{x} - x\dot{y}} = \frac{xyy}{xx + yy \times \sqrt{xx + yy}} \times \frac{y\dot{x} - x\dot{y}}{yy}$. Make $\frac{y\dot{x} - x\dot{y}}{yy} = \dot{p}$, and, by integration, $\frac{x}{y} = p$. Whence, making the ſubſtitution, we ſhall have $\frac{-\ddot{x}}{y\dot{x} - x\dot{y}} = \frac{xyy\dot{p}}{xx + yy \times \sqrt{xx + yy}}$, from whence we can expunge x or y, by means of the equation $\frac{x}{y} = p$. Expunge x from the ſecond member, by putting it's value py in it's place, and we ſhall have $\frac{-\ddot{x}}{y\dot{x} - x\dot{y}} = \frac{p\dot{p}}{1 + pp \times \sqrt{1 + pp}}$; and, proceeding to the integration, it will be $\frac{-\dot{x}}{y\dot{x} - x\dot{y}} = -\frac{1}{\sqrt{1 + pp}}$, that is, $\frac{-\dot{x}}{y\dot{x} - x\dot{y}} = \frac{-y}{\sqrt{1yy + xx}}$, inſtead of p, by reſtoring it's value $\frac{x}{y}$.

In this integration the conſtant $y\dot{x} - x\dot{y}$ might have been added; but whether it be added or omitted, the reduction of firſt differences to finite quantities, in each caſe, will always give the conic ſections.

55. I ſaid before, at § 52, that when the differentio-differential equations contain both the variables, there is no general method to reduce them. One, however, may be aſſigned, which, though it does not ſerve in all caſes, yet is very general in it's kind, and comprehends all the infinite number of equations, which may be referred to theſe three following canons. By the help of this method, the given equations are transformed into others, in which one of the two variables is wanting, and conſequently they may be managed by the method of § 49.

The firſt canon comprehends thoſe which are of two terms only, and are expreſſed by the general formula $ax^m\dot{x}^p = y^n\dot{y}^{p-2}\ddot{y}$, in which let $\dot{x}$ be taken as conſtant. To reduce this equation, make $x = c^{hu}$, and $y = c^u t$, where c is a number, the logarithm of which is unity, and h is an arbitrary quantity to be determined afterwards, and u, t, are two new variables. Now, ſince $x = c^{hu}$, and $y = c^u t$, by the rules of the exponential calculus it will be $\dot{x} = hc^{hu}\dot{u}$, $\ddot{x} = hc^{hu} \times \overline{\ddot{u} + h\dot{u}\dot{u}}$, $\dot{y} = c^u\dot{t} + c^u t\dot{u}$, $\ddot{y} = c^u \times \overline{\ddot{t} + 2\dot{t}\dot{u} + t\dot{u}\dot{u} + t\ddot{u}}$.

But,

But, making $\dot{x}$ constant, it is $\ddot{x} = 0$, and therefore $bc^{bu} \times \overline{\ddot{u} + b\dot{u}\dot{u}} = 0$, or $\ddot{u} = -b\dot{u}\dot{u}$. This, being substituted, instead of $\ddot{u}$, in the value of $\ddot{y}$, will be $\ddot{y} = c^{u} \times \overline{\ddot{t} + 2\dot{t}\dot{u} + \overline{1-b} \times t\dot{u}\dot{u}}$. In the proposed equation, substituting the respective values instead of x, y, and their differentials, it will be changed into this other, $ac^{bmu} \times b^{p} \times c^{bpu}\dot{u}^{p} = c^{nu}t^{n} \times \overline{c^{u}\dot{t} + c^{u}t\dot{u}}|^{p-2} \times c^{u} \times \overline{\ddot{t} + 2\dot{t}\dot{u} + \overline{1-b} \times t\dot{u}\dot{u}}$, that is, $ac^{bu \times \overline{m+p}}b^{p}\dot{u}^{p} = c^{\overline{n+p-1} \times u}t^{n} \times \overline{\dot{t} + t\dot{u}}|^{p-2} \times \overline{\ddot{t} + 2\dot{t}\dot{u} + \overline{1-b} \times t\dot{u}\dot{u}}$.

Now, to free this equation from exponential quantities, that is, to take c out of it, it will be necessary that $n + p - 1 = bm + bp$, by which the value of the assumed quantity b will be determined, that is, $b = \frac{n+p-1}{m+p}$. Whence the equation will be $\frac{a \times \overline{n+p-1}|^{p} \times \dot{u}^{p}}{\overline{m+p}|^{p}} = t^{n} \times \overline{\dot{t} + t\dot{u}}|^{p-2} \times \overline{\ddot{t} + 2\dot{t}\dot{u} + \frac{m-n+1}{m+p} \times t\dot{u}\dot{u}}$, which, because it contains only one of the finite variables, that is, t, will now be subject to the above-cited rule.

Now, since we have found the value of $b = \frac{p+n-1}{p+m}$, it easily appears what substitutions might have been made at the beginning, that is, $x = c^{\frac{n+p-1}{m+p} \times u}$, and $y = c^{u}t$, in order to obtain our intention.

To go on with the operation according to the method of § 49, make $\dot{u} = z\dot{t}$, and therefore $\ddot{u} = z\ddot{t} + \dot{z}\dot{t}$. But the supposition of $\dot{x}$ constant has given us $\ddot{u} = -b\dot{u}\dot{u}$, that is, $\ddot{u} = \frac{1-n-p}{m+p} \times zz\dot{t}\dot{t}$. Therefore we shall have $\frac{1-n-p}{m+p} \times zz\dot{t}\dot{t} = z\ddot{t} + \dot{t}\dot{z}$, whence $\ddot{t} = \frac{1-n-p}{m+p} \times z\dot{t}\dot{t} - \frac{\dot{t}\dot{z}}{z}$. Wherefore, substituting in the equation their respective values, instead of $\dot{u}$ and $\ddot{t}$, it will be $a \times \frac{\overline{p+n-1}|^{p}}{\overline{m+p}|^{p}} \times z^{p}\dot{t}^{p} = t^{n} \times \overline{\dot{t} + zt\dot{t}}|^{p-2} \times \overline{\frac{1-n-p}{p+m} \times z\dot{t}\dot{t} - \frac{\dot{t}\dot{z}}{z} + 2z\dot{t}\dot{t} + \frac{m-n+1}{m+p} \times zzt\dot{t}\dot{t}}$; or, dividing by $\dot{t}^{p-1}$, and multiplying by z, it will be $a \times \frac{\overline{n+p-1}|^{p}}{\overline{m+p}|^{p}} \times z^{p+1}\dot{t} = t^{n} \times \overline{1+tz}|^{p-2}$

×

$\times \overline{\frac{1 + 2m - n + p}{m + p} \times zz\dot{t} + \frac{m - n + 1}{m + p} \times tz^3\dot{t} - \dot{z}}$; which equation is now reduced to first fluxions. It is easy to perceive, that, to reduce the equation, it would be sufficient to make $x = c^{\frac{n+p-1}{m+p} \times \int z\dot{t}}$, and $y = c^{\int z\dot{t}} \times t$.

In this general equation, which I have now reduced, I supposed the fluxion $\dot{x}$ to be constant; yet it would make no difficulty in the method, that, in any proposed equation, some other fluxion different from $\dot{x}$ should be made constant. For, by § 53, the proposed equation may be changed into another equivalent to it, in which no fluxion is constant, and then the said $\dot{x}$ may be made constant.

EXAMPLE I.

Let the equation be $x\dot{x}\dot{y} = y\ddot{y}$, in which $\dot{x}$ is constant. I write it thus, $\dot{x}x = y\dot{y}^{-1}\ddot{y}$. This being compared with the canonical equation, it will be $a = 1$, $m = 1$, $p = 1$, $n = 1$; whence, these values being substituted in the general differential equation of the first degree found above, we shall have $\frac{1}{2}zz\dot{t} = \frac{t}{1 + tz} \times \overline{\frac{3}{2}zz\dot{t} + \frac{1}{2}tz^3\dot{t} - \dot{z}}$.

EXAMPLE II.

Let $p = 1$, $n = -1$, $m = -1$, or the equation $ax^{-1}\dot{x} = y^{-1}\dot{y}^{-1}\ddot{y}$, or $\frac{a\dot{x}}{x} = \frac{\ddot{y}}{y\dot{y}}$, in which $\dot{x}$ is a constant fluxion. In respect of this, the method will be of no use, for we shall have $p + m = 0$, and consequently every one of the terms of the general differential equation of the first degree, except the last, will be infinite.

But, in this case, the reduction is easy, without any further artifice. I write the equation thus, $x\ddot{y} = ay\dot{y}\dot{x}$. Now the integral of the first member is $x\dot{y} - y\dot{x}$, that of the second is $\frac{1}{2}ayy\dot{x}$. Therefore the equation is $x\dot{y} - y\dot{x} = \frac{1}{2}ayy\dot{x} \pm b\dot{x}$.

56. The

56. The ſecond canon comprehends all thoſe equations, in which the ſum of the exponents of the indeterminates, and of their differentials, is the ſame in every term. Suppoſing x and y the two indeterminates, and $\dot{x}$ to be conſtant, theſe are reduced to the caſe of § 49, by putting $x = c^u$, and $y = c^u t$; c being ſtill the number, the logarithm of which is unity, and u, t, are new indeterminates. To ſhow the method, I ſhall take the equation $ax^m y^{-m-1} \dot{x}^p \dot{y}^{2-p} + bx^n y^{-n-1} \dot{x}^q \dot{y}^{2-q} = \ddot{y}$, which, though it be but of one dimenſion only, and of three terms only, yet the method is general notwithſtanding, and will ſerve for any number of terms and dimenſions, if the conditions be obſerved.

Therefore I make $x = c^u$, $y = c^u t$; it will be $\dot{x} = c^u \dot{u}$; and, becauſe $\dot{x}$ is conſtant, we ſhall have $c^u \ddot{u} + c^u \dot{u}\dot{u} = 0$, that is, $\ddot{u} = -\dot{u}\dot{u}$. It will be alſo $\dot{y} = c^u \dot{t} + c^u t\dot{u}$, and $\ddot{y} = c^u \times \overline{\ddot{t} + 2\dot{u}\dot{t} + t\dot{u}\dot{u} + t\ddot{u}}$. But $\ddot{u} = -\dot{u}\dot{u}$; therefore $\ddot{y} = c^u \times \overline{\ddot{t} + 2\dot{u}\dot{t}}$. Wherefore, theſe values being ſubſtituted in the propoſed equation, it will be $at^{-m-1} \dot{u}^p \times \overline{\dot{t} + t\dot{u}}|^{2-p} + bt^{-n-1} \dot{u}^q \times \overline{\dot{t} + t\dot{u}}|^{2-q} = \ddot{t} + 2\dot{u}\dot{t}$. And, becauſe in this the indeterminate u is wanting, we may proceed by the method of § 49.

Make $\dot{u} = z\dot{t}$; it will be $\ddot{u} = \dot{z}\dot{t} + z\ddot{t}$. But $\ddot{u} = -\dot{u}\dot{u} = -zz\dot{t}\dot{t}$; therefore $\ddot{t} = -\frac{\dot{z}\dot{t}}{z} - z\dot{t}\dot{t}$. Wherefore, ſubſtituting theſe values, we ſhall have $at^{-m-1} z^p \dot{t}^p \times \overline{\dot{t} + zt\dot{t}}|^{2-p} + bt^{-n-1} z^q \dot{t}^q \times \overline{\dot{t} + zt\dot{t}}|^{2-q} = -\frac{\dot{z}\dot{t}}{z} + z\dot{t}\dot{t}$, or $at^{-m-1} z^p \dot{t} \times \overline{1 + zt}|^{2-p} + bt^{-n-1} z^q \dot{t} \times \overline{1 + zt}|^{2-q} = -\frac{\dot{z}}{z} + z\dot{t}$, a differential equation of the firſt degree. From hence it may be ſeen, that the propoſed equation might have been reduced at the beginning, by putting $x = c^{\int z\dot{t}}$, and $y = c^{\int z\dot{t}} t$.

For example, let the equation be $x\dot{x}\dot{y} - y\dot{x}\dot{x} = yy\ddot{y}$. To bring this to the canonical equation, I write it thus, $xy^{-2}\dot{x}\dot{y} - y^{-1}\dot{x}\dot{x} = \ddot{y}$. Then it will be $a = 1$, $m = 1$, $p = 1$, $n = 0$, $b = -1$, $q = 2$. Wherefore, theſe values being ſubſtituted in the differential canonical equation, here before found, we ſhall have the equation reduced, $t^{-2} z\dot{t} \times \overline{1 + zt} - t^{-1} zz\dot{t} = -\frac{\dot{z}}{z} + z\dot{t}$; or $\frac{z\dot{t} + zzt\dot{t}}{tt} - \frac{zz\dot{t}}{t} = \frac{-\dot{z} + zz\dot{t}}{z}$, that is, $zz\dot{t} - zzt\dot{t} = -tt\dot{z}$.

If

If we proceed on to the integration, it will be $\frac{tt\dot{t} - \dot{t}}{tt} = \frac{\dot{z}}{zz}$, and therefore, by integrating, $t + \frac{1}{t} = -\frac{1}{z} + f$, (where f is a conſtant to complete the integral,) that is, $ttz + z = -t + ftz$. But, by the ſubſtitutions, $z = \frac{\dot{u}}{\dot{t}}$, $x = c^u$, $y = c^u t$, it will be $\dot{u} = \frac{\dot{x}}{x}$, $t = \frac{y}{x}$, $\dot{t} = \frac{x\dot{y} - y\dot{x}}{xx}$, and therefore $z = \frac{x\dot{x}}{x\dot{y} - y\dot{x}}$; wherefore, ſubſtituting the values of t and z, we ſhall have $\frac{x\dot{x} + y\dot{y}}{y\dot{x}} = f$.

57. The third canon comprehends all thoſe equations, in which one of the two variables, whatever it may be, together with it's differentials, always makes in every term the ſame number of dimenſions. But we muſt here diſtinguiſh two caſes. One is, when the differential of that variable is conſtant, which forms the ſame number of dimenſions. The other caſe is, when the differential of the other is conſtant.

As to the firſt caſe, let the canonical equation be $Px^m\dot{y}^{m+2} + Qx^{m-n}\dot{x}^n\dot{y}^{m+2-n} = \dot{x}^m\ddot{y}$, in which the ſum of the exponents of x and $\dot{x}$ is the ſame in every term. P and Q are any functions of y, and $\dot{x}$ is conſtant. To reduce this equation, make $x = c^u$, where alſo c is a number, the logarithm of which is unity, and u is a new variable. Therefore it will be $\dot{x} = c^u\dot{u}$; and differencing again, making $\dot{x}$ conſtant, it will be $c^u\ddot{u} + c^u\dot{u}\dot{u} = 0$, that is, $\ddot{u} = -\dot{u}\dot{u}$. Theſe values being ſubſtituted in the equation, we ſhall have $P\dot{y}^{m+2} + Q\dot{u}^n\dot{y}^{m+2-n} = \dot{u}^m\ddot{y}$, which, becauſe it does not contain u, will be under the canon of § 49.

Therefore I put $\dot{u} = z\dot{y}$, and it will be $\ddot{u} = \dot{z}\dot{y} + z\ddot{y}$; but $\ddot{u} = -\dot{u}\dot{u} = -z^2\dot{y}^2$; therefore we ſhall have $z\ddot{y} + \dot{z}\dot{y} = -zz\dot{y}\dot{y}$; and thence $\ddot{y} = \frac{-zz\dot{y}\dot{y} - \dot{z}\dot{y}}{z}$. Wherefore, theſe values of $\dot{u}$ and $\ddot{y}$ being ſubſtituted in the equation before found, it will be $P\dot{y}^{m+2} + Qz^n\dot{y}^{m+2} = -z^{m+1}\dot{y}^{m+2} - z^{m-1}\dot{y}^{m+1}\dot{z}$; and, dividing by $\dot{y}^{m+1}$, it will be $P\dot{y} + Qz^n\dot{y} = -z^{m+1}\dot{y} - z^{m-1}\dot{z}$, an equation of the firſt degree. Therefore we might at firſt have made $x = c^{\int z\dot{y}}$, and thus have reduced the equation at one ſtroke.

For example, let the equation be $2a\dot{x}\dot{x}\dot{y} + ax\dot{x}\ddot{y} = 2x\dot{x}\dot{y}\dot{y} + 2xx\dot{y}\ddot{y}$, in which let $\dot{x}$ be conftant. Put $x = c^{\int z\dot{y}}$, and therefore $\dot{x} = z\dot{y}c^{\int z\dot{y}}$, and $\ddot{x} = c^{\int z\dot{y}} \times \overline{z^2\dot{y}^2 + z\ddot{y} + \dot{y}\dot{z}}$. But $\dot{x}$ is conftant, and therefore $zz\dot{y}\dot{y} + z\ddot{y} + \dot{y}\dot{z} = 0$, whence $\ddot{y} = \frac{-zz\dot{y}\dot{y} - \dot{z}\dot{y}}{z}$. Now, the values of x and $\dot{x}$ being fubftituted in the equation, we fhall have $2az^2\dot{y}^3 + az\dot{y}\ddot{y} = 2z\dot{y}^3 + 2\dot{y}\ddot{y}$; and, fubftituting the value of $\ddot{y}$, it is $2az^2\dot{y}^3 + az\dot{y} \times \frac{-z^2\dot{y}^2 - \dot{z}\dot{y}}{z} = 2z\dot{y}^3 + 2\dot{y} \times \frac{-zz\dot{y}^2 - \dot{z}\dot{y}}{z}$, that is, dividing by $\dot{y}\dot{y}$, $az^3\dot{y} - az\dot{z} = -2\dot{z}$, or $a\dot{y} = \frac{az\dot{z} - 2\dot{z}}{z^3}$. And, by integration, $ay = -\frac{a}{z} + \frac{1}{zz}$. Laftly, reftoring the value of z, which is given from the fuppofition made of $x = c^{\int z\dot{y}}$, that is, $z = \frac{\dot{x}}{x\dot{y}}$, we fhall have the equation reduced, $ay\dot{x}\dot{x} = xx\dot{y}\dot{y} - ax\dot{x}\dot{y}$.

58. As to the fecond cafe, let the canonical equation be $Px^m\dot{y}^{m+1} + Qx^{m-n}\dot{x}^n\dot{y}^{m-n+1} = \dot{x}^{m-1}\ddot{x}$, in which let $\dot{y}$ be conftant, and P, Q, any functions of y.

Put, as above, $x = c^u$, and therefore $\dot{x} = c^u\dot{u}$, $\ddot{x} = c^u\ddot{u} + c^u\dot{u}\dot{u}$. Make the fubftitutions in the canonical equation, and we fhall have $P\dot{y}^{m+1} + Q\dot{u}^n\dot{y}^{m-n+1} = \dot{u}^{m+1} + \dot{u}^{m-1}\ddot{u}$, which, becaufe it does not involve u, is fubject to the canon of § 49. Therefore I put $\dot{u} = z\dot{y}$; and, as $\dot{y}$ is conftant, it will be $\ddot{u} = \dot{z}\dot{y}$; and then making the fubftitutions, we fhall have $P\dot{y}^{m+1} + Qz^n\dot{y}^{m+1} = z^{m+1}\dot{y}^{m+1} + z^{m-1}\dot{y}^m\dot{z}$; and, dividing by $\dot{y}^m$, it will be $P\dot{y} + Qz^n\dot{y} = z^{m+1}\dot{y} + z^{m-1}\dot{z}$, an equation of the firft degree; which might have been reduced at once, by putting, as above, $x = c^{\int z\dot{y}}$.

For an example, let the equation be $2\dot{x}\dot{y} = a\ddot{x} - y\ddot{x}$, in which let $\dot{y}$ be conftant. Therefore, putting $x = c^{\int z\dot{y}}$, thence $\dot{x} = z\dot{y} \times c^{\int z\dot{y}}$, and $\ddot{x} = c^{\int z\dot{y}} \times \overline{z^2\dot{y}^2 + z\ddot{y} + \dot{y}\dot{z}}$. But $\dot{y}$ is fuppofed conftant, and therefore $\ddot{y} = 0$, and thence $\ddot{x} = c^{\int z\dot{y}} \times \overline{zz\dot{y}\dot{y} + \dot{z}\dot{y}}$. Wherefore, making the fubftitutions in the propofed equation, we fhall have $2z\dot{y}\dot{y} = azz\dot{y}\dot{y} + a\dot{z}\dot{y} - zzy\dot{y}\dot{y} - y\dot{y}\dot{z}$; and, dividing by $\dot{y}$, it will be $2z\dot{y} = azz\dot{y} + a\dot{z} - zzy\dot{y} - y\dot{z}$, which is a differential equation of the firft degree.

To go on to the integration, I divide the equation by $az - yz$, whence it is $\frac{2\dot{y}}{a-y} = z\dot{y} + \frac{\dot{z}}{z}$, or $\frac{2\dot{y}}{a-y} - \frac{\dot{z}}{z} = z\dot{y}$. And now, if you pleafe, making ufe of the method in § 24, by integrating, we fhall have $\frac{-1}{\overline{a-y}|^2 \times z} = \frac{-1}{a-y} + m$; and, laftly, by reftoring the value of $z = \frac{\dot{x}}{x\dot{y}}$, we fhall have the equation reduced, $y\dot{x} + x\dot{y} = a\dot{x}$, where the conftant m is neglected, which was introduced in the integration.

This example has ferved to fhow the application of the method; for otherwife fo many operations would have been unneceffary. Indeed, the equation itfelf, $2\dot{x}\dot{x} = a\ddot{x} - y\ddot{x}$, might have been reduced in an inftant, by only tranfpofing the term $y\ddot{x}$, and writing it thus: $2\dot{x}\dot{y} + y\ddot{x} = a\ddot{x}$; for, as $\dot{y}$ is conftant, the integral of the firft member is $y\dot{x} + x\dot{y}$, as plainly appears.

59. To what has been already faid, concerning differentio-differential equations, in which no firft fluxion was taken for conftant; another method may be added which is more univerfal, and which will ferve for all fuch as are comprehended under this canonical formula, $z^{m+1}\dot{x}^m\ddot{x} + \frac{\dot{z}}{z}\dot{y}^{m+1} = \dot{y}^m\ddot{y}$; in which z is any how given by the functions of x and y.

To reduce this, appoint the fluxion $\frac{\dot{x}}{q}$ for conftant, where q is any how given by the functions of x and y. Then make $\frac{\dot{x}}{q} = \dot{p}$. Now, becaufe $\frac{\dot{x}}{q}$ is conftant, it will be, by differencing, $q\ddot{x} - \dot{x}\dot{q} = 0$, that is, $\ddot{x} = \frac{\dot{x}\dot{q}}{q}$; or, inftead of $\frac{\dot{x}}{q}$, writing it's value $\dot{p}$, it will be $\ddot{x} = \dot{q}\dot{p}$. Befides, make $\dot{y} = u\dot{p}$, and taking the fecond fluxions, fuppofing $\dot{p}$ conftant, as being equal to $\frac{\dot{x}}{q}$, which is conftant, it will be $\ddot{y} = \dot{u}\dot{p}$. Therefore, in the canonical equation, fubftituting the values thus determined, inftead of $\dot{x}$, $\ddot{x}$, $\dot{y}$, and $\ddot{y}$, we fhall have the equation $z^{m+1}q^m\dot{q}\dot{p}^{m+1} + \frac{u^{m+1}\dot{z}\dot{p}^{m+1}}{z} = u^m\dot{u}\dot{p}^{m+1}$; and, dividing by $\dot{p}^{m+1}$, it will be $z^{m+1}q^m\dot{q} + \frac{u^{m+1}\dot{z}}{z} = u^m\dot{u}$, or $q^m\dot{q} = \frac{zu^m\dot{u} - u^{m+1}\dot{z}}{z^{m+2}}$. And, by integration, $\frac{q^{m+1}}{m+1} + g = \frac{u^{m+1}}{\overline{m+1} \times z^{m+1}}$, and therefore $u = z \times$

$\overline{q^{m+1} + \overline{m+1} \times g}^{\frac{1}{m+1}}$. But $u = \frac{\dot{y}}{\dot{p}} = \frac{q\dot{y}}{\dot{x}}$. Then $\frac{q\dot{y}}{\dot{x}} = z \times \overline{q^{m+1} + \overline{m+1}\, g}^{\frac{1}{m+1}}$, an equation reduced to firſt fluxions.

60. Concerning this laſt equation we are to obſerve, that, if the quantity z be given by x and y in ſuch manner, that to the quantity q ſuch a value may be aſſigned, alſo given by x and y, that the indeterminates may be ſeparable in the equation, and therefore that it may be conſtructible, either algebräically, or, at leaſt, by quadratures, we may have the curve, on which the differentio-differential equation depends. And, becauſe the values are many which may be aſſigned to q, the curves may be many alſo, and every value of q will ſupply us with a different curve, either tranſcendent or algebräical, which will ſatisfy the queſtion. Let the equation be $\frac{x^4y^2\dot{x}\dot{x}}{a^2} + \frac{2aay\dot{x}\dot{y}\dot{y} + aax\dot{y}^3}{xy} = aa\dot{y}\ddot{y}$. Now, applying this to the canonical equation, it will be $m = 1$, $z = \frac{xxy}{aa}$; therefore the reduced equation is $\frac{q\dot{y}}{\dot{x}} = \frac{xxy}{aa} \times \overline{qq + 2g}^{\frac{1}{2}}$. I take $q = x$; it will be $\frac{x\dot{y}}{\dot{x}} = \frac{xxy}{aa}\sqrt{xx + 2gg}$, that is, $\frac{aa\dot{y}}{y} = x\dot{x}\sqrt{xx + 2g}$; the integral of which plainly depends on the quadrature of the hyperbola, and the curve will be tranſcendent.

61. In paſſing from firſt to ſecond fluxions, either we aſſume no fluxion for conſtant, or we aſſume ſuch an one as is moſt eligible, as ſaid before. Wherefore, in finding the integrals of formulæ of the ſecond degree, becauſe we know what fluxion had been ſo taken, we know alſo how to proceed, and the rules for it have been explained.

But there are an infinite number of problems, which require ſecond fluxions, without our knowing what conſtants are involved in the formulæ thence ariſing. It often happens, that we cannot arrive at the analytical expreſſion without the aſſiſtance of the conſtants; and likewiſe, it ſucceeds ſometimes, that the equation may be reſolved without recurring to the conſtants. Theſe two caſes, therefore, ought to be examined, and we ſhould ſeek for ſome criterion, to diſtinguiſh one from the other. And, becauſe examples will perform this better than any thing elſe, I ſhall take this following.

It is required to find ſuch a curve, that it's abſciſs, raiſed to any dignity, may be directly as the ſecond difference of the ordinate, and reciprocally as the ſecond difference of the ſame abſciſs. Therefore we ſhall have this analogy,

x^m

$x^m \,.\, \frac{\dot y}{\dot x} :: a \,.\, b$. And confequently $bx^m \ddot x = a\ddot y$. In this equation I find the fecond differences both of the abfcifs and of the ordinate; but I cannot know what conftant was affumed, or whether any conftant was affumed or no; fo that I cannot know what courfe I am to purfue.

I fay, in the cafe of this equation, that no poffible curve will fatisfy the Problem, fince we pafs from firft to fecond fluxions, without the affiftance of conftants. On the contrary, the conftants being determined, we may find curves that will fulfil the conditions of the Problem, but they are infinite in number, and different in their nature, as varying by the change of the arbitrary conftant which is affumed.

To diftinguifh one fpecies from another of thefe equations, we may make ufe of the method, or canon, which will arife from the following Examples, and which will ferve in all fuch cafes, wherein the Integral Calculus does not forfake us.

EXAMPLE I.

Let this equation, $z^{m+1}\dot x^m \ddot x + \frac{\dot z}{z} \times \dot y^{m+1} = \dot y^m \ddot y$, be propofed. I fay, this is one of thofe formulæ to which we may attain, without taking any quantity by way of a conftant. Let the variable z be any how given by x and y.

The demonftration will be made general, as far as that can be done, by taking the fluxion $\frac{\dot x}{q}$ as conftant, in which q is a function of x and y, any how combined. Wherefore I put $\frac{\dot x}{q} = \dot p$; and, becaufe the firft member of this equation is conftant, the fecond $\dot p$ will be fo too. And, as it is $\dot x = q\dot p$, if we pafs to fecond fluxions, it will be $\ddot x = \dot q \dot p$.

Now make $\dot y = u\dot p$; and, taking the fecond fluxions, on the fuppofition of $\dot p$ being conftant, we fhall have $\ddot y = \dot u \dot p$. Wherefore, fubftituting, in the principal equation, the values thus determined, there will arife the equation $z^{m+1} q^m \dot q \dot p^{m+1} + \frac{u^{m+1}\dot z \dot p^{m+1}}{z} = u^m \dot u \dot p^{m+1}$; and, dividing by $\dot p^{m+1}$, an equation will arife which is free from the unknown quantity p, and from it's functions, that is, $z^{m+1} q^m \dot q + \frac{u^{m+1}\dot z}{z} = u^m \dot u$. Taking the fluent, therefore, by

by the rules before explained, not omitting to add the constant g, it will be $\frac{q^{m+1}}{m+1} + g = \frac{u^{m+1}}{\overline{m+1} \times z^{m+1}}$, which equation gives us $u = z \times$

$\overline{q^{m+1} + gm + g}|^{\frac{1}{m+1}}$. And, because $\dot{y} = u\dot{p} = \frac{u\dot{x}}{q}$, making the necessary substitutions, we sha'l have the equation reduced to it's simplest state, that is,

$$\dot{y} = \frac{z\dot{x}}{q} \times \overline{q^{m+1} + gm + g}|^{\frac{1}{m+1}}.$$

From the foregoing manner of operation, we may deduce the following Corollaries.

I. The quantity z being determined, if the last equation can be constructed, even by quadratures, so that it may but be executed, it is plain that infinite curves will agree to our formula, which will change their nature by changing the assumed constant fluxion $\frac{\dot{x}}{q}$. And every value of the quantity q will supply us with a new local equation, either algebräical or transcendental.

II. Although, if the value of the symbol q be altered, different curves will arise; yet it is certain, that, if we make the additional constant $g = 0$, we shall always have the equation $\dot{y} = z\dot{x}$. In which case, it matters not what fluxion $\frac{\dot{x}}{q}$ is taken for constant; because, the given quantity g vanishing, the variable q also vanishes.

III. Here, then, is a token by which it may be known, that we shall arrive at our primary equation, without assuming any fluxion as constant, and that, in such a supposition, it's integral is $z\dot{x} = \dot{y}$. For, recalling to our view the expression $z^{m+1}\dot{x}^{m}\ddot{x} + \frac{\dot{z}}{z} \times \dot{y}^{m+1} - \dot{y}^{m}\ddot{y} = 0$, and again differencing the integral $z\dot{x} = \dot{y}$, without assuming any constant; thence we shall have $z\ddot{x} + \dot{z}\dot{x} = \ddot{y}$; if, by means of these two last equations, we should make to vanish out of the principal formula, first y, then x, with their functions, we shall find $z^{m+1}\dot{x}^{m}\ddot{x} + z^{m}\dot{z}\dot{x}^{m+1} - z^{m+1}\dot{x}^{m}\ddot{x} - z^{m}\dot{z}\dot{x}^{m+1} = 0$, and $\dot{y}^{m}\ddot{y} - \frac{\dot{z}}{z}\dot{y}^{m+1} + \frac{\dot{z}}{z}\dot{y}^{m+1} - \dot{y}^{m}\ddot{y} = 0$.

IV. The

IV. The primary formula being managed as above, and the equation being found reduced to the firſt degree, that is, $\dot{y} = \frac{z\dot{x}}{q} \times \overline{q^{m+1} + gm + g}|^{\frac{1}{m+1}}$, we ſhould paſs on to the integrations, which ſometimes will be out of our power, according to the various values of the exponent m of the fraction z given by x and by y, and of the quantity $\frac{\dot{x}}{q}$, which is taken for conſtant. However the reſt may proceed, the aforeſaid values being determined in infinite particular caſes, the local equation of the curve is alſo diſcovered in finite terms; when we proceed to the firſt, and thence to ſecond differences, keeping ſtill the conſtant $\frac{\dot{x}}{q}$, which our principal formula will preſent us with. But, changing the conſtant, different formulæ will be found. I can aſſure nothing further, but this is very manifeſt, by turning back again the ſteps of the Analyſis.

V. The ſame thing happens by taking the firſt fluxion $\frac{\dot{y}}{q}$ for conſtant. For, making the operation according to the method, (which I ſhall omit for the ſake of brevity,) we ſhould arrive at the reduced equation $\dot{x} = \frac{\dot{y}}{z} - \frac{\dot{y}}{q} \times \overline{mg + g}|^{\frac{1}{m+1}}$; in which it may be obſerved, in like manner, that, making $g = 0$, it concludes by reſtoring the equation $\dot{x} = \frac{\dot{y}}{z}$, expreſſed by firſt differences.

VI. Aſſuming ſome limitations that are more ſimple, that is, $m = 1$, $z = xx$, and $q = x$; if we make uſe of the conſtant $\frac{\dot{x}}{q}$, as in Cor. IV, the formula $\dot{y} = \frac{z\dot{x}}{q} \times \overline{q^{m+1} + gm + g}|^{\frac{1}{m+1}}$ will be changed into this following, $\dot{y} = x\dot{x}\sqrt{xx + 2g}$, which admits of analytical integration. Now, making uſe of the expreſſion contained in Corol. V, that is, $\dot{x} = \frac{\dot{y}}{z} - \frac{\dot{y}}{q} \times \overline{mg + g}|^{\frac{1}{m+1}}$, ariſing from the aſſumed conſtant $\frac{\dot{y}}{q}$, and keeping ſtill the limitations of $m = 1$, $z = xx$, and $q = x$, there reſults the expreſſion $\frac{xx\dot{x}}{1 - x\sqrt{2g}} = \dot{y}$, which is not integrable without the help of the logarithms, and conſequently gives us none but tranſcendent curves.

Therefore

Therefore it is plain that we may arrive at the differential formula of the ſecond order, $z^{m+1}\dot{x}^{m}\ddot{x} + \frac{\dot{z}}{z} \times \dot{y}^{m+1} = \dot{y}^{m}\ddot{y}$, without taking any conſtant; in which caſe the integral $z\dot{x} = \dot{y}$ will take place; or, fixing for conſtant the fluxions $\frac{\dot{x}}{q}$, $\frac{\dot{y}}{q}$, for example-ſake, and then the ſame integrations will be made as before, that were found in theſe ſuppoſitions.

EXAMPLE II.

Let us take the equation $x^{m}\ddot{x} = \ddot{y} + \dot{y}\dot{y}$. I ſay, we cannot arrive at it, without taking ſome conſtant, except in one caſe, in which it is $m = -1$. To ſhow this plainly, I ſhall manage the formula in the manner following.

Firſt, I take $\dot{x}$ for conſtant, and thence $\ddot{x} = 0$. Then $-\frac{\ddot{y}}{\dot{y}} = \dot{y}$, and by integrating, $l\frac{\dot{x}}{\dot{y}} = y$*, or $\frac{\dot{x}}{\dot{y}} = c^{y}$. Make $c^{y} = z$, it will be $ylc = lz$, and therefore $\dot{y} = \frac{\dot{z}}{z}$; and, inſtead of $\dot{y}$, ſubſtituting this value, we ſhall have $\frac{z\dot{x}}{\dot{z}} = c^{y}$. But $c^{y} = z$, therefore $\dot{x} = \dot{z}$, and $x = z = c^{y}$; and therefore $\frac{\dot{x}}{x} = \dot{y}$, an equation to the logarithmic.

Secondly, I propoſe to inveſtigate how it may ſucceed on the ſuppoſition of another conſtant, $\dot{y}$ for example, whence $\ddot{y} = 0$. I make $\dot{x} = s\dot{y} + c\dot{y}$, where s is a new variable, and c a given quantity. I go on to ſecond differences, and it will be $\ddot{x} = \dot{s}\dot{y}$; and, making the ſubſtitution, it is $x^{m}\dot{s}\dot{y} = \dot{y}\dot{y}$, or $x^{m}\dot{s} = \dot{y}$. But $\dot{y} = \frac{\dot{x}}{s+c}$; then $s\dot{s} + c\dot{s} = x^{-m}\dot{x}$; and integrating (omitting to add a conſtant), $\frac{1}{2}ss + cs = \frac{x^{-m+1}}{-m+1}$, or $s + c = \sqrt{\frac{2x^{-m+1}}{-m+1} + cc}$. But $\dot{x} = \overline{s+c} \times \dot{y} = \dot{y}\sqrt{\frac{2x^{-m+1}}{-m+1} + cc}$; therefore $\frac{\dot{x}}{\sqrt{\frac{2x^{-m+1}}{-m+1} + cc}} = \dot{y}$.

* See § 46. EDITOR.

I proceed to inquire if poſſibly the logarithmic curve may be concealed under the laſt formula, which being found above, in the hypotheſis of $\dot{x}$ being conſtant, it may likewiſe have place in the other ſuppoſition of $\dot{y}$ being conſtant. Making $c = 0$, it is neceſſary that the equation $\sqrt{\frac{2x^{-m+1}}{-m+1}} = x$ ſhould be verified, or elſe $2x^{-m+1} = \overline{-m+1} \times xx$. And, that the equation may be found, the ſame quantity $-m+1$, both in the co-efficient and the exponent, ought to be $= 2$; for this to obtain, it follows, that it muſt be $m = -1$.

Therefore, in the formula $x^m\ddot{x} = \ddot{y} + \dot{y}\dot{y}$, by limiting the value of the exponent to $m = -1$, we come to a differential equation of the ſecond degree, without aſſuming a conſtant, the integral of which is the logarithmic expreſſion $\frac{\dot{x}}{x} = \dot{y}$. In any other caſe we could not obtain the foreſaid expreſſion, without fixing upon ſome infiniteſimal quantity of the firſt order as a conſtant.

EXAMPLE III.

It remains that we ſhould propoſe a differential equation of the other claſs, at which we cannot arrive without aſſuming a conſtant.

I reſume the problem: To conſtruct a curve, in which any dignity whatever of the abſciſs may be in a direct ratio of the ſecond fluxion of the ordinate, compounded with the inverſe ratio of the ſecond fluxion of the abſciſs.

The equation is $bx^m\ddot{x} = a\ddot{y}$. Make $\dot{x} = q\dot{p}$, $\dot{y} = u\dot{p}$; and perform the operations, as in the firſt Example. Taking the ſecond fluxions, we ſhall have $\ddot{x} = \dot{p}\dot{q}$, $\ddot{y} = \dot{u}\dot{p}$; and, ſubſtituting theſe values, it will be $bx^m\dot{q} = a\dot{u}$; and by integration, $\int bx^m\dot{q} = au \pm g$. But $\dot{y} = u\dot{p} = \frac{u\dot{x}}{q}$; then $a\dot{y} = \frac{\dot{x}}{q}\int bx^m\dot{q} \mp \frac{g\dot{x}}{q}$. Making $g = 0$, in this caſe, whatever be the value of the ſymbol q, it gives us a different curve, if alſo we do not put the exponent $m = 0$, by which the hypotheſis will be deſtroyed, and the problem changed. The ſame thing may be ſaid if we make conſtant the fraction $\frac{\dot{y}}{q}$; and from

 hence

hence we may conclude, that it is not poſſible a differential equation of the firſt degree, without the benefit of a conſtant, ſhall reſtore our formulæ, when it is differenced again; for, if it were ſo, it would be manifeſted in any aſſumption of a conſtant; and alſo, the analyſis evidences the contrary.

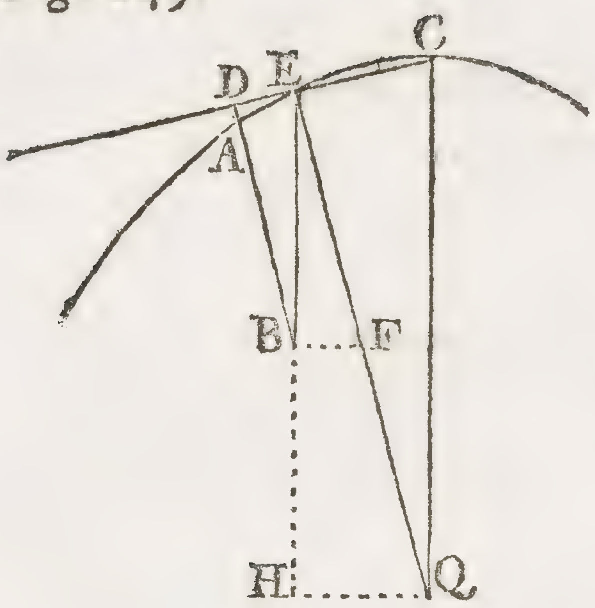
PROBLEM I.

62. The radius of curvature being given, any how expreſſed by the ordinate of a curve, to find the curve itſelf.

As, when the curve is given, to find it's radius of curvature, it is called the Direct Method, or Problem of the Radii of Curvature, of which we have treated already; ſo, when the radius of curvature is given, to find what curve it is to which it belongs, is called the Inverſe Problem of the Radii of Curvature. Wherefore, let the radius of curvature be $= r$, and be any how given by y, the ordinate of the curve; and we may take any one of the formulæ for the radii of curvature, which we pleaſe; but, firſt, for the curves referred to a *focus*; as, for example, $\frac{y\dot{s}^3}{\dot{x}\dot{s}\dot{s} - y\dot{x}\ddot{y}}$, in which $\dot{x}$ is conſtant, and $\dot{s}$ is the element of the curve. Then we ſhall have the equation $r = \frac{y\dot{s}^3}{\dot{x}\dot{s}\dot{s} - y\dot{x}\ddot{y}}$; or elſe, it being $\dot{s}\dot{s} = \dot{x}\dot{x} + \dot{y}\dot{y}$, it is $\dot{s}\ddot{s} = \dot{y}\ddot{y}$, becauſe of $\dot{x}$ conſtant, and $r = \frac{y\dot{y}\dot{s}^2}{\dot{x}\dot{y}\dot{s} - y\dot{x}\ddot{s}}$.

To reduce this equation, I make uſe of the method of § 49; and therefore I make $\dot{s} = p\dot{x}$, whence $\ddot{s} = \dot{p}\dot{x}$. Then, making the ſubſtitutions in the equation, it will be $r = \frac{ppy\dot{y}}{p\dot{y} - y\dot{p}}$, or elſe $\frac{p\dot{y} - y\dot{p}}{pp} = \frac{y\dot{y}}{r}$; and then, by integration, becauſe r is given by y, it will be $\frac{y}{p} = \int\frac{y\dot{y}}{r} \pm b$. But $p = \frac{\dot{s}}{\dot{x}}$ $= \frac{\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}}{\dot{x}}$; therefore the curve will be $\frac{y\dot{x}}{\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}} = \int\frac{y\dot{y}}{r} \pm b$, an equation reduced to firſt fluxions, becauſe, r being given by y, the integral $\int\frac{y\dot{y}}{r}$ may always be had, at leaſt tranſcendentally.

Fig. 149.



Another way. I write the equation, $r = \frac{y\dot{s}^3}{\dot{x}\dot{s}^2 - y\dot{x}\ddot{y}}$, in this manner, $\frac{y\dot{s}^3}{r} = \dot{x}\dot{s}^2 - y\dot{x}\ddot{y}$. Then, from the point B, (Fig. 149.) from whence proceed the ordinates BE of the curve required AEC, I draw BF perpendicular to EB, terminated at the radius of curvature EQ; and, making $BF = p$, $EF = q$, by the known formulæ of the normal and subnormal, it will be $q = \frac{y\dot{s}}{\dot{x}}$, $p = \frac{y\dot{y}}{\dot{x}}$, or $\dot{y} = \frac{p\dot{x}}{y}$. And, by taking the fluxions, on the supposition of $\dot{x}$ being constant, it will be $\ddot{y} = \frac{y\dot{p}\dot{x} - p\dot{x}\dot{y}}{yy}$. And, making the substitutions in the principal equation, it will be $\frac{y\dot{s}^3}{r} = \dot{x}\dot{s}^2 - \dot{p}\dot{x}^2 + \frac{p\dot{x}^2\dot{y}}{y}$. But $\dot{s} = \frac{q\dot{x}}{y}$; therefore $\frac{q^3\dot{x}}{r} = qq\dot{x} - yy\dot{p} + py\dot{y}$. And, because it is $\dot{x} = \frac{y\dot{y}}{p}$, it will be $\frac{q^3\dot{y}}{r} = qq\dot{y} + pp\dot{y} - yp\dot{p}$. But, because of the right angle EBF, it is $pp = qq - yy$, and $p\dot{p} = q\dot{q} - y\dot{y}$. Wherefore, making the substitution, we shall have $\frac{qq\dot{y}}{r} = 2q\dot{y} - y\dot{q}$; and, multiplying by y, and dividing by qq, it will be $\frac{y\dot{y}}{r} = \frac{2qy\dot{y} - yy\dot{q}}{qq}$; and, by integration, it is $\int \frac{y\dot{y}}{r} \pm b = \frac{yy}{q}$. But $q = \frac{y\dot{s}}{\dot{x}}$; therefore $\int \frac{y\dot{y}}{r} \pm b = \frac{y\dot{x}}{\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}}$.

It may be done thus more simply, by avoiding second fluxions.

Taking the infinitely little arch EC, let the chord CED be produced, to which let BD be perpendicular. Now, if we make $BD = p$, by what has been said at § 115, Sect. V, B. II, $QE = r = \frac{y\dot{y}}{\dot{p}}$, and therefore $\frac{y\dot{y}}{r} = \dot{p}$; and by integration, because r is given by y, it is $\int \frac{y\dot{y}}{r} \pm b = \frac{y\dot{x}}{\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}}$; for $p = \frac{y\dot{x}}{\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}}$, by the place now quoted.

Let it be $r = \frac{y}{b}\sqrt{aa + bb}$; then it will be $\int \frac{b\dot{y}}{\sqrt{aa + bb}} \pm b = \frac{y\dot{x}}{\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}}$, and by actual integration, (omitting the constant b for greater simplicity,) $\frac{b}{\sqrt{aa + ob}} = \frac{\dot{x}}{\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}}$, and therefore $b^2\dot{x}^2 + b^2\dot{y}^2 = a^2\dot{x}^2 + b^2\dot{x}^2$, that is, $b\dot{y} = a\dot{x}$, which is the logarithmic spiral of Example V, § 128, Book II.

Instead

Inſtead of the radius QE, let the co-radius HE $= z$ be any how given by the ordinate y. Becauſe of ſimilar triangles, EBD, QEH, it will be EB . BD :: QE . EH; that is, $y \,.\, p :: \frac{y\dot{y}}{\dot{p}} \,.\, z$, and therefore $z = \frac{p\dot{y}}{\dot{p}}$, or $\frac{\dot{y}}{z} = \frac{\dot{p}}{p}$; and by integration, $\int\frac{\dot{y}}{z} \pm b = lp$. Make $z = y$, then $\int\frac{\dot{y}}{y} \pm b = \int\frac{\dot{p}}{p}$; and by integrating, $ly = lp + l\frac{m}{b}$*, that is, $y = \frac{pm}{b}$. But $p = \frac{y\dot{x}}{\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}}$, then $b\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}} = m\dot{x}$, and therefore $b\dot{y} = \dot{x}\sqrt{mm - bb}$, which is the logarithmic ſpiral; and, when $b = b$, $m = \sqrt{aa + bb}$ is the ſame as the above-cited.

63. For curves referred to an axis, the formula of the radius of curvature is $\frac{\dot{s}^3}{-\dot{x}\ddot{y}}$, putting $\dot{x}$ conſtant; and therefore the equation will be $r = \frac{\dot{s}^3}{-\dot{x}\ddot{y}}$.

I put $\dot{y} = q\dot{x}$, whence $\ddot{y} = \dot{q}\dot{x}$; and, making the ſubſtitutions, it is $r = \frac{\overline{\dot{x}\dot{x} + \dot{y}\dot{y}}|^{\frac{3}{2}}}{-\dot{x}\dot{x}\dot{q}}$; and, inſtead of $\dot{x}$, putting it's value $\frac{\dot{y}}{q}$, it will be $r = \frac{\dot{y} \times \overline{1+qq}|^{\frac{3}{2}}}{-q\dot{q}}$, that is, $\frac{\dot{y}}{r} = -\frac{q\dot{q}}{\overline{1+qq}|^{\frac{3}{2}}}$. And, by integration, $\int\frac{\dot{y}}{r} \pm b = \frac{1}{\sqrt{1 + qq}}$. But $q = \frac{\dot{y}}{\dot{x}}$; therefore $\int\frac{\dot{y}}{r} \pm b = \frac{\dot{x}}{\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}}$.

Let $r = \frac{\overline{4yy + aa}|^{\frac{3}{2}}}{2aa}$; then it will be $\int\frac{2aa\dot{y}}{\overline{4yy + aa}|^{\frac{3}{2}}} \pm b = \frac{\dot{x}}{\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}}$. And, by actual integration, omitting the conſtant b, it is $\frac{2y}{\sqrt{4yy + aa}} = \frac{\dot{x}}{\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}}$, that is, $2y\dot{y} = a\dot{x}$; and by integration, $yy = ax$, which is the parabola of the firſt Example, § 122, Sect. V, Book II.

Inſtead of the radius, let the co-radius be given, which make $= z$, the formula of which (ſuppoſing $\dot{x}$ to be conſtant,) is $\frac{\dot{x}\dot{x} + \dot{y}\dot{y}}{-\ddot{y}}$. Then $\frac{\dot{x}\dot{x}+\dot{y}\dot{y}}{-\ddot{y}} = z$; and making $\dot{y} = q\dot{x}$, $\ddot{y} = \dot{q}\dot{x}$, and making the ſubſtitutions of theſe values of $\ddot{y}$ and $\dot{x}$, it will be $\frac{\dot{y} \times \overline{1 + qq}}{-q\dot{q}} = z$, that is, $\frac{\dot{y}}{z} = \frac{-q\dot{q}}{1 + qq}$. And, by integration, $\int\frac{\dot{y}}{z} \pm b = -l\sqrt{1 + qq}$. Whence, if z, or the co-radius, be in ſuch manner given by y, as that $\int\frac{\dot{y}}{z}$ be a logarithmic expreſſion, we ſhall have a

* This equation, as well as the ſubſequent work, would have been clearer and ſimpler, if m had been put for the conſtant number of which the logarithm is b. EDITOR.

differential

differential equation of the firſt degree expreſſed after the uſual manner; in any other caſe, it will be expreſſed by logarithmic quantities.

Let it be $z = \frac{4y^3 + aay}{aa}$; we ſhall have the equation $\int \frac{aa\dot{y}}{4y^3 + aay} \pm b = - l\sqrt{1 + qq}$. And, by actual integration, (omitting the conſtant b,) it is $l\frac{y}{\sqrt{yy + \frac{1}{4}aa}} = l\frac{1}{\sqrt{1 + qq}}$, and therefore $\frac{yy}{yy + \frac{1}{4}aa} = \frac{1}{1 + qq}$. And, ſubſtituting the value of q, it is $2y\dot{y} = a\dot{x}$, and, by integration, it is $yy = ax$, the ſame parabola as before.

64. In the ſecond place, let the radius, or co-radius, of curvature be any how given by the abſciſs x; it is plain that, in this caſe, we cannot make uſe of the ſame reductions we did in the firſt, becauſe we cannot have the fluents $\int \frac{\dot{y}}{r}$, or $\int \frac{\dot{y}}{z}$, if r and z are given by x.

Taking, therefore, the formula of the radius of curvature, in which $\dot{x}$ is conſtant, that is, $\frac{\overline{\dot{x}\dot{x} + \dot{y}\dot{y}}|^{\frac{3}{2}}}{- \dot{x}\ddot{y}}$ for curves referred to an axis, (for, in thoſe referred to a *focus*, the radius, or co-radius, cannot be given by the abſciſs,) it will be $r = \frac{\overline{\dot{x}\dot{x} + \dot{y}\dot{y}}|^{\frac{3}{2}}}{- \dot{x}\ddot{y}}$, and therefore, in the ſame manner as before, I put $\dot{y} = q\dot{x}$, whence $\ddot{y} = \dot{q}\dot{x}$, $\dot{y}\dot{y} = qq\dot{x}\dot{x}$; and, making the ſubſtitutions, $r = \frac{\overline{\dot{x}\dot{x} + qq\dot{x}\dot{x}}|^{\frac{3}{2}}}{- \dot{x}\dot{x}\dot{q}}$, that is, $\frac{\dot{x}}{r} = \frac{- \dot{q}}{\overline{1 + qq}|^{\frac{3}{2}}}$, and, by integration, $\int \frac{\dot{x}}{r} \pm b = \frac{- q}{\overline{1 + qq}|^{\frac{1}{2}}}$, which is an equation reduced to firſt fluxions; becauſe r, being given by x, the fluent $\int \frac{\dot{x}}{r}$ may always be had, at leaſt tranſcendentally. And, ſubſtituting the value of q, it is $\int \frac{\dot{x}}{r} \pm b = \frac{- \dot{y}}{\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}}$.

Let it be $r = 2\sqrt{4aa - 2ax}$; then it will be $\int \frac{\dot{x}}{2\sqrt{4aa - 2ax}} \pm b = \frac{- \dot{y}}{\sqrt{\dot{x}\dot{x} + \dot{y}\dot{y}}}$. And, by actual integration, omitting the conſtant b, it will be $\frac{- \sqrt{4aa - 2ax}}{2a} = \frac{- \dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}}$. And, by ſquaring, and reducing to a common denominator, it is $4aa\dot{x}\dot{x} - 2ax\dot{x}\dot{x} - 2ax\dot{y}\dot{y} = 0$, that is, $\dot{y} = \dot{x}\sqrt{\frac{2a - x}{x}}$, an equation to the cycloid of § 131, Sect. V, B. II.

Inſtead

Inſtead of the radius, let the co-radius be given; then $z = \frac{\dot{x}\dot{x} + \dot{y}\dot{y}}{-\ddot{y}}$. And putting, in like manner, $\dot{y} = q\dot{x}$, it is $\ddot{y} = \dot{q}\dot{x}$, $\dot{y}\dot{y} = qq\dot{x}\dot{x}$; and making the ſubſtitutions, inſtead of $\ddot{y}$ and $\dot{y}\dot{y}$, it will be $z = \frac{\dot{x}\dot{x} + qq\dot{x}\dot{x}}{-\dot{q}\dot{x}}$, that is, $\frac{\dot{x}}{z} = \frac{-\dot{q}}{1 + qq}$; and, by integration, $\int\frac{\dot{x}}{z} \pm b = \int\frac{-\dot{q}}{1 + qq}$. But the integral of the *homogeneum comparationis* is the arch of a circle; therefore, if the co-radius ſhall be given in ſuch manner, as that $\int\frac{\dot{x}}{z}$ is alſo the arch of a circle, and theſe arches ſhall ſo correſpond, as to be to each other as number to number, we ſhall have the equation reduced to firſt fluxions, and expreſſed in common quantities.

Let $z = 2\sqrt{2ax - xx}$; then it will be $\int\frac{\dot{x}}{2\sqrt{2ax - xx}} = \int\frac{-\dot{q}}{1 + qq}$. But the integral of the firſt member is the arch of a circle, the tangent of which is $\frac{\sqrt{2ax - xx}}{x}$; and of the ſecond, is the arch of a circle, the tangent of which is q. Then it will be $\frac{\sqrt{2ax - xx}}{x} = q = \frac{\dot{y}}{\dot{x}}$; therefore $\dot{y} = \dot{x}\sqrt{\frac{2a - x}{x}}$, an equation to the ſame cycloid.

PROBLEM II.

65. The radius of curvature being given in any manner, in a curve referred to an axis, to find the ſaid curve.

The formula for the radius of curvature is $\frac{\dot{x}\dot{s}}{\ddot{y}}$, making $\dot{s}$ the element of the curve conſtant; whence the equation will be $r = \frac{\dot{x}\dot{s}}{\ddot{y}}$. Call the tangent of the curve t, and the ſubtangent p. It will be $\frac{y\dot{s}}{\dot{y}} = t$, and, differencing in the hypotheſis of $\dot{s}$ conſtant, it will be $\dot{t} = \frac{\dot{y}\dot{y}\dot{s} - y\dot{s}\ddot{y}}{\dot{y}\dot{y}}$, that is, $\ddot{y} = \frac{\dot{y}\dot{y}\dot{s} - \dot{y}\dot{y}\dot{t}}{y\dot{s}}$. Wherefore, making the ſubſtitutions, it will be $r = \frac{y\dot{x}\dot{s}\dot{s}}{\dot{y}\dot{y}\dot{s} - \dot{y}\dot{y}\dot{t}}$. But, becauſe

we

we have $p = \frac{y\dot{x}}{\dot{y}}$, and $t = \frac{y\dot{s}}{\dot{y}}$, it will be $\dot{x} = \frac{p\dot{y}}{y}$, $\dot{s} = \frac{t\dot{y}}{y}$. Then, fubftituting thefe values in the equation above, we fhall have $r = \frac{pt\dot{s}}{t\dot{y} - y\dot{t}}$. But $p = \sqrt{tt - yy}$; therefore $r = \frac{t\dot{s}\sqrt{tt - yy}}{t\dot{y} - y\dot{t}}$, or $\frac{\dot{s}}{r} = \frac{t\dot{y} - y\dot{t}}{t\sqrt{tt - yy}}$.

The firft member of this laft equation is in our power, at leaft tranfcendentally, becaufe r is a function of s. Then, in the fecond, the indeterminates will be eafily feparated, if we make $q = \frac{y}{t}$, by which we fhall have a very fimple equation, $\frac{\dot{s}}{r} = \frac{\dot{q}}{\sqrt{1 - qq}}$.

In the formula $r = \frac{pt\dot{s}}{t\dot{y} - y\dot{t}}$, if, inftead of t, we had taken it's value $\sqrt{pp + yy}$, we fhould have found $r = \frac{\overline{pp + yy} \times \dot{s}}{p\dot{y} - y\dot{p}}$; and, making $\frac{y}{p} = z$, we fhould alfo have had a very fimple equation, $\frac{\dot{s}}{r} = \frac{\dot{z}}{1 + zz}$.

The two differential quantities $\frac{\dot{q}}{\sqrt{1 - qq}}$ and $\frac{\dot{z}}{1 + zz}$ are the expreffions of the element of the arch of a circle. Whence, if the integral $\int\frac{\dot{s}}{r}$ fhall be algebräical, or fhall depend on the logarithms, or on higher quadratures, the rectification of the curves required, and the value of the radius of curvature, will fuppofe the quadrature of the circle. But, on the contrary, each of them may be algebräical, if the integral $\int\frac{\dot{s}}{r}$ agrees with a formula of the circular arch.

Retaining one of the two equations, for example the fecond, $\frac{\dot{s}}{r} = \frac{\dot{z}}{1 + zz}$; becaufe $\dot{s} = \frac{t\dot{y}}{y} = \frac{\dot{y}}{y}\sqrt{pp + yy}$, and $p = \frac{y}{z}$, it will be $\dot{s} = \frac{\dot{y}}{z}\sqrt{1 + zz}$. Then, fubftituting this value into the equation, we fhall have $\dot{y} = \frac{rz\dot{z}}{1+zz \times \sqrt{1+zz}}$. Now, it being $\dot{s} = \frac{\dot{y}}{z}\sqrt{1 + zz}$, we fhall have alfo $\dot{s}\dot{s} = \dot{x}\dot{x} + \dot{y}\dot{y} = \frac{\dot{y}\dot{y} + zz\dot{y}\dot{y}}{zz}$, and therefore $\dot{x} = \frac{\dot{y}}{z}$.

Make

Make the given radius of curvature $r = 1 + ss$. Then the equation $\frac{\dot{z}}{1 + zz} = \frac{\dot{s}}{r}$ will be changed into this, $\frac{\dot{z}}{1 + zz} = \frac{\dot{s}}{1 + ss}$; from whence we obtain $z = s$, and therefore $r = 1 + zz$. Subſtitute this value in the equation $\dot{y} = \frac{rz\dot{z}}{1+zz \times \sqrt{1+zz}}$, and it will be $\dot{y} = \frac{z\dot{z}}{\sqrt{1 + zz}}$. And, by integration, omitting the conſtant, it is $y = \sqrt{1 + zz}$, whence $z = \sqrt{yy - 1}$. Then, becauſe I retained $\dot{x} = \frac{\dot{y}}{z}$, it will be finally $\dot{x} = \frac{\dot{y}}{\sqrt{yy - 1}}$, an equation of the curve required, on the aſſumed ſuppoſition of the radius of curvature. It's conſtruction depends on the quadrature of the hyperbola.

I take the formula of the radius of curvature, $\frac{\dot{s}}{r} = \frac{\dot{s}\ddot{y} - \dot{y}\ddot{s}}{\dot{x}\dot{s}}$, in which no firſt fluxion is conſtant. I diſpoſe the equation thus, $\frac{\dot{y}}{\dot{x}} \times \overline{\frac{\ddot{y}}{\dot{y}} - \frac{\ddot{s}}{\dot{s}}} = \frac{\dot{s}}{r}$. The integral of $\frac{\ddot{y}}{\dot{y}} - \frac{\ddot{s}}{\dot{s}}$ is $l\dot{y} - l\dot{s}$, which I make equal to lp. Then it will be $\frac{\ddot{y}}{\dot{y}} - \frac{\ddot{s}}{\dot{s}} = \frac{\dot{p}}{p}$; and $\frac{\dot{y}}{\dot{s}} = p$, and then the equation will be $\frac{\dot{s}}{r} = \frac{\dot{y}}{\dot{x}} \times \frac{\dot{p}}{p}$. But $p = \frac{\dot{y}}{\dot{s}}$, and $\frac{\dot{y}\dot{y}}{pp} = \dot{s}\dot{s} = \dot{x}\dot{x} + \dot{y}\dot{y}$; therefore $\dot{x} = \frac{\dot{y}\sqrt{1 - pp}}{p}$. And, ſubſtituting this value, it will be $\frac{\dot{s}}{r} = \frac{\dot{p}}{\sqrt{1 - pp}}$, an equation in which the variables are ſeparated, and conſequently may be treated in the manner made uſe of before.

Let the formula of the radius of curvature be $\frac{\dot{s}}{r} = -\frac{\dot{y}\ddot{s}}{\dot{s}\dot{x}}$, in which $\dot{y}$ is conſtant. Make $\dot{s} = q\dot{y}$, and therefore $\ddot{s} = \dot{q}\dot{y}$. Then $\frac{\dot{s}}{r} = -\frac{\dot{y}\dot{y}\dot{q}}{\dot{s}\dot{x}}$; but $\dot{s}\dot{s} = \dot{x}\dot{x} + \dot{y}\dot{y} = qq\dot{y}\dot{y}$. Whence we have $\dot{x} = \dot{y}\sqrt{qq - 1}$, and $\dot{x}\dot{s} = q\dot{y}^2\sqrt{q^2 - 1}$. Wherefore, making this ſubſtitution, it will be $\frac{\dot{s}}{r} = -\frac{\dot{q}}{q\sqrt{qq - 1}}$.

Laſtly, let the formula of the radius of curvature be $\frac{\dot{s}}{r} = -\frac{\dot{x}\ddot{y}}{\dot{s}\dot{s}}$, in which $\dot{x}$ is conſtant. Make $z = \frac{\dot{x}}{\dot{y}}$, and therefore $\dot{z} = -\frac{\dot{x}\ddot{y}}{\dot{y}\dot{y}}$. Then $\frac{\dot{s}}{r} = \frac{\dot{y}\dot{y}\dot{z}}{\dot{s}\dot{s}}$. But $\dot{x} = z\dot{y}$, and $\dot{s}\dot{s} = \dot{x}\dot{x} + \dot{y}\dot{y} = zz\dot{y}\dot{y} + \dot{y}\dot{y}$. Whence $\frac{\dot{s}}{r} = \frac{\dot{z}}{1 + zz}$.

Therefore, after whatever manner we operate, the integral $\int \frac{\dot{s}}{r}$ will always be brought, either to the rectification or quadrature of the circle.

Let the co-radius u be any how given, to find the curve. Take one of the three formulæ before, that, for example, in which $\dot{y}$ is taken for conſtant; that is, $\frac{\dot{s}}{r} = -\frac{\dot{q}}{q\sqrt{qq - 1}}$, in which it is put $\dot{s} = q\dot{y}$. The radius will be $r = \frac{u\dot{s}}{\dot{x}}$; and, putting this value in the formula, we ſhall have $\frac{\dot{s}}{u} = -\frac{\dot{s}\dot{q}}{q\dot{x}\sqrt{qq - 1}}$. But $\dot{s} = q\dot{y}$, and $\dot{x} = \dot{y}\sqrt{qq - 1}$. Whence, making the ſubſtitutions, it will be $\frac{\dot{s}}{u} = -\frac{\dot{q}}{qq - 1}$. But u is given by s; therefore, &c.

Here it may be obſerved, that, as the integral $\int \frac{\dot{s}}{r}$ is equal to an expreſſion of a circular arch; ſo the other integral $\int \frac{\dot{s}}{u}$ will be referred to the quadrature of the hyperbola, or to the logarithms.

66. By like artifices and expedients, or but little different from theſe, many equations, or formulæ, may be reduced to ſecond differentials, which are expreſſed by third, fourth, or higher degrees of fluxions. And, firſt, the method of § 49 may be extended, (yet within certain limitations,) to differential equations of the third, fourth, &c. order. That is to ſay, equations of the third order may always be reduced to the firſt order, provided that either one or the other of the finite variables, x or y, is wanting in them. Thoſe of the fourth order may be reduced, if, beſides one or other of the two finite variables, x or y, one or other of the firſt fluxions, $\dot{x}$ or $\dot{y}$, be wanting, together with their reſpective functions. Thoſe of the fifth may be reduced, if both the finite variables, and both their firſt fluxions, be wanting in them. Thoſe of the ſixth, if, beſides all this, one or other of their ſecond fluxions be wanting. And ſo on.

Let the equation be $\dot{x}\dddot{y} + \dot{x}\dot{x}\ddot{y} = \dot{x}^4 + \dot{y}^4$, in which $\dot{x}$ is taken for conſtant. I make, as uſual, $p\dot{x} = \dot{y}$, and therefore $\dot{p}\dot{x} = \ddot{y}$, and $\ddot{p}\dot{x} = \dddot{y}$. Wherefore, making

making the ſubſtitutions, we ſhall have $\dot{x}\dot{x}\ddot{p} + \dot{x}^3\dot{p} = \dot{x}^4 + \dot{y}^4$. But $\dot{y}^4 = p^4\dot{x}^4$; therefore it will be $\ddot{p} + \dot{x}\dot{p} = \dot{x}\dot{x} + p^4\dot{x}\dot{x}$, an equation reduced to the ſecond order. Make further $q\dot{x} = \dot{p}$, retaining $\dot{x}$ as conſtant, and therefore $\dot{q}\dot{x} = \ddot{p}$. Then, by ſubſtitution, it will be $\dot{q}\dot{x} + \dot{p}\dot{x} = \dot{x}\dot{x} + p^4\dot{x}\dot{x}$, that is, $\dot{q} + \dot{p} = \dot{x} + p^4\dot{x}$. But $\dot{x} = \frac{\dot{p}}{q}$; therefore $\dot{q} + \dot{p} = \frac{\dot{p}}{q} + \frac{p^4\dot{p}}{q}$; which equation is now reduced to firſt fluxions.

Let there be a fluxional equation of the fourth order, $\ddddot{y} + \dot{x}\dddot{y} - \dot{x}\dot{x}\ddot{y} = 0$, in which let $\dot{x}$ be conſtant. Therefore I make $p\dot{x} = \dot{y}$, and thence $\dot{p}\dot{x} = \ddot{y}$, and $\ddot{p}\dot{x} = \dddot{y}$, and $\dddot{p}\dot{x} = \ddddot{y}$. Therefore, making the ſubſtitutions, we ſhall have $\dddot{p} + \dot{x}\ddot{p} - \dot{x}\dot{x}\dot{p} = 0$; an equation which is a caſe of the foregoing Example, and which therefore we know how to manage; and which will eaſily be reduced to firſt fluxions.

The method of § 49, found ſome time ago by S. Count *James Riccati*, was now firſt known to me; but the foregoing application, as alſo the ſecond inverſe Problem concerning Radii of Curvature, I have learned of him only ſince the ſecond Tome of the Commentaries of the Inſtitute of *Bolonia* is fallen into my hands. And, indeed, ſomething too late for me, becauſe I was now at the cloſe of the impreſſion of this my Work; nor could I take the advantage of the other learned Diſſertations, neither of P. *Vincent Riccati*, ſon of the aforeſaid gentleman, nor of S. *Gabriel Manfredi*, therein inſerted. Therefore it muſt ſuffice that I have juſt named them to the readers, that they may there find them, and be improved and inſtructed by them.

67. Having ſhown the aforeſaid application, or improvement of the method of § 49, I ſhall go on to other equations, and to other expedients. Therefore let the equation be $p\dot{y}\ddot{y}\ddot{y} = p\dot{x}\dot{x}\dddot{y} - 2p\dot{x}\ddot{x}\ddot{y} - \dot{p}\dot{x}\dot{x}\ddot{y}$, in which p is any how given by x and y, and now the element of the curve, $\dot{s}$, is taken for conſtant. Becauſe $\dot{s}$ is conſtant, it will be $\dot{x}\ddot{x} = -\dot{y}\ddot{y}$; then, ſubſtituting this value inſtead of $\dot{x}\ddot{x}$, it will be $p\dot{y}\ddot{y}\ddot{y} = p\dot{x}\dot{x}\dddot{y} + 2p\dot{y}\ddot{y}\ddot{y} - \dot{p}\dot{x}\dot{x}\ddot{y}$, that is, ſtriking out the ſuperfluous terms, $\dot{p}\dot{x}\dot{x}\ddot{y} = p\dot{y}\ddot{y}\ddot{y} + p\dot{x}\dot{x}\dddot{y}$, or $\frac{\dot{p}}{p} = \frac{\dot{y}\ddot{y}}{\dot{x}\dot{x}} + \frac{\dddot{y}}{\ddot{y}}$. And, inſtead of $\dot{y}\ddot{y}$, putting it's value $-\dot{x}\ddot{x}$, it will be $\frac{\dot{p}}{p} = -\frac{\ddot{x}}{\dot{x}} + \frac{\dddot{y}}{\ddot{y}}$. And laſtly, integrating by the logarithms, $lp = l\ddot{y} - l\dot{x} - l\dot{s}$, $\dot{s}$ being conſtant; and therefore $p = \frac{\ddot{y}}{\dot{x}\dot{s}}$: which equation is reduced to ſecond fluxions.

Let the equation be $h\dot{z}\dddot{x} - 3h\ddot{z}\ddot{x} - \dot{h}\dot{z}\ddot{x} = 0$, in which h is any how given by x and z. Let us aſſume the following fictitious equation, $h^m\dot{z}^n\ddot{x}^r =$ conſtant; where m, n, r, are unknown exponents of powers, to be determined by the proceſs. Then, by taking the fluxions, we ſhall have $rh^m\dot{z}^n\ddot{x}^{r-1}\dddot{x} + nh^m\ddot{x}^r\dot{z}^{n-1}\ddot{z} + mh^{m-1}\dot{h}\dot{z}^n\ddot{x}^r = 0$, which, being divided by $h^{m-1}\dot{z}^{n-1}\ddot{x}^{r-1}$, will be reduced to $rh\dot{z}\dddot{x} + nh\ddot{x}\ddot{z} + m\dot{h}\dot{z}\ddot{x} = 0$. This equation being compared, term by term, with the principal equation propoſed, we ſhall have $r = 1$, $n = -3$, $m = -1$; wherefore, inſtead of the fictitious equation $h^m\dot{z}^n\ddot{x}^r =$ conſtant, we ſhall have the true one, $\frac{\ddot{x}}{h\dot{z}^3} =$ conſtant, which is the integral of the propoſed equation.

Alſo, by the way of the logarithms, we may obtain the ſame integration. I reſume the equation $h\dot{z}\dddot{x} - 3h\ddot{z}\ddot{x} - \dot{h}\dot{z}\ddot{x} = 0$. I divide it by $h\dot{z}\ddot{x}$; it will be $\frac{\dddot{x}}{\ddot{x}} - \frac{3\ddot{z}}{\dot{z}} - \frac{\dot{h}}{h} = 0$, and by integration, $l\ddot{x} - l\dot{z}^3 - lh =$ to a conſtant logarithm. Therefore $\frac{\ddot{x}}{h\dot{z}^3}$ is equal to a conſtant quantity.

ADVERTISEMENT.

68. I SHALL finiſh theſe Inſtitutions with an Advertiſement, which is this; that the ingenious Analyſt muſt endeavour, with all his ſkill, in the ſolution of Problems, to avoid ſecond fluxions, and much more thoſe of a higher order; and this by means of various expedients, which will offer themſelves commodiouſly on the ſpot. Such artifices may be ſeen, as they are made uſe of by famous Mathematicians, in the Problems of the Elaſtic Curves, the Catenaria, the Velaria, in that of Iſoperimetral Curves, and in others of this kind; the ſolutions of which may be ſeen in the *Leipſic* Acts, and other works of this nature: by which a learner may acquire ſuch ſkill and dexterity, as will be very beneficial to him.

END OF THE FOURTH BOOK.

AN

AN ADDITION

TO THE FOREGOING

ANALYTICAL INSTITUTIONS;

Being a Paper of Mr. *Colfon's*, containing a Specimen of the Manner in which Two or more Perfons may entertain themfelves, by propofing and anfwering curious Queftions in the Mathematicks.

THE Manufcript of this little piece appears to be a firft draught, and only a part, of what Mr. *Colfon* intended to draw up: yet, I perfuade myfelf, it is fufficient to point out to the readers of it the way in which feveral perfons may amufe themfelves with propofing and anfwering Queftions of this kind. Thofe readers, who wifh to fee more of this, may find it in the VIth Section of Mr. *Colfon's* Comment on Sir ISAAC NEWTON'S *Fluxions*. They may alfo, with a little attention, propofe and folve, in the fame manner, any of the Queftions in thefe Volumes.

" A *Problem* is fuppofed to be managed between two perfons, the *Querift* and the *Refpondent*: the *Data* are fuch numbers or quantities as are given or fupplied by the Querift; the *Affumpta* or *Quæfita* are fuch as are affumed or found by the Refpondent."

PROBLEM I.

" QUERIST. *I give you three numbers*, 4, 5, *and* 10; *I require a fourth*,

RESPONDENT. I affume x to denote that fourth.

Q. *So that, if from the product of this into the third, the firft be fubtracted*,

R. Then

R. Then the remainder will be denoted by $10x - 4$.

Q. *And if the remainder be divided by the firſt,*

R. The quotient will be denoted by $\frac{10x - 4}{4}$;

Q. *The Quotient will be equal to the ſecond number.*

R. Then the equation is $\frac{10x - 4}{4} = 5$; whence $10x - 4 = 20$, and $10x = 24$, and $x = \frac{24}{10} = 2.4$."

PROBLEM II.

" Q. *A certain number of ſhillings,*

R. That number ſhall be denoted by x;

Q. *Was to be diſtributed among a certain number of poor people;*

R. The number of poor ſhall be y.

Q. *Now if three ſhillings were given to each, there would be 8 wanting;*

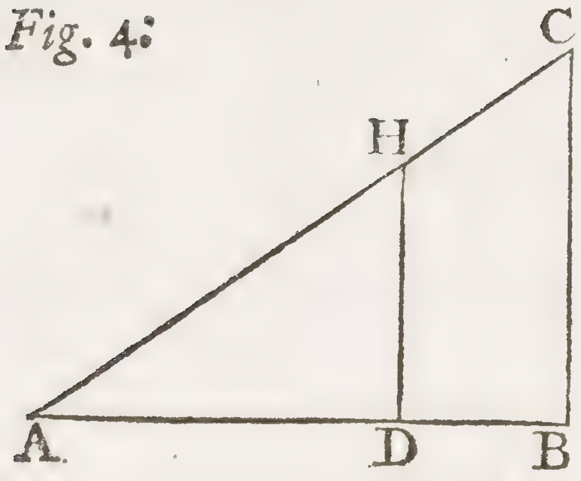
R. Then $x = 3y - 8$.

Q. *But if two were given to each, there would be 3 to ſpare.*

R. Then $x = 2y + 3 = 3y - 8$, or $y = 11$, the number of poor; and thence $x = 2y + 3 = 22 + 3 = 25$, the number of ſhillings."

PROBLEM III.

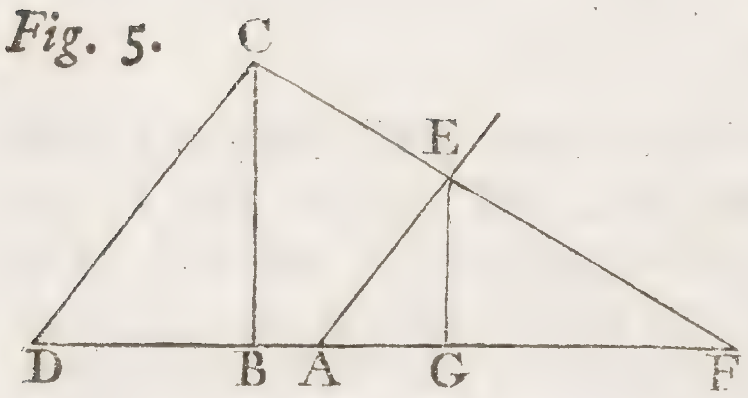
Fig. 4:



" Q. *In the triangle* ABC, *I give you the ſides* AC $= a$, BC $= b$, *and the baſe* AB $= c$; *you are to find in this ſuch a point* D, R. I will aſſume AD $= x$; then DB $= a - x$; Q. *That drawing* DH *parallel to* BC, R. Then it will be AB (c) . BC (b) :: AD (x) . DH $= \frac{bx}{c}$; Q. *The ſquare of* DH *may be equal to the rectangle of* AD *and* DB. R. Then $\frac{bbxx}{cc} = x \times \overline{a - x}$, and $\frac{bbx}{cc} = a - x$, and $bbx = acc - ccx$, and $bbx + ccx = acc$, and $x = \frac{acc}{bb + cc}$."

PROBLEM IV.

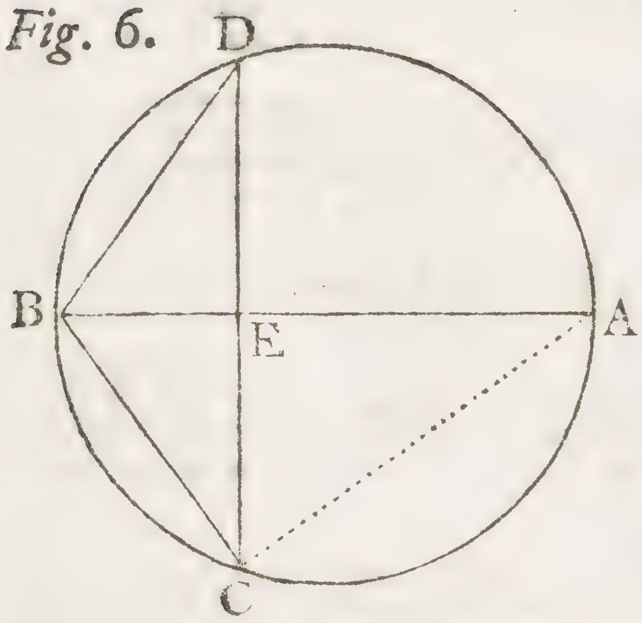
Fig. 5.



" Q. *I give you in position the two right lines* AF, AE, *and a point* C *in neither of those lines*; R. Then I can continue AF to D, and draw CD parallel to AE; and as AD will be given, I shall make AD $= a$. And I can let fall the perpendicular CB, which will be given also; and therefore I will make CB $= b$. Q. *You are to draw the line* CEF *in such a manner, as that it shall cut off the triangle* AEF *equal in area to the given plane cc.* I will let fall the perpendicular EG, and make the base AF $= x$. And then, by similar triangles, it will be DF $(a + x)$. AF (x) :: DC . AE :: CB (b) . EG $= \frac{bx}{a + x}$. But the area of the triangle AEF is $\frac{1}{2}$AF $\times$ EG $=$ $\frac{1}{2}x \times \frac{bx}{a + x}$. Therefore $\frac{bxx}{2a + 2x} = cc$." [From which quadratick equation the value of x is easily obtained by § 74, Sect. II, Book I.]

PROBLEM V.

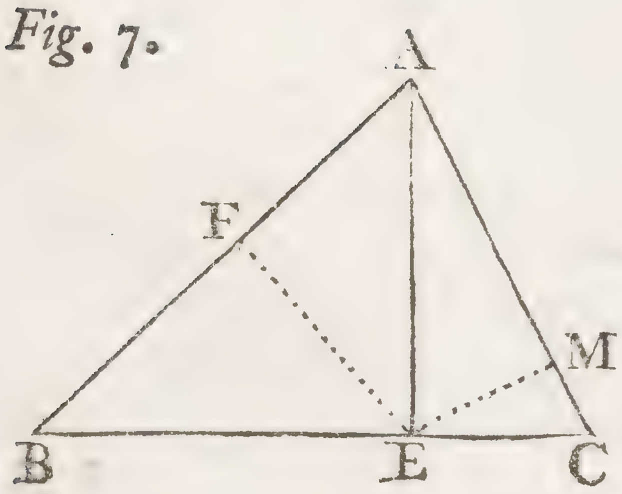
Fig. 6.



" Q. *I give you the isosceles triangle* CDB; R. Then I will make CD $= a$, BC $= b$; I will bisect CD in E, and draw the indefinite line BEA. Q. *The diameter of the circle is required in which it may be inscribed.* R. Let AB $= x$ be the diameter, and the circle ACBD. Now, because of similar triangles, it is AB (x) . BC (b) :: BC (b) . BE $= \frac{bb}{x}$. But BE $= \sqrt{BCq - CEq} = \sqrt{bb - \frac{1}{4}aa}$. Therefore $\frac{bb}{x} = \sqrt{bb - \frac{1}{4}aa}$, and $x = \frac{bb}{\sqrt{bb - \frac{1}{4}aa}}$."

PROBLEM VI.

Fig. 7.

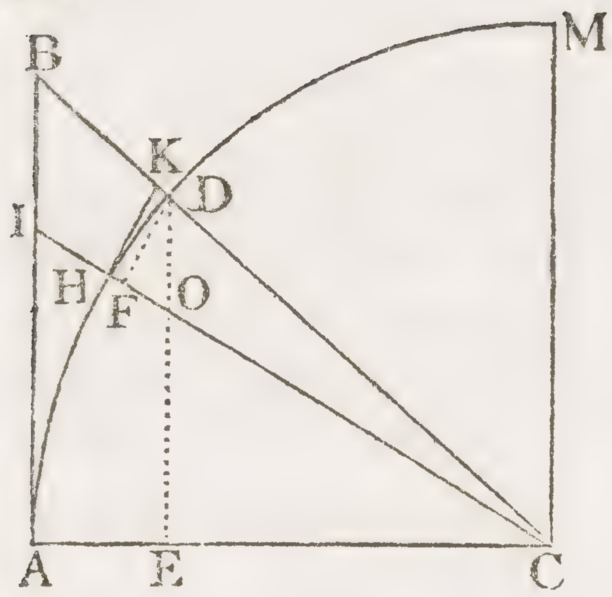


" Q. *In the triangle* ABC, *I give you the three sides,* AB $= a$, AC $= b$, *and* BC $= c$; *and letting fall the perpendicular* AE, *I require the segments of the base,* BE *and* EC. R. I make BE $= x$; then is EC $= c - x$. But ABq $-$ BEq $=$ AEq $=$ ACq $-$ ECq; that is, $aa - xx = (\text{AE}q) = bb - cc + 2cx - xx$; from which $x = \frac{aa - bb + cc}{2c}$."

PRO-

PROBLEM VII.

Fig. 8.



" Q. *In the quadrantal arch* AM, *described with center* C, *and radius* AC, *the tangent* AI *of the arch* AH, *and also the tangent* HK *of the arch* HD, *are given*; R. I will make AC $= a$, AI $= b$, and HK $= c$. Q. *You are to find* AB, *the tangent of* AD, *which is the sum of those two arches.* R. I will make AB $= x$, and let fall the perpendiculars DF and DE; and then, from similar triangles, I shall have

$$CB\ (\sqrt{aa+xx})\ .\ AB\ (x)\ ::\ CD\ (a)\ .\ DE = \frac{ax}{\sqrt{aa+xx}}.$$

and $CB\ (\sqrt{aa+xx})\ .\ CA\ (a)\ ::\ CD\ (a)\ .\ CE = \dfrac{aa}{\sqrt{aa+xx}}.$

and $BC\ (\sqrt{aa+xx})\ .\ DC\ (a)\ ::\ AC\ .\ EC\ ::\ AI\ (b)\ .\ EO = \dfrac{ab}{\sqrt{aa+xx}}.$"

[and $BC\ (\sqrt{aa+xx})\ .\ DC\ (a)\ ::\ IC\ (\sqrt{aa+bb})\ .\ OC = \dfrac{a\sqrt{aa+bb}}{\sqrt{aa+xx}}.$

and $KC\ (\sqrt{aa+cc})\ .\ DC\ (a)\ ::\ KH\ (c)\ .\ DF = \dfrac{ac}{\sqrt{aa+cc}}.$

and $CE\ \left(\dfrac{aa}{\sqrt{aa+xx}}\right).\ CO\ \left(\dfrac{a\sqrt{aa+bb}}{\sqrt{aa+xx}}\right)\ ::\ DF\ \left(\dfrac{ac}{\sqrt{aa+cc}}\right).\ DO = \dfrac{c\sqrt{aa+bb}}{\sqrt{aa+cc}}.$

But $DO = DE - OE$; therefore I have $\dfrac{c\sqrt{aa+bb}}{\sqrt{aa+cc}} = \dfrac{ax-ab}{\sqrt{aa+xx}}$, an equation which differs from that in § 108, Sect. II, Book I, only in notation, and which therefore may be solved in the same manner.]

AN

AN INDEX,

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ERRATA.

E R R A T A.

Note. When the letter *b* is joined to the number of any line, it is counted from the bottom of the page.

VOLUME I.

In the Plan of the Lady's System of Analyticks.

Page.	Line.	
xl.	11.	*After the word* branch, *insert a comma.*

In the Body of the Work.

Page.	Line.	
41.	3.*b*.	*Dele* as.
125.	7.	*Instead of* 2*aaccx*, *read* 2*aacx*.
		And in the head-lines, on the right-hand pages, from p. 209 to p. 223, *instead of* Sect. IV., *read* Sect. V.

VOLUME II.

Page.	Line.	
9.		*In fig.* 11, *the perpendicular to* AC *is drawn from the point* G, *instead of* E.
11.		*The small letter* i *is wanting in fig.* 15.
15.	4.*b*.	*Instead of* each, *read* one the.
16.	9.	*Instead of* EG, read EF.
24.		*In the head-line, instead of* Book I., *read* Book II.
64.	7.*b*.	*After the letter* a, *instead of* —, *read* $=$.
113.		*Instead of art.* 9, *read* 10. N. B. All the articles from 9 to 22 are numbered too little by 1.

125.

Page.	Line.
125.	20. *Towards the end of the line, after the word* radius, *dele the comma; and instead of* adding, *read* added to.
189.	9.*b.* *After* =, *insert the letter* a.
205.	8. *Instead of* $\overset{t}{x}$, *read* $\frac{\overset{t}{x}}{t}$.
216.	6.*b.* *After* =, *instead of* a, *read* 1.
295.	13. *Instead of* in, *read* is.
317.	3.*b.* *Instead of* $\frac{\overline{y\dot{x} - xy}\vert^2}{xx + yy}$, *read* $\frac{\overline{y\dot{x} - x\dot{y}}\vert^2}{xx + yy}$.
339.	3. *Instead of* qx, *read* $q\dot{x}$.

N. B. The name of the city *Bologna* is in a few places printed *Bolonia*, as it was found in the Translator's Manuscript, but I take it to be erroneous.

EDITOR.

A LETTER

A LETTER

FROM

PHILALETHES CANTABRIGIENSIS.

Reprinted from the Gentleman's Magazine for November 1801.

IN the Gentleman's Magazine for November last, pages 997 and 998, is a Letter ſigned *Philalethes Cantabrigienſis*, the deſign of which is ſo laudable, that I gladly embrace this opportunity of contributing my mite to it by reprinting the Letter; conceiving that it cannot fail of the approbation of all the ſober and diſcerning part of mankind, and that, if the ſuggeſtions of it be duly attended to, it will prove very beneficial to thoſe who are of a different character, as well as to the public in general. EDITOR.

Dec. 10, 1801.

'Mr. URBAN, *Oct.* 7.

'THE following paſſage, taken from the preface to the fourth volume of the "Scriptores Logarithmici," lately publiſhed by Mr. Baron Maſeres, appears to be written with ſo benevolent a deſign, and points out

out to the Great objects ſo worthy of their attention, that I wiſh it were more generally known; and therefore ſhall be glad to ſee it in the Gentleman's Magazine.

'The paſſage begins in the ixth page of the preface, where, ſpeaking of Dr. James Wilſon's "Hiſtorical Diſſertation of the Riſe and Progreſs of the Modern Art of Navigation," the Baron ſays,'

"It is full of curious hiſtorical matter, and has ſuggeſted to my mind a wiſh that ſome perſon of affluence, fond of the ſubject of navigation, and who ſhould have been indebted to it, perhaps, for his rank or fortune, would cauſe a collection of all the authors on that ſubject, whoſe works are mentioned in this Diſſertation, to be made, and reprinted in a handſome manner in a ſet of quarto volumes, of the ſize of theſe volumes of the *Scriptores Logarithmici*, under the title of *Scriptores Nautici.* Such collections of learned tracts on particular ſubjects, under various titles ſuited to the ſeveral ſubjects of which they treated, would be very convenient in the preſent ſtate of ſcience; which is extended to ſuch a variety of ſubjects, and diſperſed in ſuch a number of different books, that it is very difficult and very expenſive for a perſon, fond of any particular branch of ſcience, to procure himſelf all the books that relate to it. Beſides the collection called *Scriptores Nautici*, relating to navigation, there might be a collection called *Scriptores Statici*, relating to the doctrine of *ſtaticks*, or bodies at reſt that form an equilibrium, or counterpoiſe to each other; under which head all the books of merit that treat of the *lever*, the *inclined plane*, and the other mechanical powers, would be comprized, and thoſe that treat of the catenary curve, and of the partial immerſion and the poſitions of bodies floating in liquids of greater ſpecifick gravity than themſelves, and of many other curious ſubjects of the like nature. And there might be another collection called *Scriptores Phoronomici*, relating to the doctrine of bodies in motion; under which head would be comprized Galileo's Mechanical Dialogues, of which the 3d and 4th contain the doctrine of the fall of heavy bodies to the earth with the law of their acceleration, and of their motion on inclined planes,

and of the motion of pendulums in circular arches, and of the motion of projectiles, which (abſtracting from the reſiſtance of the air,) would deſcribe parabolas; and under the ſame head would be comprized Mr. Huygens's tract on the motions of perfectly elaſtic bodies ſtriking againſt each other, and his admirable treatiſe *De Horologio Oſcillatorio*, or on the motion of a pendulum-clock, and his tract on central forces; and all Sir Iſaac Newton's moſt profound, but very difficult work, called the *Principia*, or *Mathematical Principles of Natural Philoſophy*, with the ſeveral commentators on it, and Herman's *Phoronomia*, and Euler's work *De Motu*. Another collection might relate to the finding the centres of gravity of different bodies; which is, I believe, a more ſubtle and difficult ſubject than is generally ſuppoſed. This collection might be called *Scriptores Centrobarici*. And another collection might conſiſt of all the writers on opticks, under the title of *Scriptores Optici*. This collection ſhould comprize the work of Euclid, or that which has been aſcribed to him, on this ſubject, and thoſe of Alhazen, and Vitellio, and Roger Bacon (the learned Engliſh monk), and *Antonio De Dominis*, and Willebrord Snell, and Des Cartes, and Huygens's Dioptricks, and his treatiſe *De Lumine*, and other works of his on the ſubject of opticks, and James Gregory's *Optica Promota*, and Dr. Barrow's *Lectiones Opticæ*, and Sir Iſaac Newton's *Lectiones Opticæ*, and his Treatiſe of Opticks, or Experiments on Light and Colours, and Molineux's Dioptricks, and Dr. Smith's Compleat Syſtem of Opticks, and Harris's Opticks, and many papers in the Philoſophical Tranſactions relating to the ſame ſubject. If ſuch ſeparate collections of authors were publiſhed, every perſon who was devoted to any particular branch of theſe ſciences, (and no man can attend to all of them, or even to many of them, with any great proſpect of becoming maſter of them,) might buy the collection which related to his particular branch at a moderate expence."

'On this occaſion I beg leave to make another remark or two.

'The importance of the art of navigation to this iſland, in times of peace as well as of war, is generally acknowledged; yet it may be juſtly doubted whether it has been encouraged here in a degree ſuitable to its

 importance,

importance, or equal to what it has received, in the laſt fifty years, from other nations; certainly not ſo as to excite equal emulation amongſt men of ſcience *. In ſupport of this aſſertion, I might enumerate the prizes which, from time to time, have been given by foreign academies for improvements in navigation and aſtronomy, and recount the learned tracts which have been produced in conſequence of that encouragement; but I ſhall at preſent wave this ſubject.

'In all civilized nations, arts and ſciences have been conſidered as making a part of the education of the Great, and as being under their patronage. Amongſt the men of rank in this country, in former ages, are to be found the names of *Napier*, *Bacon*, *Boyle*, *Newton*, *Macclesfield*, and *Stanhope*; men who excelled in ſcience, and patronized it in others. May I then be allowed to ſuggeſt to the nobility and gentry who, of late, have made a conſpicuous figure in *Weſtminſter-Hall*, and to all others of rank and fortune, who, although their names have not yet *graced* the columns of the London *news-papers*, are waſting their time and money in the ſeduction of the *wives* and *daughters* of their *friends*, or in other idle and vicious amuſements, that, if they would exchange thoſe vicious amuſements for the innocent and rational ones purſued by the men whoſe names I have mentioned, and, inſtead of ſquandering away thouſands on *courteſans*, lay out a few hundreds in printing ſuch *ſcientific tracts* as the worthy baron has mentioned, and in the ſupport of *Genius ſtruggling with poverty*, it would undoubtedly be much more

* I am aware of the rewards which have been offered by acts of parliament for the diſcovery of the longitude at ſea, and not unacquainted with the manner in which 20,000l. has been beſtowed.

for

for their present honour and future satisfaction, as well as for the good of mankind.'

'PHILALETHES CANTABRIGIENSIS.'

Omne animi vitium tanto conspectius in se
Crimen habet, quanto major, qui peccat, habetur.

— — — — — — — — — —

Tota licet veteres exornent undique ceræ
Atria, NOBILITAS *sola est atque unica* VIRTUS.

JUV.

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THE SOCIETY OF WRITERS TO THE SIGNET